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Directed percolation in the two-dimensional continuum

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We report the first study of directed percolation in the continuum. The percolation threshold is found to be at $B_c = 5.0 \pm 0.1$, where B_c is the average number of intersections per diode at the threshold. This is to be compared with $B_c = 3.2 \pm 0.1$ in the corresponding nondirected problem. It is found that the critical exponents of this system are $\nu_{\parallel} = 0.74 \pm 0.05$, $\nu_{\perp} = 0.46 \pm 0.08$, $\beta = 0.33 \pm 0.07$, and $\beta' = (2.00 \pm 0.05)\beta$. The good agreement with values found for directed lattices appears to be a confirmation of universality for these systems as well as a demonstration that geometrical and physical properties of directed systems in the continuum can be computed.

The multitude of interesting phase transitions and the variety of applications for mathematical, physical, chemical, and biological sciences has made directed percolation a very active field.[1,2] In directed percolation, unlike the nondirected case,[3] bonds on a lattice are randomly occupied by diodes rather than by resistors. The diodes are aligned in orientations which provide a preferred (or a "time") direction in the system. Two diodes are said to be connected if there is at least one continuous path between them such that in every step along the path one traverses only along the direction permitted by the diode. This implies that in directed percolation the clusters will be smaller than the ensemble of diodes touching each other and that (unlike nondirected percolation), the backbone probability is equal to the square of the percolating cluster probability.[4,5]

The study of directed percolation has been extended thus far to various lattices[6,7] and to lattices which contain diodes and resistors.[1,8] The percolation thresholds and the critical behavior of the percolating cluster have been studied for different dimensions,[9,10] while the critical behavior of the electrical conductance was limited to two dimensions.[4,5,11] As far as we know, no attempt was made to study directed percolation in the continuum, i.e., where the diodes are not put on a lattice but are randomly distributed in space. In such a case the diodes intersect each other anywhere along their extent (rather than touch at their ends) and the "flow" is allowed only along the part of the diode which is consistent with the diode direction. In addition to the unknown critical behavior of such a system, it appears that the continuum systems describe more closely some of the systems modeled previously by directed percolation on lattices. This applies to systems where the steps traversed are not equal in their length and to systems where the number of nearest neighbors is not fixed as in lattices. Examples include an epidemic spread in a wild forest rather than in a cultivated orchard,[12] a drainage area of river networks[13] and electrical, communication, and transport networks, which do not have the relatively simple "Manhattan" geometry.[1]

In this Rapid Communication we present the first attempt to study continuum-directed percolation. For this purpose we have carried out a Monte Carlo study of the simplest possible system of equal-length diodes implanted in a two-dimensional sector of unity radius. We have found the percolation threshold dependence on the length of the diodes and the critical behavior of the directed path (which starts at the origin), both below and above the critical diode concentration N_c. For diode concentration N for which $N < N_c$ we studied the characteristic lengths of directed animals,[10] $\xi_{\parallel}$ and $\xi_{\perp}$, while for $N > N_c$ we have studied the percolating path and backbone probabilities. A study of other geometrical and physical properties will be reported later.

The kind of sample used in the present work is shown in Fig. 1. For the sake of clarity we deliberately show a very small ($N = 13$) ensemble of diodes. The first and second diodes are intentionally put at the origin (the "source"), while the centers of all subsequent diodes are randomly selected in the sector. The orientations of the diodes are alternately vertical ("up") and horizontal ("right"). All diodes are of the same length L (in sector's radius units) and each diode is given a number according to the implantation sequence. The intersection of two diodes is very different than in lattices, not only because of the correlations

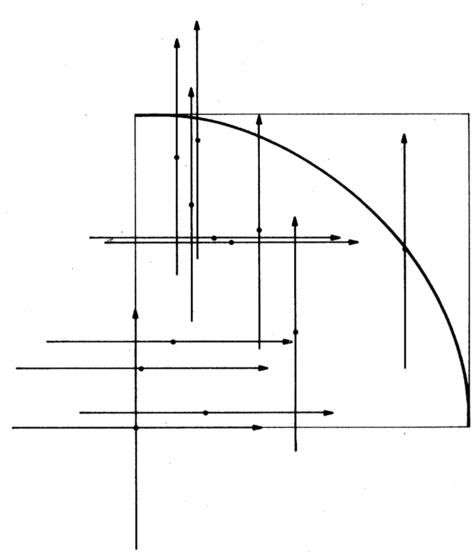

FIG. 1. A small ensemble illustration of the samples used in this study. Here $N = 13$, $N_p = 9$, and $N_b = 3$.

(if one views the continuum as a highly correlated lattice problem), but also because a diode here has components on both sides of an intersecting diode so that part of it may "conduct" while part of it may not.

Since the present problem has not been dealt with previously and since both the system and the algorithm used are quite different from the lattice ones, we sketch here the principle of the algorithm. A more detailed and graphical description will be presented later.

The first two diodes implanted in the origin are assumed to start the path. Let us assume that after many diodes are present in the system a new horizontal diode j is added. The intersection of this diode with each of the already present vertical diodes is checked (as in the nondirected case[14]). The diodes which intersect j, $i_1(j), \ldots, i_n(j)$, are recorded according to their left-to-right intersecting sequence. If all these diodes do not belong to the path, j is not added to the path (and another diode is thrown into the sample). If some of the above diodes belong to the path, we check which one of them makes j join the path (i.e., conducts into j). From all diodes which can make j join the path, we register the leftmost diode $i_p(j)$ and call it the "parent" of j. Now, j can make all the i diodes which it intersects, and which lie to the right of $i_p(j)$, belong to the path. For those which did not belong to the path previously, j becomes the parent, while for those which did belong, it is checked whether j replaces (i.e., lies below) their old parent. The updating process continues for each diode which gets a parent or has its parent changed until no more changes are recorded. A similar procedure is applied to vertical diodes. While the algorithm indicates a chainlike process, the computing time is not as long as it may seem. The reason is that in our largest samples the average number of intersections per diode at $N=2N_c$ is about 10 (the maximum per diode is less than 30) and the largest path size contained about 7000 diodes.

Once the diodes belonging to the path are known, one can "measure" the desired geometrical and physical quantities. For the check of the percolation threshold, we find for each diode joining the path whether, as a result of this, there is a diode which belongs to the path and crosses the boundary opposite to the origin (a segment of the arc in our case). If intersection occurs after N_c diodes have been thrown into the sector, we say that percolation is obtained and that N_c is the critical diode concentration. (The program continues, however, until the path is fully updated for the N_c diodes.) In the present work we considered the portion of the arc, which lies within $\pm 1°$ of the (1,1) direction, as the other end (or the "drain") of the percolating path. We have registered, then, N_c as well as B_c, the average number of intersections per diode[15] at the threshold. The longitudinal correlation length $\xi_\parallel$ was considered to be the distance between the farthest diode in the path and the origin, while the transverse correlation length $\xi_\perp$ was the average of the distances of the two diodes which are farthest from and on opposite sides of the (1,1) direction. We should remark that with the larger ensembles studied ($N \gtrsim 5000$), the $\xi_\parallel$ diode was essentially on the (1,1) direction and the $\xi_\perp$ diodes were symmetrically located about it. For finding the backbone we have considered the diodes which belong both to the path from the origin to the arc and from the arc to the origin when the preferred direction was reversed. Then, dangling diodes were eliminated by comparing the positions of each diode's parents in the two paths.

For illustration let us go back to Fig. 1 and examine the ensemble of $N=13$ diodes. While 12 diodes belong to the nondirected cluster, only $N_p=9$ belong to the directed path, or directed animal. Here, the backbone consists of $N_b=3$ diodes, and the percolation is achieved by the 13th diode ($N_c=13$), which in this case is the only diode which intersects the arc and belongs to the percolating-directed path.

Turning to the results, we have checked first the dependence of N_c on the diode length L and found that $N_c \propto L^{-2}$ as is to be expected from an exluded-area argument,[15] which was confirmed[14,15] for the nondirected problem. If this argument is correct, B_c should be a constant. Indeed (apart from finite scaling effects to be discussed elsewhere), we found that $B_c=5.0\pm0.1$ (for samples from $N_c \approx 5000$ to the largest samples studied of $N_c \approx 9000$). This value of B_c is higher than the $B_c=3.2$ value obtained for the corresponding nondirected case,[15] as is to be expected from the lower connectivity of the directed system.

The other property checked in the present study is the critical behavior of directed animals, i.e., the dependence of $\xi_\parallel$ and $\xi_\perp$ on N_p. This dependence is expected to yield, for $N < N_c$, the critical exponents $\nu_\parallel$ and $\nu_\perp$, derived from the fit of the data[16] to the power laws,

$$\xi_\parallel \propto N_p^{\nu_\parallel}, \quad \xi_\perp \propto N_p^{\nu_\perp} \; . \tag{1}$$

In order to obtain better statistics over the range of N_p studied, we have used three samples to derive the above exponent. The data shown in Fig. 2 were fitted to Eq. (1). The results found were

$$\nu_\parallel = 0.74 \pm 0.05$$

and

$$\nu_\perp = 0.46 \pm 0.08 \; .$$

These results are in good agreement with the results obtained on two-dimensional directed lattices[17–21] yielding the first indication that continuum-directed percolation and lattice-directed percolation belong to the same universality class.

In fact, the above values of $\nu_\parallel$ and $\nu_\perp$ deserve some dis-

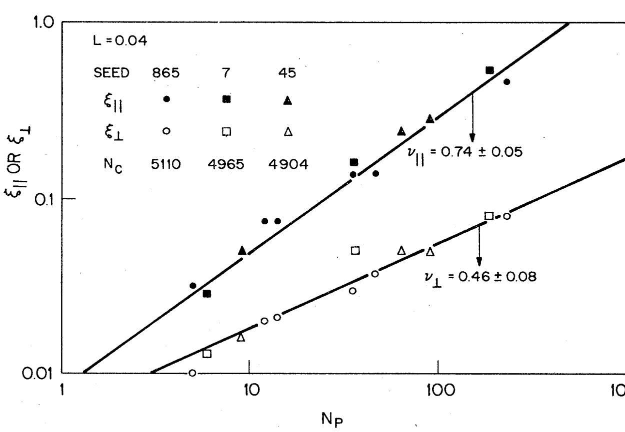

FIG. 2. The dependence of the directed animal length $\xi_\parallel$ and width $\xi_\perp$ on the number of diodes in the animal cluster, N_p.

cussion, since for directed lattices exact[18,20] ($\nu_{\parallel}^{LA}=\frac{9}{11}$ and $\nu_{\perp}^{LA}=\frac{1}{2}$) and highly accurate[20,21] values (e.g., $\nu_{\parallel}^{LA}=0.818$, $\nu_{\perp}^{LA}=0.498$) have been obtained. It may seem then that our $\nu_{\parallel}$ differs from $\nu_{\parallel}^{LA}$. In this context one has to realize that our error bars are associated only with the statistical errors of the data points and they do not account for the systematic errors associated with finite-size scaling.[1,21] The study of continuum systems requires much more computing time than the study of lattices and thus, in this preliminary study, we have used samples of rather limited size, and we did not follow the values of $\nu_{\parallel}$ and $\nu_{\perp}$ as a function of sample size. However, such a study has been carried out for lattices.[20] For the lattice model which is the most similar to our model (the directed bonds model–"model C" in Ref. 20), it was found that the value of $\nu_{\parallel}^{LA}$ increases asymptotically with sample size (strip width) from 0.79 towards the above-quoted $\nu_{\parallel}^{LA}$ value. In view of this there should be very little doubt that, within the accuracy of finite-size scaling, the $\nu_{\parallel}$ value and the $\nu_{\parallel}^{LA}$ value are the same.

The above suggestion of the lattice and the continuum belonging to the same class is strengthened when one considers the percolating path-size dependence on the departure from N_c above the directed percolation threshold. The expected dependence is

$$(N_p/N)\propto(N/N_c-1)^{\beta} \quad . \tag{2}$$

Here, a fit of the data obtained for the two samples (seeds), shown in Fig. 3, yielded the exponent

$$\beta=0.33\pm0.07$$

in good agreement with the accepted lattice value[6] of $\beta=0.28$.

As pointed out above, in directed percolation the backbone is made of diodes which belong to the origin-arc percolating path as well as to the arc-origin percolating path when the preferred direction is reversed. In our case the probability that a diode belongs to the path is N_p/N and the probability that it belongs to the backbone is N_b/N. Hence, one should expect that

$$(N_b/N)=(N_p/N)^2 \quad . \tag{3}$$

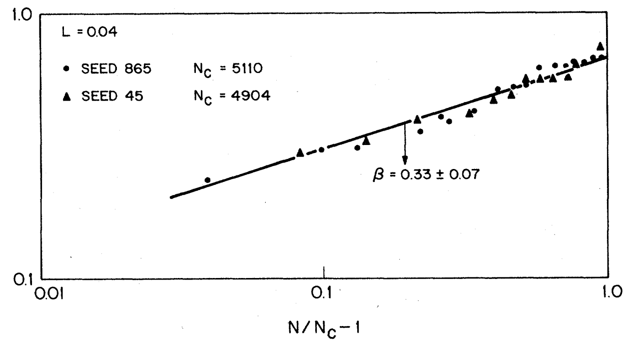

FIG. 3. The dependence of the percolating-path probability N_p/N on the departure from threshold, N/N_c-1.

We have indeed confirmed this relationship for the wide range of N, $0.013\leq N/N_c-1\leq0.75$, to a very high accuracy. The exponent in Eq. (3) is found to be 2.00 ± 0.05 for samples of $N_c>4000$. This confirmation means that the backbone exponent β' is just 2β as in the directed-lattice problem. One notes that the general property of directed percolation [Eq. (3)] is confirmed to a much higher accuracy than the value of β. This is not surprising, since for Eq. (3) the small relative fluctuations in N_p and N_b (over the above N/N_c-1 range) determine the uncertainty limits, while for Eq. (2) the very wide error bars on the $\log(N/N_c-1)$ scale (due to fluctuations in N_c, see, e.g., Fig. 2 and Ref. 16) cause the relatively large errors in the value of β. Our result shows then that the β'/β ratio is determined to a much higher accuracy than either β or β'. This kind of behavior is well known, and, for example, in nondirected percolation the determination[22,23] of γ/ν or t/ν is much more accurate than the determination[3] of γ or t.

In summary, we have studied a directed percolation problem in the continuum, defined its percolation threshold, computed its geometrical properties, and found that it belongs to the same universality class as the lattice-directed percolation.

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