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Localization and scattering of acoustic waves in a one-dimensional random system: A proposed experiment

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An experimental arrangement suitable for the observation of acoustic-wave localization in a system where the excitations are one dimensional is proposed. Explicit predictions about the outcome of such an experiment are made. Some useful information about third-sound dissipation can also be obtained.

The localization of acoustic waves in a two- or three-dimensional fluid containing a random distribution of hard-disk or hard-sphere scattering centers has been discussed by one of us.[1] First the acoustical boundary value problem was transformed into a multiple-scattering one by using standard techniques.[2,3] The resulting multiple scattering theory for sound-wave propagation was identical in structure to the theory of electron motion in a disordered solid. A discussion of sound-wave localization was then given by using the techniques developed by Vollhardt and Wölfle[4] for the electron problem.

Previously a field theory and renormalization-group formalism had been used by John, Sompolinsky, and Stephen[5] to describe the transition to localized phonon modes in a disordered elastic medium in $2+\epsilon$ dimensions.

The general conclusion of localization theory is that sound or acoustic waves are always localized in $d \leq 2$, regardless of the amount of disorder or of the impurity density. However, the effects of localization become important only when the localization length is smaller than the size of the experimental system. Recently, an experiment was proposed and analyzed for the observation of localization in a two-dimensional system.[6] The system considered was a superfluid helium film and the localized excitations were third-sound waves. Both here and in Ref. 6 the techniques used to describe sound-wave localization are identical to those given in Ref. 1. We know from the analogous electron problem[7] that localization effects are strongly enhanced by lowering the dimensionality; here, we propose an experimental arrangement suitable for the observation of wave localization in one dimension and make explicit predictions about the outcome of such an experiment.

The system we consider is also a superfluid ^{4}He film and the excitations are also third-sound waves.[8] The temperature is assumed to be low enough that the normal fluid component can be neglected. On the substrate we set up an array of thin parallel strips of another material whose van der Waals interaction with the film is as different as possible from that of the original substrate. The strips will be assumed to have a uniform width $2a$, while the distance between them is random. We only specify the strip density n (see Fig. 1). Our treatment allows for overlapping strip configurations.

The equilibrium thicknesses on the substrate, h_1, and on the strips, h_2, are related by equating the van der Waals potentials as seen by the film surfaces on both regions. To a good approximation, $h_2/h_1 = (\alpha_2/\alpha_1)^{1/3}$, where α_1 and α_2 are the van der Walls force constants corresponding to the original substrate and the strips, respectively. The unperturbed speed of sound C is the same in both regions.

The shape of the film profile in the boundary between two substrates has been studied by Cole and Vittoratos.[9] The region where the film thickness changes is very narrow ($\lesssim 100$ Å) for thin films. Consequently, for a reasonable strip width (say, $a \gtrsim 1$ μm), the waves essentially see a sudden change in film thickness. This allows us to derive boundary conditions on the strip sides and evaluate the transition matrix corresponding to the scattering of a wave by a single strip. Using well-known multiple-scattering techniques,[1-3,10,11] the complicated boundary value problem arising from considering the random distribution of strips is

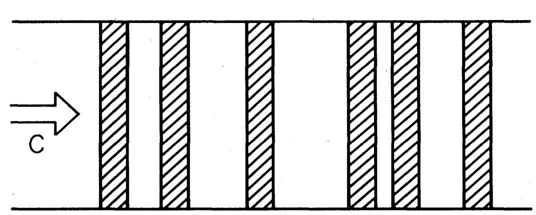

FIG. 1. Sketch of the random arrangement of parallel strips of a different substrate. A third-sound wave with speed C on the unperturbed substrate is coming from the left.

tansformed into a tractable many-body problem.

The calculation of the Green's function averaged over the random distribution of scatterers leads to explicit formulas for the renormalized speed of sound and the added dissipation due to the disordered substrate. On the other hand, the dissipation of third-sound waves on "clean" substrates[12-14] is not yet completely understood in spite of the several different mechanisms that have been considered.[15-20] By varying the size or density of the scatterers in an experiment like the one put forward here, it should be possible to describe quantitatively the contribution of substrate disorder to third-sound attenuation. Such a description would shed some light on the complex dissipation problem.

For reasons discussed elsewhere,[7] the analysis of localization effects requires the calculation of functions involving the averaged product of two Green's functions. We carry out this evaluation using a self-consistent diagrammatic theory, which will allow us to determine the localization length as a function of the physical parameters involved.

Of course, in a scattering experiment, the third-sound wave should be created in such a way that it propagates perpendicularly to the strips. Similarly, a one-dimensional localized excitation would have its "crests" and "valleys" parallel to the strips.

Under the conditions described above, the problem consists of solving the wave equation

$$\frac{\partial^2\phi}{\partial x^2}-\frac{1}{C^2}\frac{\partial^2\phi}{\partial t^2}=0 \quad , \tag{1}$$

for the velocity potential $\phi(x,t)$ ($v_s=\partial\phi/\partial x$, where v_s is the velocity of the superflow). The initial conditions can be taken to be identical to those in Refs. 1 and 5,

$$\phi(x,t\leqslant 0)=0 \quad , \tag{2}$$

$$\frac{\partial\phi}{\partial t}(x,t=0)=f(x) \quad , \tag{3}$$

for a given initial perturbation $f(x)$. The corresponding Green's function is defined by

$$\phi(x,t)=\int dx'\, G(x,t|x')f(x') \quad , \tag{4}$$

and it satisfies the differential equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{C^2}\frac{\partial^2}{\partial t^2}\right)G(x,t|x')=0 \quad , \tag{5}$$

together with the appropriate initial conditions. We have already commented on the validity of considering a sudden change in film thickness at the strip sides. Although the velocity potential is continuous, mass conservation requires a discontinuity in its first derivative at the locations $x=X_i\pm a$ of the strip sides, where the ith strip is supposed to be centered at X_i. Hence, the boundary conditions on $\phi(x,t)$ and $G(x,t|x')$ are, for the ith scatterer,

$$\begin{aligned}
&\phi(x,t)|_1=\phi(x,t)|_2 \ \text{ at } x=X_i-a \quad ,\\
&h_1\frac{\partial}{\partial x}\phi(x,t)|_1=h_2\left.\frac{\partial\phi(x,t)}{\partial x}\right|_2 \ \text{ at } x=X_i-a \quad ,\\
&\phi(x,t)|_2=\phi(x,t)|_3 \ \text{ at } x=X_i+a \quad ,\\
&h_2\frac{\partial}{\partial x}\phi(x,t)|_2=h_1\left.\frac{\partial}{\partial x}\phi(x,t)\right|_3 \ \text{ at } x=X_i+a \quad ,
\end{aligned} \tag{6}$$

where 1, 2, and 3 denote, respectively, the regions $x<X_i-a$, $X_i-a<x<X_i+a$, and $X_i+a<x$.

In the operator notation used in Ref. 1 the N-scatterer ($N\to\infty$) Green's function can be expressed in terms of an infinite series of $\hat{T}_i$ operators describing scattering from the ith scatterer,

$$\begin{aligned}\hat{G}_E^{\pm}=\hat{G}_{E,0}^{\pm}&+\sum_{i=1}^{N}\hat{G}_{E,0}^{\pm}\hat{T}_i^{\pm}(E)\hat{G}_{E,0}^{\pm}\\
&+\sum_{i\neq j}^{N}\hat{G}_{E,0}^{\pm}\hat{T}_i^{\pm}(E)\hat{G}_{E,0}^{\pm}\hat{T}_j^{\pm}(E)\hat{G}_{E,0}^{\pm}+\cdots \quad .\end{aligned} \tag{7}$$

Here the subscript 0 denotes the free Green's operator and ($\epsilon\to 0$)

$$G_E^{+}(x|x')=\int_0^{\infty}dt\, e^{i(E+i\epsilon)t}G(x,t|x')=\langle x|\hat{G}_E^{+}|x'\rangle \tag{8}$$

is the E-dependent Green's function in a coordinate representation, circumflexes denoting operators. In the momentum representation one has

$$\langle p|\hat{G}_E^{\pm}|p'\rangle=\frac{1}{2\pi}\int dx dx'\exp(-ipx+ip'x')\langle x|\hat{G}_E^{\pm}|x'\rangle \quad . \tag{9}$$

The transition matrices in Eq. (7) can be readily computed from Eqs. (5) and (6). The final result has the general form

$$\langle p|\hat{T}_i^{\pm}(E)|p_1\rangle=\exp[i(p_1-p)X_i]\langle p|\hat{T}^{\pm}(E)|p_1\rangle \quad , \tag{10}$$

where the X_i dependence appears only in the exponential factor. We give $\langle p|\hat{T}^{\pm}(E)|p_1\rangle$ only in the long-wavelength limit, which is the relevant case under most experimental conditions

$$\begin{aligned}\langle p|\hat{T}^{\pm}(E)|p_1\rangle\cong\, &p_1(ph_1-p_1h_2)\frac{a(h_1-h_2)C^2}{\pi h_1h_2}\\
&\pm iC(p_1a)(Ea)\frac{(ph_1^2+p_1h_2^2)}{\pi h_1^2h_2^2}(h_1-h_2)^2 \quad ,\end{aligned} \tag{11}$$

when pa, p_1a, $Ea/C\ll 1$. As for the general result, we only point out that it satisfies the relevant Ward identity,

$$\mathrm{Im}\langle p|\hat{T}^{\pm}|p\rangle=\pm(\pi/2EC)(|\langle p|\hat{T}^{\pm}|p\rangle|^2+|\langle p|\hat{T}^{\pm}|-p\rangle|^2)$$

for $p=E/C$ ("particle number conservation") and that, for N integer, $\langle N\pi/2a|\hat{T}^{\pm}|N\pi/2a\rangle=0$, this last result coinciding with the condition for a transparent barrier.

Since space is homogeneous on the average, the Green's function averaged over the random location of the scatterers (the averaging is denoted by $\langle\cdots\rangle_{\mathrm{av}}$) is diagonal in the wave-number representation and can be written in terms of a self-energy Σ as

$$\begin{aligned}\langle\langle p|\hat{G}_E^{\pm}|p_1\rangle\rangle_{\mathrm{av}}&=\delta(p-p_1)\langle G(p,E\pm i\epsilon)\rangle_{\mathrm{av}}\\
&=\delta(p-p_1)[C^2p^2-(E\pm i\epsilon)^2-\Sigma_p^{\pm}(E)]^{-1} \quad .\end{aligned} \tag{12}$$

The diagrams corresponding to the self-energy, which can be written as $\Sigma_p^{\pm}(E)=\Gamma_p(E)\pm i\sigma_p(E)$, are identical to those in Ref. 1. Taking into account only the leading contribution for $na<1$ yields $\Sigma_p^{\pm}(E)=2\pi n\langle p|\hat{T}^{\pm}(E)|p\rangle$;

the renormalized speed of sound is

$$\tilde{C}^2 = C^2 + 2\pi np^{-2}\,\mathrm{Re}\langle p|\hat{T}^{\pm}|p\rangle = C^2[1-2na(h_1-h_2)^2/h_1h_2] \quad . \qquad (13)$$

We observe that $\tilde{C} < C$, as expected, since interference in the scatterer generates a lag in the wave transmission. This renormalization effect should be measurable: a conservative estimate ($\alpha_1 = 2\alpha_2$, $na \sim 0.2$) gives a correction of the order of one percent. The imaginary part of the self-energy is, on the other hand,

$$\sigma_p(E) = \pm 2\pi n\,\mathrm{Im}\langle p|\hat{T}^{\pm}(E)|p\rangle = 2na^2\tilde{C}E\frac{(h_1-h_2)^2(h_1^2+h_2^2)}{(h_1h_2)^2}p^2 \quad . \qquad (14)$$

Carrying out the inverse Fourier transform of Eq. (2.12), we find an attenuation length which is $L_a = 2C^3p^2/E\sigma_p(E)$. The attenuation constant $\chi = L_a^{-1}$ can be evaluated from Eq. (14). For $\alpha_1 = 2\alpha_2$, $na \sim 0.2$, $\chi \approx a/\lambda^2$, where λ is the excitation wavelength. Note the strong effect of an increase in strip width. The attenuation constant has the same frequency dependence as the normal hydrodynamic attenuation constant. The frequency dependence is stronger for higher dimensionalities: for example, the two-dimensional model of Ref. 6 leads to $\chi \sim \lambda^{-3}$. Writing $C^2 = \alpha_1 h_1^{-3}$, we also see that the attenuation increases essentially with the cube of the equilibrium thickness, a behavior quite different from that predicted by the other proposed attenuation mechanisms and which should be easy to detect experimentally.[14]

As mentioned above, an analysis of localization effects involves the average of the squared Green's function. We define,

$$P_\omega(x|x') = \int_0^\infty dt\,\exp[i(\omega+i\epsilon)t]\,G^2(x,t|x') = \int_{-\infty}^{+\infty}\frac{dE}{2\pi}G^+_{E+\omega/2}(x|x')\,G^-_{E-\omega/2}(x|x') = \int_{-\infty}^{+\infty}\frac{dE}{2\pi}P_{E,\omega}(x|x') \quad . \qquad (15)$$

The quantity we calculate is

$$\langle P_E(k,\omega)\rangle_{\rm av} = \int d(x-x')\exp[-ik(x-x')]\langle P_{E,\omega}(x|x')\rangle_{\rm av} . \qquad (16)$$

$\langle P_E(k,\omega)\rangle_{\rm av}$ is the k and ω Fourier component of the average intensity resulting from the E Fourier component of a pulse excited at the origin. We call $\langle P_E(k,\omega)\rangle_{\rm av}$ the intensity propagator.

The intensity propagator can be calculated as in Ref. 1. In the long-wavelength ($k \to 0$) small frequency ($\omega \to 0$) limit one obtains

$$\langle P_E(k,\omega)\rangle_{\rm av} \simeq \frac{1}{2\tilde{C}^3}\left(\frac{\tilde{C}}{E}\right)^2\frac{1}{-i\omega+k^2D(E,\omega)} \quad . \qquad (17)$$

Physically Eq. (17) implies that the intensity of a pulse created at the origin diffuses in this random system with a frequency-dependent diffusion coefficient $D(E,\omega)$. The localization of third sound is described by the behavior of $D(E,\omega\to 0)$.

Following Vollhardt and Wölfle,[4] $D(E,\omega)$ can be calculated diagrammatically. Retaining the lowest-order (or Boltzmann) contribution and the maximally crossed diagrams important in the theory of electron localization, the self-consistent result is

$$D(E,\omega) = D_B(E)\left[1-\frac{\sigma_{E/\tilde{C}}(E)}{2\pi\tilde{C}^2}\frac{\tilde{C}}{E}\int_0^\infty dq\,\frac{1}{q^2-i\omega/D(E,\omega)}\right] \quad . \qquad (18)$$

Here $D_B(E)$ is the lowest-order contribution:

$$D_B(E) = \frac{\tilde{C}}{2na^2}\left(\frac{\tilde{C}}{E}\right)^2\left(\frac{h_1h_2}{h_1^2-h_2^2}\right)^2 \quad . \qquad (19)$$

Although we have used different boundary conditions, the frequency (E) and scatterer-size dependence of $D_B(E)$ follow the same rule as in the case of hard-core scatterers in higher dimensions:[1] $D_B(E) \sim n^{-1}a^{-2d}\tilde{C}(\tilde{C}/E)^{d+1}$. We also note a power-law divergence, $D_B(E) \sim (h_1-h_2)^{-2}$, when $h_1 \to h_2$, indicating that in the absence of impurities $\langle P_E(k,\omega)\rangle_{\rm av}$ is no longer diffusive.

The fact that one-dimensional excitations are always localized in the presence of impurities is reflected in the second term in Eq. (18). The only solution to Eq. (18) as $\omega \to 0$ is $D(E,\omega) = -i\omega\xi^2(E) + 0(\omega^2)$. Inserting it in Eq. (18) yields (in space-time language) an intensity propagator that is constant in time for long times and localized in a distance ξ. Consequently, ξ is called the localization length. It can be explicitly determined from Eq. (18). We obtain

$$\xi(E) \cong \frac{4}{n}\left(\frac{\tilde{C}}{Ea}\right)^2\left(\frac{h_1h_2}{h_1^2-h_2^2}\right)^2 \quad . \qquad (20)$$

The frequency dependence of the localization length is identical to that predicted by John, Sompolinsky, and Stephen.[5] We note that $\xi \sim (\delta V_s)^{-2}$, where $\delta V_s = [2a|h_1-h_2|]$ is the excess volume of ^{4}He on the strip per unit width. To get reasonably short localization lengths, it is necessary to use relatively high frequencies ($\gtrsim 2\times 10^5$ Hz) and low speeds of sound ($\lesssim 10^3$ cm/sec) (which means the film cannot be too thin), together with strips which are not too narrow ($a \gtrsim 10^{-3}$ cm). The use of nuclepore, a substrate on which C has been shown to be considerably reduced[21] should be considered. For thick films, surface tension causes the film profile at the edge of the strip to change slowly[9] and thus our approximation of considering a sudden change in film thickness is no longer strictly correct. However, we believe that Eq. (20) would still give a good description of localization, since (i) we are still in a long-wavelength region and $Ea/C \ll 1$ for a wide range in the parameters, and (ii) δV_s will be essentially conserved, even if its spatial distribution is modified.

We conclude with some remarks.

(1) The main results of this paper are given by Eqs. (13), (14), and (20). Equations (13) and (14) give the renormalization to the speed of third sound and attenuation constant in the average Green's functions. For experimental situations where the localization length is very large, the averaged Green's function adequately describes the propagation of third sound. Equation (20) gives the localization length in terms of known quantities. When ξ is less than the size of the system, the third-sound excitation is localized and does not propagate. The localization effect is stronger than in higher dimensionalities and we estimate that it is experimentally observable, as are the renormalizations mentioned above.

(2) Generazio and co-workers[22,23] have recently studied experimentally the effects that an isolated "groove" or "edge" has on the propagation and attenuation of third-sound waves on an otherwise smooth substrate. This suggests the possibility of considering a collection of parallel, identical grooves distributed at random distances from each other on a given substrate. Since the isolated scatterer effects can be computed by the methods used in Ref. 23, it should be possible to analyze quantitatively the modifications introduced by the disorder in the distance between irregularities. The individual groove modifies the behavior of the film through a thickening due to capillary condensation. As for the result of a third-sound experiment carried out on a substrate with a random distribution of grooves, we expect the $(nE^2)^{-1}$ dependence of the localization length to be universal for all one-dimensional acoustic random systems. Furthermore, we expect $\xi \sim (\delta V_g)^{-2}$, where δV_g is the excess volume of ^{4}He in the groove per unit width. Since, typically,[23] $\delta V_g \gg \delta V_s$, grooved substrates should be extremely good candidates for short localization lengths.

(3) It is easy to create a flow in the superfluid film. It would be interesting to study how a flow pattern affects the localization properties, both from an experimental and a theoretical point of view.

(4) In the theory presented here we have neglected intrinsic dissipation which is analogous to inelastic scattering in the electron problem. Although dissipation destroys localization on a very long time scale, we estimate that under usual experimental conditions it can be neglected in the third-sound experiments we are proposing.

For our one-dimensional system an exact analysis of localization may be possible using the ideas of Berezinskii.[24] However, on the basis of its success with the electron localization theory[4] we believe that the self-consistent formalism we have used here, being far more tractable, gives a correct description of the problem.

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