Equation of state for the infinite cluster and backbone in the anisotropic square lattice

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We use a real-space renormalization-group procedure recently developed for calculating equations of state for geometrical problems, to treat bond percolation in the anisotropic square lattice. By choosing a convenient self-dual cluster, we calculate, *for all values of the occupancy probabilities* p_x and p_y (along the x and y axes, respectively), the order parameters $P_\infty(p_x,p_y)$ and $P_\infty^B(p_x,p_y)$, respectively, associated with the complete percolating infinite cluster and with its backbone. An interesting difference appears between these two quantities whenever one of the occupancy probabilities, for example p_y, equals unity: $\lim_{p_y\to 1}P_\infty(p_x,p_y)$ is *discontinuous* at $p_x=0$ (where P_∞ jumps from 0 to 1), whereas $\lim_{p_y\to 1}P_\infty^B(p_x,p_y)$ *continuously increases* from 0 to 1 when p_x increases from 0 to 1. Through a convenient extrapolation procedure which includes the use of the best available values for the critical exponents β and β^B, we obtain values for P_∞ and P_∞^B which are believed to be numerically quite reliable. In particular, $P_\infty(p,p)\sim A(p-\frac{1}{2})^\beta$ ($\beta=\frac{5}{36}$ and $A\simeq 1.25$) and $P_\infty^B(p,p)\sim A^B(p-\frac{1}{2})^{\beta^B}$ ($\beta^B\simeq 0.53$ and $A^B\simeq 1.92$).

I. INTRODUCTION

The real-space renormalization-group (RG) techniques are employed mainly to evaluate critical points (or more generally critical frontiers) and exponents. However, in principle, nothing precludes their use to calculate various thermodynamical quantities (free energy, specific heat, order parameter, susceptibility, etc.) for the *entire range* of the external parameters (typically temperature). As an example, we can mention the work by Niemeijer and van Leeuwen[1] where a RG formalism is developed for calculating several thermodynamical quantities.

Recently, some of us[2] have developed a RG procedure which provides equations of state corresponding to statistical geometrical problems (not necessarily related to Hamiltonian formalisms). The approach is as simple as a mean-field approximation, preserving nevertheless the criticality of the problem. It has been used for calculating the order parameters (site mass density) for the complete percolating infinite cluster and for its backbone, noted $P_\infty(p)$ and $P_\infty^B(p)$, respectively, for all values of the bond occupancy probability p on an isotropic square lattice.

The aim of the present paper is to extend the above type of treatment to the anisotropic square lattice for arbitrary occupancy probabilities p_x and p_y (along the x and y axes, respectively). By using a convenient self-dual cluster (see Ref. 3 and references therein) we calculate $P_\infty(p_x, p_y)$ and $P_\infty^B(p_x, p_y)$. In Sec. II we present the RG formalism and the results; in Sec. III we apply a quite efficient extrapolation procedure (first introduced[4] to improve RG results for surface tension in Ising systems) to obtain numerically reliable results for P_∞ and P_∞^B; we finally conclude in Sec. IV.

II. FORMALISM AND RESULTS

We consider a square lattice whose bonds are randomly and independently occupied (or "active") with probability p_x for the x axis and p_y for the y axis. To construct the RG recursive relations in the (p_x,p_y) space we adopt the treatment[3,5] based on the self-dual clusters indicated in Fig. 1, which have proved to be extremely performative for the anisotropic square lattice. Note in Fig. 1(c) the existence in the cluster of two entry points and two exit points; if the configuration is a spanning one, it will simulate the infinite percolating cluster and all four entry and/or exit points will be considered to belong to it. The cluster of Fig. 1(a) [Fig. 1(c)] contains one (nine) relevant bonds, and presents consequently 2 (2^9) different occupancy configurations. The analysis of these configurations shows that only half of them percolate, and are responsible for the following RG recursive relations:[3,5]

$$\begin{aligned}p_x' =& p_x^5p_y^4+4p_x^5p_y^3(1-p_y)+6p_x^5p_y^2(1-p_y)^2+4p_x^5p_y(1-p_y)^3+p_x^5(1-p_y)^4+5p_x^4(1-p_x)p_y^4+20p_x^4(1-p_x)p_y^3(1-p_y)\\&+27p_x^4(1-p_x)p_y^2(1-p_y)^2+14p_x^4(1-p_x)p_y(1-p_y)^3+2p_x^4(1-p_x)(1-p_y)^4+10p_x^3(1-p_x)^2p_y^4\\&+36p_x^3(1-p_x)^2p_y^3(1-p_y)+40p_x^3(1-p_x)^2p_y^2(1-p_y)^2+14p_x^3(1-p_x)^2p_y(1-p_y)^3\\&+P_x^3(1-p_x)^2(1-p_y)^4+9p_x^2(1-p_x)^3p_y^4+26p_x^2(1-p_x)^3p_y^3(1-p_y)+20p_x^2(1-p_x)^3p_y^2(1-p_y)^2\\&+4p_x^2(1-p_x)^3p_y(1-p_y)^3+3p_x(1-p_x)^4p_y^2(1-p_y)^2+6p_x(1-p_x)^4p_y^3(1-p_y)+3p_x(1-p_x)^4p_y^2(1-p_y)^2\equiv f(p_x,p_y)\end{aligned}\tag{1}$$

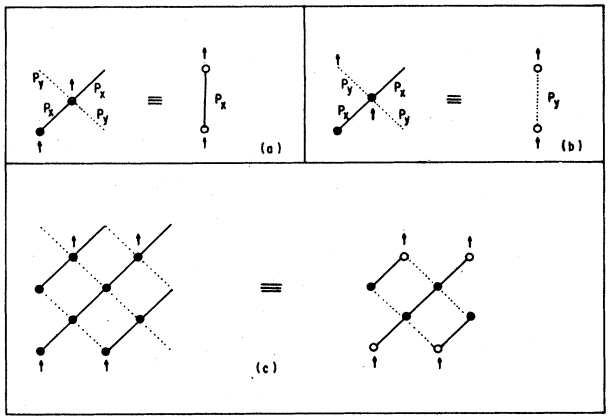

FIG. 1. Self-dual cells (the arrows represent the entrances to and exits from the cell). The closed circles (open circle) denotes internal (terminal) site.

and

$$p_y' = f(p_y, p_x) \ . \tag{2}$$

In Eq. (2) we have taken into account the $p_x \rightleftarrows p_y$ symmetry of the problem [see Fig. 1(b)]. The flow diagram determined by Eqs. (1) and (2) is shown in Fig. 2. The *exact* critical line[6] $p_x + p_y = 1$ is recovered, as well as the correct universality classes (one-dimensional percolation for $p_x = 0$ or $p_y = 0$, and two-dimensional percolation otherwise). The "correlation" length critical exponent ν corresponding to anisotropic square lattice is given by

$$\nu = \frac{\ln b}{\ln \lambda} = \frac{\ln 2}{\ln(249/2^7)} \simeq 1.042 \ , \tag{3}$$

where $b=2$ is the RG linear scale factor and $\lambda \equiv [df(p,p)/dp]_{p=1/2}$.

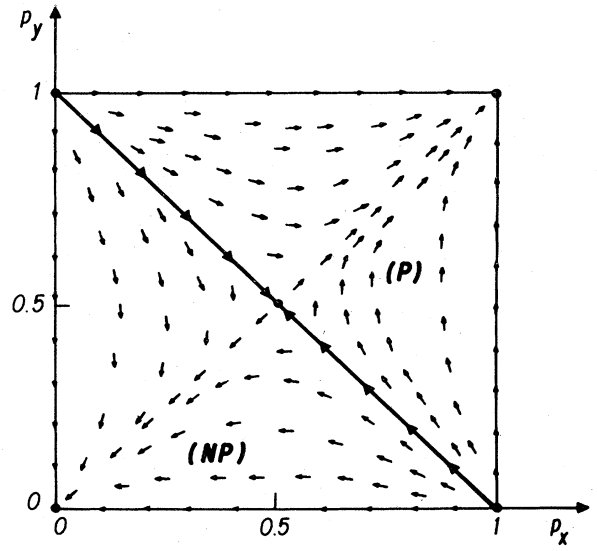

FIG. 2. RG flow diagram. P (NP) denotes the percolating (nonpercolating) phase.

The percolation "order parameter" is defined as follows:

$$P_\infty(p_x, p_y) \equiv \lim_{L \to \infty} \frac{N_L(p_x, p_y)}{L^2} \ , \tag{4}$$

where L is the (dimensionless) linear size of a (finite) square lattice, and N_L is the average number of sites which belong to the biggest cluster (which, in the $L \to \infty$ limit, generates the unique infinite percolating cluster). Consequently P_∞ is the probability of randomly choosing a site which belongs to the infinite cluster. Following along the lines of Ref. 2, we associate a dimensionless "mass" m_0 with each site. The parameter m_0 changes under renormalization as follows:

$$m_0' = g(p_x, p_y) m_0 \ , \tag{5}$$

where $g(p_x, p_y)$ satisfies $g(1,1) = b^d$ (d is the dimension and equals 2 in our case) and has to be established. The order parameter we are looking for is given by[2]

$$P_\infty(p_x, p_y) = \lim_{n \to \infty} \frac{m_0^{(n)}(p_x, p_y)}{b^{nd} m_0} \ , \tag{6}$$

where $m_0^{(n)}(p_x, p_y)$ is the nth iterated mass value, through Eqs. (1), (2), and (5), starting from arbitrary values for m_0, p_x, and p_y. In short, the procedure consists in choosing, for given p_x and p_y such that $p_x + p_y \geq 1$, an arbitrary initial value for m_0 (e.g., $m_0 = 1$), and then performing the recurrence determined by Eqs. (1), (2), and (5) up to arrival to the fixed point $(p_x, p_y, m_0) = (1, 1, m_0^{(\infty)})$ [which is always warranted by the fact that $g(1,1) = b^d$]: $P_\infty(p_x, p_y)$ is proportional to $m_0^{(\infty)}$. For p_x and p_y such that $p_x + p_y \leq 1$, the procedure automatically yields $P_\infty = 0$.

Let us now determine $g(p_x, p_y)$. We impose[2] the average mass of the spanning cluster to be preserved through renormalization. The set of configurations of cluster of Fig. 1(a) provides $2 m_0' p_x'$. With respect to the cluster of Fig. 1(c), two typical configurations are indicated in Figs. 3(a) and 3(b), and their respective contributions are $7 m_0 p_x^3 (1-p_x)^2 p_y^3 (1-p_y)$ and $6 m_0 p_x^3 (1-p_x)^2 p_y^2 (1-p_y)^2$. When all the configurations are taken into account, the preservation of the average mass yields

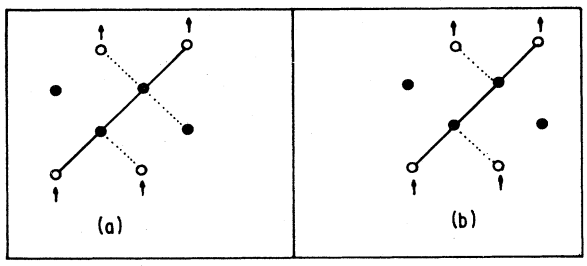

FIG. 3. Two typical configurations appearing in the calculation of the function $g(p_x, p_y)$ in Eq. (5).

$$2m'_0p'_x=[8p_x^5p_y^4+32p_x^5p_y^3(1-p_y)+48p_x^5p_y^2(1-p_y)^2+32p_x^5p_y(1-p_y)^3+8p_x^5(1-p_y)^4+40p_x^4(1-p_x)p_y^4$$
$$+158p_x^4(1-p_x)p_y^3(1-p_y)+210p_x^4(1-p_x)p_y^2(1-p_y)^2+106p_x^4(1-p_x)p_y(1-p_y)^3+14p_x^4(1-p_x)(1-p_y)^4$$
$$+80p_x^3(1-p_x)^2p_y^4+280p_x^3(1-p_x)^2p_y^3(1-p_y)+298p_x^3(1-p_x)^2p_y^2(1-p_y)^2+96p_x^3(1-p_x)^2p_y(1-p_y)^3$$
$$+6p_x^3(1-p_x)^2(1-p_y)^4+72p_x^2(1-p_x)^3p_y^4+194p_x^2(1-p_x)^3p_y^3(1-p_y)+134p_x^2(1-p_x)^3p_y^2(1-p_y)^2$$
$$+24p_x^2(1-p_x)^3p_y(1-p_y)^3+24p_x(1-p_x)^4p_y^4+40p_x(1-p_x)^4p_y^3(1-p_y)+18p_x(1-p_x)^4p_y^2(1-p_y)^2]m_0$$
$$\equiv h(p_x,p_y)m_0 \ . \tag{7}$$

Comparison with Eq. (5) and use of Eq. (1) provide

$$g(p_x,p_y)=h(p_x,p_y)/2f(p_x,p_y) \ , \tag{8}$$

the formalism being thus closed. The fact that for establishing Eq. (7) we have used the clusters of Figs. 1(a) and 1(c) destroys, strictly speaking, the $p_x \rightleftarrows p_y$ symmetry of the square lattice (the same problem would of course arise if we were to privilege the y axis instead of the x axis). The numerical discrepancies are, however, practically neglected over the entire range of p_x and p_y. The RG flow determined by Eqs. (1), (2), and (5) [with Eq. (8)] is illustrated, for $p_x=p_y$, on Fig. 4. The results obtained are indicated in Table I (upper row) and Fig. 5. The critical exponents (occurring along the entire two-dimensional critical line $p_x+p_y=1$) and critical amplitudes (occurring at $p_x=p_y\rightarrow\frac{1}{2}$) are indicated in Table II.

Let us now focus the backbone of the infinite percolating cluster (all dangling bonds or sets of bonds are to be eliminated). The corresponding order parameter $P_\infty^B(p_x,p_y)$ is calculated, within the present RG procedure, in precisely the same manner as for $P_\infty(p_x,p_y)$ but substituting Eq. (7) by the following one:

$$2m_0^{B'}p'_x=[8p_x^5p_y^4+30p_x^5p_y^3(1-p_y)+42p_x^5p_y^2(1-p_y)^2+26p_x^5p_y(1-p_y)^3+6p_x^5(1-p_y)^4+38p_x^4(1-p_x)p_y^4$$
$$+142p_x^4(1-p_x)p_y^3(1-p_y)+178p_x^4(1-p_x)p_y^2(1-p_y)^2+86p_x^4(1-p_x)p_y(1-p_y)^3+12p_x^4(1-p_x)(1-p_y)^4$$
$$+70p_x^3(1-p_x)^2p_y^4+234p_x^3(1-p_x)^2p_y^3(1-p_y)+246p_x^3(1-p_x)^2p_y^2(1-p_y)^2+84p_x^3(1-p_x)^2p_y(1-p_y)^3$$
$$+6p_x^3(1-p_x)^2(1-p_y)^4+58p_x^2(1-p_x)^3p_y^4+158p_x^2(1-p_x)^3p_y^3(1-p_y)+120p_x^2(1-p_x)^3p_y^2(1-p_y)^2$$
$$+24p_x^2(1-p_x)^3p_y(1-p_y)^3+18p_x(1-p_x)^4p_y^4+36p_x(1-p_x)^4p_y^3(1-p_y)+18p_x(1-p_x)^4p_y^2(1-p_y)^2]m_0^B \ . \tag{9}$$

TABLE I. RG values of P_∞ (upper value) and P_∞^B (lower value) for typical (p_x,p_y). ? refers to the fact that this value is not uniquely determined.

p_y \ p_x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	0	0	0	0	0	0	0	0	0	?
	0	0	0	0	0	0	0	0	0	0	0
0.1	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0.564
0.2	0	0	0	0	0	0	0	0	0	0.936	1
	0	0	0	0	0	0	0	0	0	0.540	0.674
0.3	0	0	0	0	0	0	0	0	0.907	0.964	1
	0	0	0	0	0	0	0	0	0.527	0.659	0.747
0.4	0	0	0	0	0	0	0	0.891	0.945	0.977	1
	0	0	0	0	0	0	0	0.519	0.649	0.736	0.802
0.5	0	0	0	0	0	0	0.884	0.935	0.964	0.984	1
	0	0	0	0	0	0	0.514	0.642	0.728	0.795	0.847
0.6	0	0	0	0	0	0.882	0.930	0.957	0.975	0.989	1
	0	0	0	0	0	0.510	0.637	0.722	0.788	0.842	0.885
0.7	0	0	0	0	0.886	0.932	0.956	0.972	0.983	0.992	1
	0	0	0	0	0.509	0.633	0.717	0.783	0.836	0.881	0.919
0.8	0	0	0	0.899	0.940	0.961	0.974	0.983	0.990	0.995	1
	0	0	0	0.511	0.633	0.715	0.779	0.831	0.877	0.915	0.948
0.9	0	0	0.928	0.960	0.974	0.982	0.988	0.992	0.995	0.997	1
	0	0	0.519	0.636	0.715	0.777	0.828	0.872	0.911	0.946	0.975
1	?	1	1	1	1	1	1	1	1	1	1
	0	0.538	0.647	0.720	0.779	0.827	0.870	0.907	0.942	0.973	1

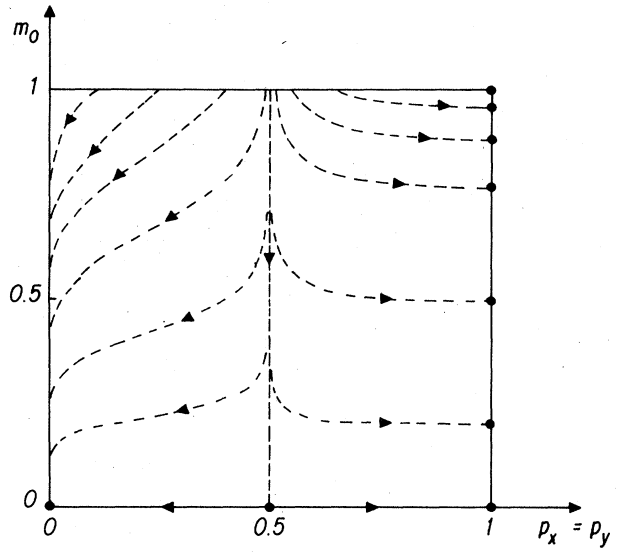

FIG. 4. Flow diagram in the m_0-p space ($p_x=p_y=p$). The dots indicate the values of the order parameter $P_\infty(p)$, where p corresponds to the starting point of the flux line.

The results we obtain are indicated in Table I (lower row) and Fig. 6; the two-dimensional critical exponents and critical amplitudes (in the limit $p_x=p_y\to\frac{1}{2}$) are indicated in Table II.

III. EXTRAPOLATION PROCEDURE

In order to numerically improve RG results for the surface tension of the Ising model, some of us developed[4] an extrapolation technique ["single extrapolation procedure"

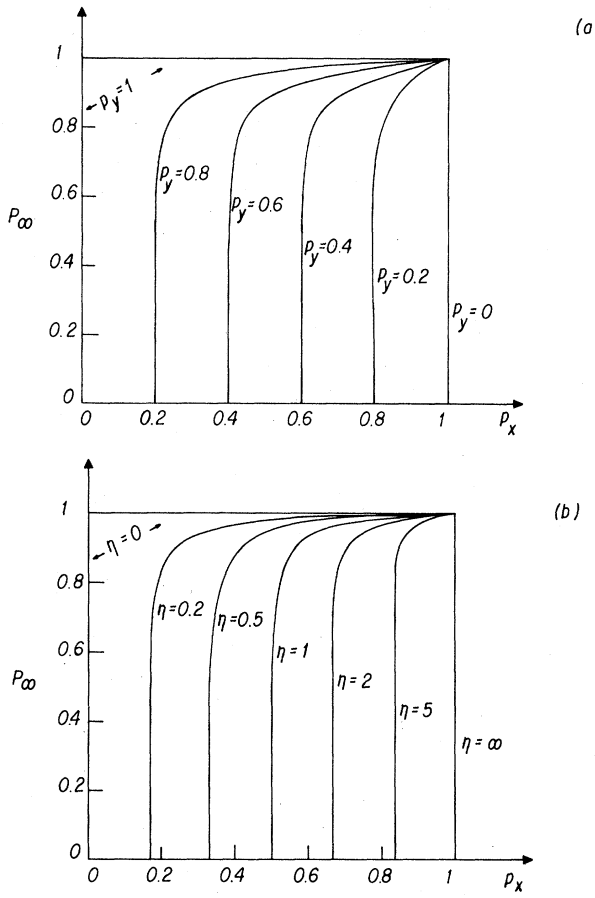

FIG. 5. Typical sections of the RG order parameter $P_\infty(p_x,p_y)$: (a) fixed p_y; (b) fixed $\eta\equiv(1-p_y)/(1-p_x)$.

TABLE II. Present RG and extrapolation, Tsallis *et al.* RG (Ref. 2), and exact or Monte Carlo results for the critical exponents (ν,β,β^B) and amplitudes (A,A^B) associated with the quantities P_∞ and P_∞^B (see the text).

	ν	β	A	β^B	A^B
Present RG ($b=2$)	$\frac{\ln 2}{\ln(249/2^7)}\simeq 1.042$[a]	$\frac{\ln(2^{10}/961)}{\ln(249/2^7)}\simeq 0.095$	1.19	$\frac{\ln(2^8/205)}{\ln(249/2^7)}\simeq 0.334$	1.41
Other RG	1.428 ($b=2$) 1.380 ($b=3$) 1.305 ($b=\frac{3}{2}$)	0.428 ($b=2$) 0.338 ($b=3$) 0.198 ($b=\frac{3}{2}$)	2.09 ($b=2$)[b] 1.84 ($b=3$) 4.0 ($b=\frac{3}{2}$)	0.550 ($b=2$)	2.45 ($b=2$)[b]
Present extrapolation	$\frac{4}{3}$[c]	$\frac{5}{36}$[c]	1.25	0.53[c]	1.92
Exact or Monte Carlo	$\frac{4}{3}\simeq 1.333$[d]	$\frac{5}{36}\simeq 0.139$[e]		0.53[f]	

[a]Reference 3 and references therein.
[b]Results obtained from Ref. 2.
[c]These (exact or almost exact) values are input (and not output) within the present extrapolation procedure.
[d]Exact: Ref. 7.
[e]Exact: Ref. 8.
[f]Monte Carlo: Ref. 9.

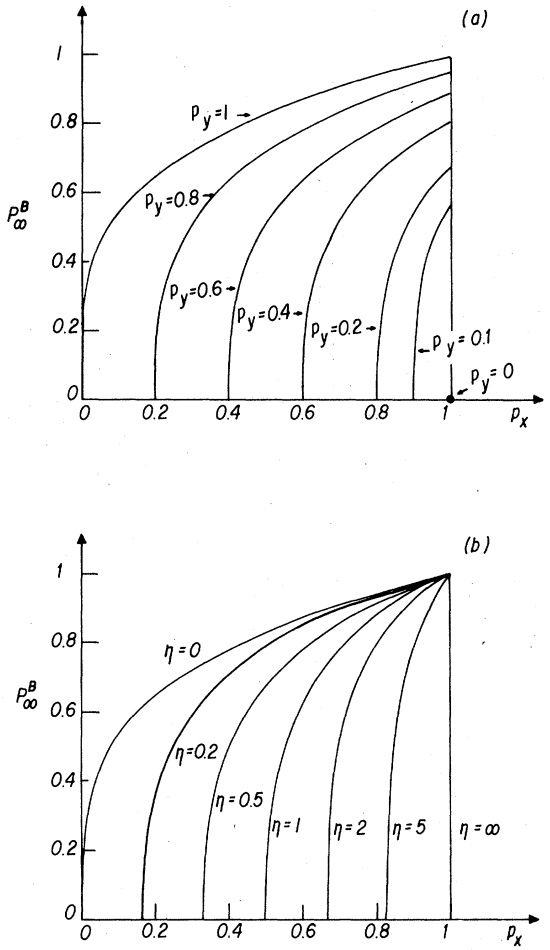

FIG. 6. Typical sections of the RG order parameter $P_\infty^B(p_x,p_y)$: (a) fixed p_y; (b) fixed $\eta\equiv(1-p_y)/(1-p_x)$.

(SEP) in Ref. 4] which proved to be quite efficient. We intend to apply here the same technique for improving the RG results for $P_\infty(p)$ and $P_\infty^B(p)$. The procedure uses, as input, the exact (or almost exact) values for the critical point (p_c), the critical exponents ν and β (or β^B) as well as that of the slope of $P_\infty(p)$ [or $P_\infty^B(p)$] at $p=1$. Its central basis is that the extrapolation should be "soft" (*polynomial* correction) if rescaled "natural" variables are introduced in the problem. More specifically the asymptotic equation $P_\infty\propto(p-p_c)^\beta$ can be rewritten as $y\propto x$ with $x\propto(p-p_c)^\nu$ and $y\propto P_\infty^{\nu/\beta}$, which defines the natural variables just mentioned (note that x and y are variables which currently appear in standard finite-size scalings). It is in the (x,y) space that the polynomial extrapolation will be performed. Let us describe it in detail. We define

$$x_b(p)\equiv\left[\frac{p-p_b}{1-p_b}\right]^{\nu_b}, \tag{10}$$

$$x(p)\equiv\left[\frac{p-p_c}{1-p_c}\right]^{\nu}, \tag{11}$$

$$y_b\equiv(P_{\infty b})^{\nu_b/\beta_b}, \tag{12}$$

$$y\equiv(\tilde{P}_{\infty b})^{\nu/\beta}, \tag{13}$$

where p_b, ν_b, β_b, and $P_{\infty b}$ are, respectively, the critical point, "correlation" length critical exponent, order-parameter critical exponent, and order parameter obtained within a RG which uses clusters corresponding to a linear scale factor b; p_c, ν, and β are the *exact* (or best available) values for the corresponding parameters (hopefully $\lim_{b\to\infty}p_b=p_c$, $\lim_{b\to\infty}\nu_b=\nu$, and $\lim_{b\to\infty}\beta_b=\beta$); $\tilde{P}_{\infty b}$ is the extrapolated order parameter we are looking for (hopefully $\lim_{b\to\infty}P_{\infty b}=\lim_{b\to\infty}\tilde{P}_{\infty b}=P_\infty^{\text{exact}}$). x_b and x vary from 0 to 1 when p varies from the critical probability to 1. We introduce the following relation:

$$y=f_b(x)y_b, \tag{14}$$

where the *correction function* $f_b(x)$ has to be found. For $p=1$ (hence $x=1$), both y and y_b are exact and equal to unity, therefore $f_b(x)$ satisfies

$$f_b(1)=1. \tag{15}$$

Furthermore, relation (14) implies

$$\frac{\nu}{\beta}\left.\frac{d\tilde{P}_{\infty b}}{dp}\right|_{p=1}=\left.\frac{df_b(x)}{dx}\right|_{x=1}\frac{\nu}{1-p_c}+\frac{\nu_b}{\beta_b}\left.\frac{dP_{\infty b}}{dp}\right|_{p=1} \tag{16}$$

hence

$$\left.\frac{df_b(x)}{dx}\right|_{x=1}=\frac{1-p_c}{\nu}\left[\frac{\nu}{\beta}C-\frac{\nu_b}{\beta_b}C_b\right] \tag{17}$$

with

$$C_b\equiv(dP_{\infty b}/dp)_{p=1} \tag{18}$$

and

$$C\equiv(d\tilde{P}_{\infty b}/dp)_{p=1}=(dP_\infty^{\text{exact}}/dp)_{p=1}, \tag{19}$$

where we have imposed that the slope of $\tilde{P}_{\infty b}$ at $p=1$ equals the *exact* one (currently known, through simple arguments, for the particular lattice under analysis). Moreover, the correcting function $f_b(x)$ has been introduced mainly to redress the possibly wrong slope of $P_{\infty b}$ at $p=1$ [$f_b(x)=1\forall x$, if the exact slope is reproduced by the RG], and we want its effects to gradually relax while approaching p_c (hence $x=0$); it seems therefore sensible to demand

$$\left.\frac{df_b(x)}{dx}\right|_{x=0}=0. \tag{20}$$

The simplest function which simultaneously satisfies conditions (15), (17), and (20) is the parabola

$$f_b(x)=1+\frac{1-p_c}{2\nu}\left[\frac{\nu_b}{\beta_b}C_b-\frac{\nu}{\beta}C\right](1-x^2). \tag{21}$$

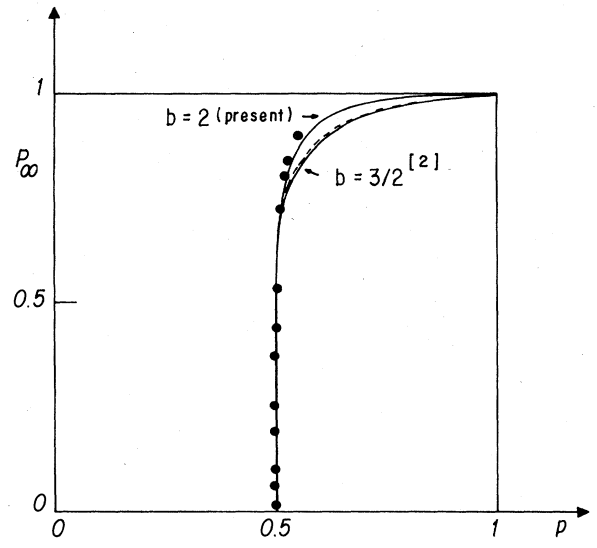

FIG. 7. RG order parameter P_∞ as a function of $p_x=p_y\equiv p$. The upper full line corresponds to our $b=2$ proposal; the dashed line is the corresponding extrapolated curve; the lower solid line is the $b=\frac{3}{2}$ result of Tsallis *et al.* (Ref. 2); the dots represent Monte Carlo data (Ref. 11).

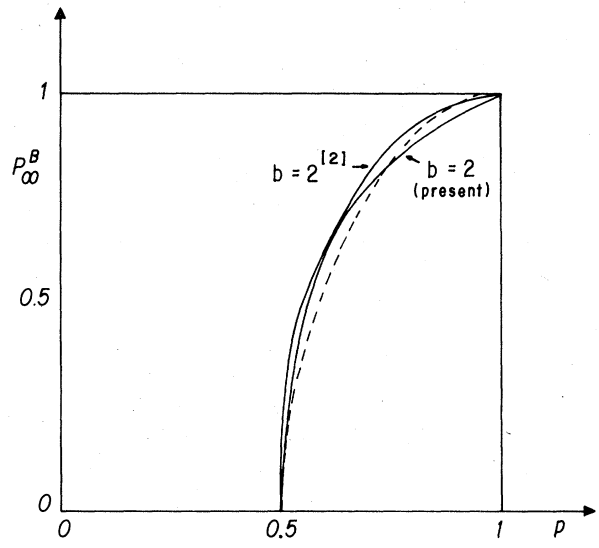

FIG. 8. RG order parameter P_∞^B as a function of $p_x=p_y\equiv p$. The solid lines correspond to our $b=2$ proposal and the $b=2$ result of Tsallis *et al.*; the dashed line is the extrapolated curve.

The expression for the correcting function is now uniquely determined; however the extrapolation procedure is not yet closed, as we still have to indicate the *arguments* of y and y_b in Eq. (14). We postulate the following transformation:

$$x_b(\bar{p})=x(p)\ . \tag{22}$$

Summarizing, Eqs. (10)—(14) and (22) lead to the following extrapolation algorithm:

$$\widetilde{P}_{\infty b}(p)=\left\{f_b([(p-p_c)/(1-p_c)]^\nu)\left[P_{\infty b}\left[p_b+(1-p_b)\left[\frac{p-p_c}{1-p_c}\right]^{\nu/\nu_b}\right]\right]^{\nu_b/\beta_b}\right\}^{\beta/\nu} \tag{23}$$

which, together with Eq. (21), completely closes the procedure.

Equation (22) is the central assumption of the present procedure and states that a kind of *law of corresponding states* holds for the RG approximate functions ($P_{\infty b}(p)$) associated with different cluster sizes. Note that Eq. (23) becomes identically satisfied if, in the $b\to\infty$ limit, $P_b\to p_c$, $\nu_b\to\nu$, $\beta_b\to\beta$, and $C_b\to C$ [in short, if $P_{\infty b}(p)$ regularly approaches $P_\infty^{\text{exact}}(p)$ for increasingly large clusters]. The present formalism essentially reproduces that used in Ref. 4 (for the surface tension) for the particular case $\beta=(d-1)\nu$ and $\beta_b=(d-1)\nu_b$ (we recall that the singularities of the surface tension and the correlation length are, in general, intimately related). For the $d=2$ Ising model the present extrapolation procedure yielded[4] errors inferior to 3% (1%) for the $b=2$ ($b=5$) RG approach, over the *whole* domain of temperatures. The main interest of the procedure comes from the fact that the knowledge of commonly available information (p_c, ν, β, and C) and a *single* RG approximate result provides a curve which is hopefully satisfactory over the entire domain of bond concentrations.

In our present $b=2$ RG approach we obtain the P_∞ results indicated in Fig. 7 and Tables II and III, where we have used that $p_b=p_c=\frac{1}{2}$, $\nu=\frac{4}{3}$, $\nu_b\simeq 1.042$, $\beta=\frac{5}{36}$, $\beta_b\simeq 0.095$, $C=0$,[10] and $C_b=0$. The results corresponding to P_∞^B are indicated in Fig. 8 and Tables II and III, where we have used that $\beta^B\simeq 0.53$,[9] $\beta_b^B\simeq 0.334$, $C^B=0$, and $C_b^B=\frac{3}{4}$.

TABLE III. RG and extrapolated order parameters P_∞ and P_∞^B for the isotropic case ($p_x = p_y \equiv p$).

	P_∞		P_∞^B	
p	RG ($b=2$)	Extrapolated	RG ($b=2$)	Extrapolated
0.5	0	0	0	0
0.51	0.764	0.655	0.303	0.167
0.52	0.814	0.721	0.380	0.241
0.53	0.844	0.786	0.434	0.335
0.54	0.865	0.791	0.476	0.346
0.55	0.882	0.814	0.512	0.389
0.56	0.895	0.833	0.543	0.427
0.57	0.906	0.849	0.570	0.462
0.58	0.915	0.863	0.594	0.495
0.59	0.923	0.875	0.616	0.525
0.60	0.930	0.886	0.637	0.553
0.65	0.956	0.926	0.720	0.674
0.70	0.972	0.952	0.783	0.769
0.75	0.982	0.969	0.834	0.845
0.80	0.989	0.982	0.877	0.905
0.85	0.994	0.990	0.913	0.951
0.90	0.997	0.996	0.946	0.982
0.95	0.999	0.993	0.974	0.999
1	1	1	1	1

IV. CONCLUSION

A real-space renormalization-group formalism (based on an appropriate cluster and extending a previous one[2] has been developed to calculate, for all bond concentrations on an anisotropic square lattice, the infinite cluster and backbone order parameters [$P_\infty(p_x,p_y)$ and $P_\infty^B(p_x,p_y)$, respectively]. They exhibit an interesting difference: while $P_\infty(p_x,p_y)$ presents discontinuities at $(p_x,p_y)=(1,0)$ and $(0,1)$, $P_\infty^B(p_x,p_y)$ continuously vanishes while approaching the critical line $p_x+p_y=1$ (exactly recovered within the present approach). The treatment slightly destroys the $p_x \rightleftarrows p_y$ symmetry of the square lattice. It should not be hard to restore it by performing *ad hoc* averages (frequently adopted in the literature) between the p_x and p_y equations: this seems, however, unworthy, the numerical breakdown being practically negligible.

In order to obtain quite reliable values along the $p_x = p_y \equiv p$ axis (isotropic square lattice), we have implemented an extrapolation procedure which has proved to be very efficient in other similar problems. Our best proposals appear in Table III (exact values for P_∞ and P_∞^B are unavailable in the literature). In the vicinity of the critical point $p_c = \frac{1}{2}$, we obtain $P_\infty \sim A(p-1/2)^\beta$ with $\beta = \frac{5}{36}$ (Ref. 8) and $A \simeq 1.25$, and $P_\infty^B \sim A^B(p-1/2)^{\beta^B}$ with $\beta^B \simeq 0.53$ (Ref. 9) and $A^B \simeq 1.92$.

In conclusion, within simple real-space renormalization-group frameworks (not harder than mean-field approaches), it has been possible to obtain numerically reliable results for "geometrical" order parameters over the entire range of the external parameters. The extension of this type of techniques to other quantities (both geometrical and thermal) and other lattices would be very welcome.

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