Finite-temperature properties of the damped one-dimensional quantum sine-Gordon model

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The finite-temperature properties of the damped one-dimensional quantum sine-Gordon model are investigated by using the semiclassical approximation. The renormalization of the phonon mass and the soliton density is calculated. It is found that the dissipative effect changes the low-temperature behaviors of these equilibrium properties by suppressing quantum fluctuations. The dynamic response function is obtained. The limit of applicability of the theory and its relation to other classical and quantum theories are discussed.

I. INTRODUCTION

The quantum statistical mechanics of soliton-bearing systems have attracted the attention of many authors.[1–3] Maki and Takayama[1] (hereafter referred to as MT) have studied the static and dynamic properties of the one-dimensional sine-Gordon ϕ^4 and double-quadratic models using the semiclassical approximation. The static properties of the sine-Gordon model have also been investigated exactly using the Bethe-ansatz method.[2,3]

On the other hand, the relevance of dissipative effects in quantum systems has been recognized recently in relation to macroscopic quantum phenomena.[4,5] This should also be important in the investigation of soliton-bearing systems, because they are usually derived by elimination of microscopic degrees of freedom, which results in a damping mechanism. For example, experimental data indicate that the motion of a charge-density wave in a quasi-one-dimensional conductor is overdamped.[6] Intrinsic damping is also observed for quasi-one-dimensional spin systems.[7]

In the one-dimensional quantum sine-Gordon model, we have shown that dissipation effects change the weak-field singularity of the quantum soliton-pair creation rate.[8] Finite-temperature effects on this phenomenon were also discussed and the features of the crossover from quantum to thermal behavior was found to be essentially affected by dissipation.[9] These phenomena were investigated at temperatures well below the phonon mass.

In the present paper, we investigate the properties of this system including the temperature range higher than the phonon mass but lower than the soliton mass. We investigate the effects of damping on the thermal and quantum renormalization of phonons and solitons. The dynamical response function is also obtained. These results are compared with previous quantum and classical calculations. It is found that the dissipation changes not only the dynamical properties but also equilibrium properties such as the soliton density in the quantum case.

Damping is introduced in the Feynmann path-integral description following the method of Caldeira and Leggett.[4] This method allows us to describe the damping effect in a compact manner.

In the next section we formulate the problem in terms of a Feynman path integral. The phonon renormalization is calculated in Sec. III. The soliton density is calculated in Sec. IV. The dynamic response function is obtained in Sec. V. The last section is devoted to concluding remarks.

II. PATH-INTEGRAL FORMULATION

The one-dimensional quantum sine-Gordon model is defined by the following Lagrangian:

$$\mathscr{L}=\tfrac{1}{2}\int_{-L/2}^{L/2}\left[(\partial_t\phi)^2-(\partial_x\phi)^2+\frac{2m_b^2}{g^2}\cos(g\phi)-\frac{2m_0^2}{g^2}\right]dx, \tag{2.1}$$

where ϕ is the bose field, m_b the bare phonon mass, m_0 the renormalized phonon mass at zero temperature, g the coupling constant, L the system size, and t and x are time and space coordinates, respectively. Planck's constant and the phonon velocity are taken to be equal to unity. In the presence of damping, the partition function Z is expressed in terms of the path integral

$$Z=C\int\!\!\!\int_{\phi(x,\beta)=\phi(x,0)} D\left[\frac{\phi(x,\tau)}{\sqrt{\beta}}\right]\exp\{-S[\phi(x,\tau)]\}\ , \tag{2.2}$$

where β is the inverse of temperature T and C is a divergent constant which is determined to give a finite result for Z. S is the Euclidian action including the effect of damping given by

$$S=S_0+S_d\ , \tag{2.3}$$

$$S_0=-\int_0^\beta d\tau\,\mathscr{L}_E\ , \tag{2.4}$$

and

$$S_d=\frac{\eta}{4\pi}\int_{-L/2}^{L/2}dx\int_0^\beta d\tau\int_0^\beta d\tau'\frac{\pi^2[\phi(x,\tau)-\phi(x,\tau')]^2}{\beta^2\sin^2[\pi(\tau-\tau')/\beta]}\ . \tag{2.5}$$

$\mathscr{L}_E$ is the Euclidian Lagrangian obtained by setting $t\rightarrow -i\tau$ in (2.1). S_d represents the Ohmic damping introduced by Cladeira and Leggett.[4,8,9] The extension to one-dimensional systems is explained in Refs. 8 and 9. In the real-time equation of motion this corresponds to the Ohmic damping force which is equal to $-\eta\dot{\phi}$.

In this paper, we confine ourselves to the weak-coupling region ($g^2/8\pi \ll 1$) and low temperatures compared to the soliton mass ($T \ll 8m_0/g^2$). In this regime thermal and quantum fluctuations can be treated by perturbation theory, and the self-consistent harmonic approximation is applicable.

II. RENORMALIZATION OF PHONON MASS

In this section we calculate the properties of the system in the absence of solitons. In the one-dimensional sine-Gordon model, the ultraviolet divergence is completely renormalized by assuming a normal product in the $\cos g\phi$ term.[10] The same is true for the dissipative case because dissipative effects are irrelevant for the ultraviolet divergence. The renormalized value itself, however, depends on the strength of dissipation. In the ground state the renormalized mass m_0 is expressed in terms of the bare mass m_b in the form

$$m_0^2 = m_b^2 \exp\left[-\frac{g^2}{2}\langle\phi^2\rangle_g\right], \tag{3.1}$$

where $\langle\ \rangle_g$ denotes the average in the ground state.[10] At finite temperatures, it is convenient to introduce the finite-temperature normal product defined by

$$:A: = A - \langle A \rangle_{\text{pair}}, \tag{3.2}$$

where $\langle\ \rangle_{\text{pair}}$ denotes all possible pairings of the ϕ field in A at finite temperatures.[1] Then we can write

$$m_b^2 \cos g\phi = m^2 :\cos g\phi:, \tag{3.3}$$

where

$$m^2 = m_b^2 \exp\left[-\frac{g^2}{2}\langle\phi^2\rangle\right]. \tag{3.4}$$

Here $\langle\ \rangle$ denotes the thermal average. The phonon mass defined as above is the self-consistent phonon mass at finite temperatures. The quantity $\langle\phi^2\rangle$ is evaluated using the linearized action around $\phi=0$ to have

$$\langle\phi^2\rangle = \int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\sum_n (\omega_n^2 + k^2 + |\omega_n|\eta + m^2)^{-1}, \tag{3.5}$$

where $\omega_n = 2\pi n T$ (n an integer), and Λ is the momentum cutoff. This is calculated as

$$\langle\phi^2\rangle = \frac{1}{2\pi}\ln\left[\frac{2\Lambda}{m}\right] + f_0(\beta m) + T\sum_n \int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}[(\omega_n^2 + m^2 + k^2 + |\omega_n|\eta)^{-1} - (\omega_n^2 + m^2 + k^2)^{-1}] \tag{3.6}$$

where f_0 is defined by

$$f_0(\beta m) = \frac{1}{\pi}\int_0^\infty dk \frac{1}{(k^2+m^2)^{1/2}} \times \{\exp[\beta(k^2+m^2)^{1/2}] - 1\}^{-1}. \tag{3.7}$$

The third term is convergent. We can, therefore, take the limit $\Lambda \to \infty$ safely in this term to obtain the finite result. We evaluate the above expression analytically in the low-temperature limit ($T \ll m$) and numerically at finite temperatures.

At $T=0$, the summation reduces to the integral and can be evaluated analytically. We have

$$m_0^2 = m_b^2 \exp\left[-\frac{1}{4\pi}g^2 \ln\left[\frac{2\Lambda}{m_0+\eta/2}\right]\right], \tag{3.8}$$

where $m_0 = m(T=0)$. This formula can be written in a form independent of the cutoff as

$$m_0^2 = m_{00}^2 \exp\left[-\frac{1}{4\pi}g^2 \ln\left[\frac{m_{00}}{m_0+\eta/2}\right]\right], \tag{3.9}$$

where $m_{00} = m(T=0, \eta=0)$. This equation is solved numerically for $g^2 = 0.2\pi$ (Fig. 1). This value is chosen to compare our results with those of MT. In the weak-damping limit ($\eta \ll m$) it reduces to

$$m_0 \cong m_{00} + \eta (g')^2/16\pi, \tag{3.10}$$

where g' is the renormalized coupling constant given by

$$(g')^{-2} = g^{-2} - 1/8\pi. \tag{3.11}$$

The temperature dependence of m is calculated analytically for $T \ll m$ to be

$$m \cong m_0 - \frac{\pi T^2}{48}\frac{g^2\eta}{m_0^2}\left[1 - \frac{g^2}{8\pi}\frac{m_0}{m_0+\eta/2}\right]^{-1}. \tag{3.12}$$

It should be noted that the correction term is proportional to T^2, while in the undamped case it is proportional to $\exp(-\beta m)$. At higher temperatures it is calculated numerically for $g^2 = 0.2\pi$, which is shown in Fig. 2 for various values of parameters. At high temperatures it tends to the value of the undamped case. This is seen clearly because the last term of (3.6) vanishes as $1/T$ for $T \gg \eta, m$.

The partition function Z_0 in the absence of solitons is calculated using the linearized action around the ground states as

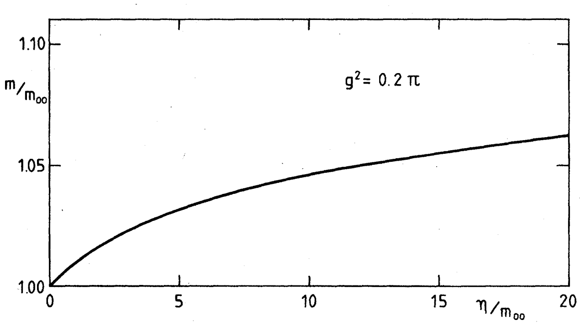

FIG. 1. η dependence of the phonon mass in the ground state for $g^2 = 0.2\pi$.

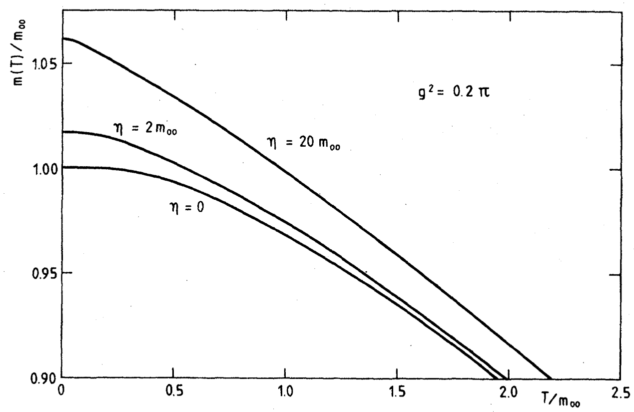

FIG. 2. Temperature dependence of the self-consistent phonon mass for $g^2=0.2\pi$ and $\eta/m_{00}=0,2,20$.

$$Z_0=\left[\prod_{l,n}\frac{(4\pi^2n^2+\delta_{n,0})T^2}{\omega_n^2+|\omega_n|\eta+m^2+k_l^2}\right]^{1/2}\times\exp\left[\beta L\left[\frac{m^2}{g^2}-\frac{m_0^2}{g^2}+\frac{m^2}{2}\langle\phi^2\rangle\right]\right],\tag{3.13}$$

where k_l satisfies

$$Lk_l=2\pi l\quad(l=\text{an integer})\ .\tag{3.14}$$

The constant but divergent normalization factor which depends on neither η nor T is chosen to give

$$Z_0=\prod_l\left[2\sinh\left[\frac{(m^2+k_l^2)^{1/2}}{2T}\right]\right]^{-1}\times\exp\left[L\beta\left[\frac{m^2}{g^2}-\frac{m_0^2}{g^2}+\frac{m^2}{2}\langle\phi^2\rangle\right]\right]\tag{3.15}$$

in the absence of dissipation, although this choice is arbitrary. Expression (3.15) is the partition function obtained by MT in the absence of solitons.

IV. SOLITON DENSITY

In this section, we evaluate the density of solitons which are thermally excited. In the present scheme, a soliton appears as the stationary point solution ϕ_S of the action S as

$$\phi_S=\frac{4}{g}\arctan\{\exp[m(x-x_S)]\}\ ,\tag{4.1}$$

where x_S is the center of the soliton. Here m is, in principle, the phonon mass renormalized in the presence of soliton. It turns out, however, to be equal to that in the absence of a soliton (Appendix A). In contrast to MT, who treated the motion of solitons in a real-time representation, the single moving soliton does not appear in our imaginary-time scheme even in the undamped case because of the periodic boundary condition along imaginary-time axis. In the presence of damping, there is no stationary moving soliton in the real-time representation. Therefore, the direct extension of the method of MT to the present case is difficult. The τ-dependent soliton-(anti)soliton pair solution is allowed. This, however, contributes to higher-order effects in the soliton density, which we neglect here.

The partition function in the presence of solitons is calculated by linearizing the action around the soliton (4.1). For this purpose, it is convenient to introduce the eigenmodes U_l of the fluctuations around the soliton in the following way:[1]

$$\left[-\frac{\partial^2}{\partial x^2}+m^2\cos(g\phi_S)\right]U_l(x)=\Omega_l^2U_l(x)\ .\tag{4.2}$$

The orthonormal set $\{U_l(x)\}$ is given by the following.

(i) For the continuum,

$$U_l(x)=\frac{k_l+im\tanh[m(x-x_S)]}{[\beta L(\Omega_l^2-2m/L)]^{1/2}}e^{ik_lx}\ ,\tag{4.3}$$

where $\Omega_l=(k_l^2+m^2)^{1/2}$ and k_l satisfies

$$\Delta(k_l)+Lk_l=2\pi l\ ,\tag{4.4}$$

where l is an integer not equal to zero and Δ is the phase shift due to a soliton given by

$$\Delta(k)=2\arctan(m/k)\ ,\tag{4.5}$$

which is defined to satisfy $-\pi<\Delta\leq\pi$.

(ii) For the zero mode,

$$U_0=\left[\frac{m}{2\beta}\right]^{1/2}\frac{1}{\cosh m(x-x_S)}\ ,\qquad \Omega_0=0\ .\tag{4.6}$$

Here all the functions are normalized as

$$\int_0^\beta d\tau\int_{-L/2}^{L/2}dx\,|U_l|^2=1\tag{4.7}$$

up to order $1/L$.

We expand the fluctuation field $\hat{\phi}=\phi-\phi_S$ as

$$\hat{\phi}=\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\phi_{nl}e^{i\omega_n\tau}U_l(x)\ .\tag{4.8}$$

Using the linearized action around ϕ_S and integrating over ϕ_{nl}, we find

$$Z_1=Z_0(m/\sqrt{2\pi})\exp(-\beta F)\int d\phi_{00}\ ,\tag{4.9}$$

where

$$\beta F=\beta E_S^c-\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_n\frac{2[k\arctan(m/k)-m]}{\omega_n^2+k^2+m^2+\eta|\omega_n|}-\tfrac{1}{2}\sum_{n\neq0}\ln\left[1+\frac{m^2}{\omega_n^2+\eta|\omega_n|}\right].\tag{4.10}$$

Here $E_S^c=8m/g^2$ is the classical soliton energy. The second term is due to the contribution from the continuum modes and the third term from the zero mode. In obtaining the second term we used (3.13), (3.14), (4.4), (4.5), and the result of Appendix A. This calculation is similar to that of MT except that the summation over n cannot be performed analytically. Therefore we do not repeat it here.

The integral over ϕ_{00} is evaluated as follows: The excitation of ϕ_{00} mode corresponds to the translational displacement of the soliton. By considering the normalization of U_0 given by (4.7), the displacement of the soliton dx_S is related to $d\phi_{00}$ by

$$dx_S = d\phi_{00}/(\beta E_S^c)^{1/2} . \tag{4.11}$$

Therefore, we have

$$\int d\phi_{00} = (\beta E_S^c)^{1/2} \int_{-L/2}^{L/2} dx_S = L(\beta E_S^c)^{1/2} . \tag{4.12}$$

Considering that the phase shift by n_S solitons and n_A antisolitons is equal to $\Delta(n_S+n_A)$, the partition function with n_S solitons and n_A antisolitons is given by

$$Z_{n_S n_A} = Z_0 (Z_1/Z_0)^{n_S+n_A}/n_S! n_A! \tag{4.13}$$

as long as the soliton density is small; $(n_S+n_A)/m \ll L$. Therefore the full partition function is given by

$$\begin{aligned} Z &= \sum_{n_S=0}^{\infty} \sum_{n_A=0}^{\infty} Z_{n_S n_A} \exp[-\mu(n_S+n_A)] \\ &= Z_0 \exp(2Z_1 e^{-\mu}/Z_0) . \end{aligned} \tag{4.14}$$

Here μ is the common chemical potential of solitons and antisolitons which is introduced in order to count the soliton number.

The soliton-antisoliton density is $N_S = (n_S+n_A)/L$ given by

$$\begin{aligned} N_S &= -\frac{1}{L} \frac{\partial}{\partial \mu} \ln Z \bigg|_{\mu=0} \\ &= 2Z_1/Z_0 L = 2m(\beta E_S^c/2\pi)^{1/2} \exp(-\beta F) . \end{aligned} \tag{4.15}$$

The quantity F is evaluated analytically for the following cases: (i) low-temperature limit $T \ll m, \eta$; (ii) intermediate-temperature regime in underdamped case ($\eta \ll T \ll m$); (iii) classical limit $\eta, m \ll T$.

(i) Low-temperature limit. Here N_S is given by

$$N_S \simeq 2(E_S^c \eta/2\pi)^{1/2} \exp(-\beta E_S^0) , \tag{4.16}$$

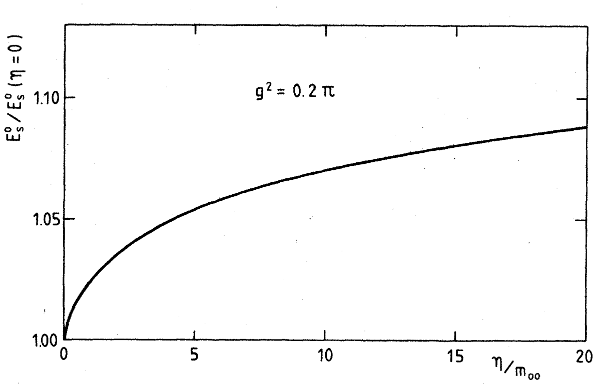

FIG. 3. η dependence of the soliton mass in the ground state for $g^2 = 0.2\pi$.

where E_S^0 is the renormalized soliton mass in the presence of dissipation at $T=0$. The η dependence of E_S^0 is calculated numerically and is shown in Fig. 3 for $g^2=0.2\pi$. For weak damping ($\eta \ll m$) the analytic expression is obtained

$$E_S^0 \cong \frac{8m_0}{(g')^2} + \frac{\eta}{2\pi} \ln \frac{2m_0}{\eta} \tag{4.17}$$

where m_0 is given by (3.10). On the other hand for strong damping E_S^0 tends to

$$E_S^0 \to 8m_0/g^2 . \tag{4.18}$$

It should be noted that the prefactor does not depend on T, in contrast to the undamped case.[1]

(ii) Intermediate temperature. In this regime, both dissipation and renormalization of the soliton by thermal phonons are irrelevant. Therefore, we obtain

$$N_S \cong (E_S^c T/2\pi)^{1/2} \exp(-\beta E_S^{00}) \tag{4.19}$$

where $E_S^{00} = 8m_{00}^2/(g')^2$. This coincides with the result of MT in the undamped case. In the present case, however, the validity of this formula is limited to this temperature range.

(iii) Classical limit. Here solitons are renormalized not only by quantum fluctuations but also by thermal phonons. This changes the temperature dependence of the prefactor of N_S. We find

$$N_S \cong 4m(\beta E_S^c/2\pi)^{1/2} \exp(-\beta E_S^{00}) , \tag{4.20}$$

which coincides with the classical result.[1,11]

In the prefactors of (4.16), (4.19), and (4.20), there appears E_S^c, while in MT it is replaced by the "inertial mass" of the soliton. This difference, however, is out of the scope of our approximation because the correction to E_S^c is of the order of m or T and this corresponds to the higher correction to F which we have neglected.

The soliton density is calculated numerically for

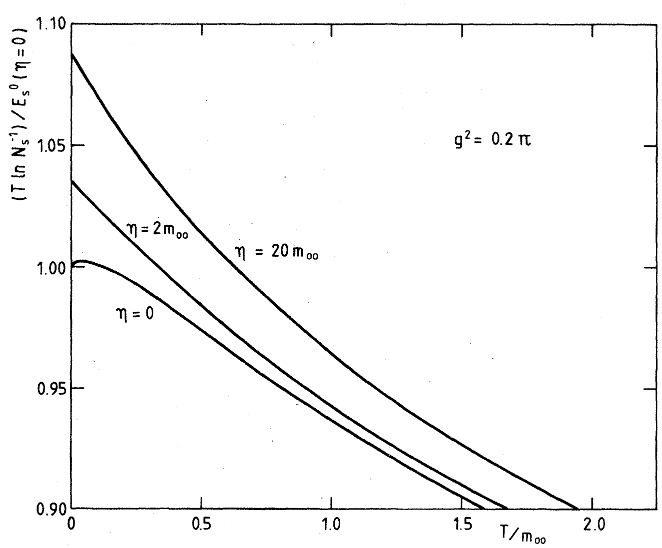

FIG. 4. Temperature dependence of the soliton density $N_S(T)$ for $g^2=0.2\pi$ and $\eta/m_{00} = 0,2,20$.

$g^2=0.2\pi$ and various values of η (Fig. 4). At low temperatures, the soliton density is sensitive to η, reflecting the property in case (i). At higher temperatures the curves for the damped case approach that for the undamped case.

V. DYNAMIC RESPONSE FUNCTION

When the external field ϵ conjugate to ϕ is applied to the system there appears a time-dependent response of ϕ which is given by

$$\langle \dot{\phi}(x,t)\rangle = \int_{-L/2}^{L/2} dx' \int_{-\infty}^{t} dt\, \chi^R(x-x',t-t')\epsilon(x',t') \tag{5.1}$$

in the real-time representation. The Fourier component of $\chi^R_{\omega q}$ of the response function is given by

$$\begin{aligned}\chi^R_{\omega q} &= -i\omega\Phi_{\omega_n q}\,|_{i\omega_n\to\omega+i0}\\ &= -i\omega\left[\int_{-L/2}^{L/2} dx \int_0^\beta d\tau\, e^{i\omega_n\tau} e^{iq(x-x')} \times\langle T_\tau \phi(x,\tau)\phi(x',0)\rangle\right]_{i\omega_n\to\omega+i0}\end{aligned} \tag{5.2}$$

where τ is the imaginary time introduced in Sec. II.

When the soliton density is small ($N_S \ll m$), the expectation value of the intensive physical quantity A is given by

$$\langle A\rangle = \langle A\rangle_0 + N_S(\langle A\rangle_1 - \langle A\rangle_0)L \ , \tag{5.3}$$

where $\langle\ \rangle_N$ denotes the average in the presence of N solitons. The intuitive derivation of this formula is given in Appendix B. In the absence of solitons, the evaluation of $\Phi_{\omega_n q}$ is straightforward and we obtain

$$\Phi_{\omega_n q} = \Phi^0_{\omega_n q} \equiv (\omega_n^2 + |\omega_n|\eta + m^2 + q^2)^{-1} \ , \tag{5.4}$$

where q satisfies

$$Lq = Lq_l \equiv 2\pi l \quad (l \text{ an integer}) \ . \tag{5.5}$$

In the presence of one soliton we have

$$\begin{aligned}\langle \phi(x,\tau)\phi(x',0)\rangle_1 &= \langle \phi_S(x-x_S)\phi_S(x'-x_S)\rangle_1\\ &\quad + \langle \hat{\phi}(x-x_S,\tau)\hat{\phi}(x'-x_S,0)\rangle_1 \ .\end{aligned} \tag{5.6}$$

The first term, however, does not contribute to the dynamic properties because it is time independent. In the following, we take only the second term. Then we have

$$\begin{aligned}\Phi_{\omega_n q} &= \Phi^1_{\omega_n q}\\ &\equiv \frac{1}{L}\int_{-L/2}^{L/2} dx_S \int_{-L/2}^{L/2} dy \int_0^\beta d\tau\, e^{i\omega_n\tau + iqy}\langle \hat{\phi}(x+y-x_S,\tau)\hat{\phi}(x-x_S,0)\rangle_1 \ .\end{aligned} \tag{5.7}$$

Here the average is take over the position of the soliton.

Using the expansion by U_l we have

$$\Phi^1_{\omega_n q} = \frac{\beta}{L}\sum_{k_l}\left|\int_{-L/2}^{L/2} U_l(x)e^{iqx}dx\right|^2 \langle \phi_{nl}\phi_{-n\,-l}\rangle \ , \tag{5.8}$$

where

$$\langle \phi_{nl}\phi_{-n\,-l}\rangle = \begin{cases}(\omega_n^2 + m^2 + k_l^2 + |\omega_n|\eta)^{-1} & \text{for } l\neq 0\\ (\omega_n^2 + |\omega_n|\eta)^{-1} & \text{for } l=0 \ .\end{cases} \tag{5.9}$$

The matrix elements are given by

$$\int_{-L/2}^{L/2} U_l(x)e^{iq_{l'}x}dx = \frac{\pi}{[L\beta(m^2+k_l^2-2m/L)]^{1/2}}\,\frac{\exp(iq_{l'}x_S)}{\sinh[(q_{l'}-k_l)\pi/2m]} \ , \tag{5.10}$$

$$\int_{-L/2}^{L/2} U_0(x)e^{iq_{l'}x}dx = \frac{\pi}{(2m\beta)^{1/2}}\,\frac{\exp(iq_{l'}x_S)}{\cosh(\pi q_{l'}/2m)} \ , \tag{5.11}$$

where the conditions (4.4) and (5.5) are used for k_l and $q_{l'}$.

After lengthy but straightforward calculations, we find

$$\Phi^1_{\omega_n q} = \Phi^0_{\omega_n q} + \delta\Phi_{\omega_n q}/L \ , \tag{5.12}$$

where

$$\delta\Phi_{\omega_n q} = -\frac{2m}{L}\,\frac{1}{\alpha(\alpha+m)}\,\frac{q^2-\alpha^2}{(q^2+\alpha^2)^2} + \frac{1}{Lm}\,\frac{1}{\alpha^2-m^2}\sum_{n=0}^{\infty}\frac{4m[4(n+\alpha/2m)^2-q^2/m^2]}{\alpha[4(n+\alpha/2m)^2+q^2/m^2]^2} \ , \tag{5.13}$$

where $\alpha^2=\omega_n^2+\eta\,|\,\omega_n\,|+m^2$. The analytic continuation $i\omega_n \to \omega+i0$ is straightforward. Thus we have

$$\chi^R_{\omega q}=\chi^{R0}_{\omega q}+N_S(\chi^{R1}_{\omega q}-\chi^{R0}_{\omega q})L$$
$$\equiv\chi^{R0}_{\omega q}+N_S\delta\chi^R_{\omega q}\ , \tag{5.14}$$

where

$$\chi^{RN}_{\omega q}=-i\omega\Phi^N_{\omega_n q}\,|_{\,i\omega_n\to\omega+i0}\ . \tag{5.15}$$

Therefore we have

$$\chi^{RO}_{\omega q}=\frac{i\omega}{\omega^2-m^2-q^2+i\omega\eta}\ . \tag{5.16}$$

The function $\delta\chi^R_{\omega q}$ is plotted in Fig. 5 for $\eta=0.5m$. Here the frequency and the wave number is normalized by m. Structures appear near $\omega\sim(m^2+q^2)^{1/2}$ and $\omega\sim q\sim 0$. The first one is due to the decrease and mixing of continuum states caused by the soliton. The second one is due to the translational mode associated with the soliton. If η is much larger than m, $\delta\chi^R_{\omega q}$ becomes small for $\omega\gg(m^2+q^2)/\eta$ and the structure around $\omega\sim(m^2+q^2)^{1/2}$ becomes very broad (width $\sim\sqrt{m\eta}\gg m$) and almost invisible. In the undamped limit both structures become very sharp.

The above result for $q\neq 0$, however, fails in the low-frequency regime. In practice for $\omega=0$ and $q\neq 0$, solitons will accumulate around the nodes of the external field and reach the equilibrium distribution in the periodic potential. Therefore, the dynamic response function should vanish, while our result gives the finite value of χ^R_{0q}. In the situation described above, the simplistic dilute soliton approximation would break down. Therefore, the validity of our formula should be restricted in the regime where external field changes sign before the solitons arrive at the nodes. This is realized if $v_0 q \ll \omega$, where v_0 is the thermal velocity of solitons. v_0 is estimated to be order of $(T/E_S^c)^{1/2}$ in the absence of dissipation. This would be also correct for $T\gg\eta$. In the present scheme it is difficult to estimate v_0 because our formulation does not include the concept of a moving soliton.

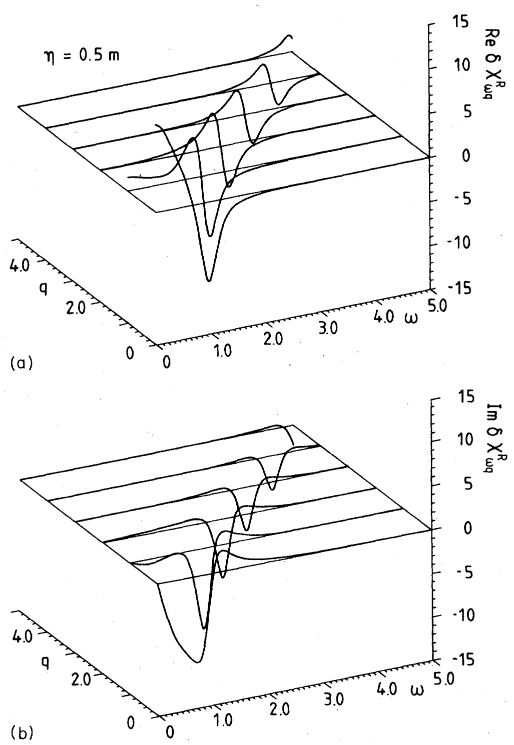

FIG. 5. Dynamic response function $\delta\chi^R_{\omega q}$ due to a soliton with $\eta=0.5m$. Wave number and frequency are measured in units of m and $\delta\chi^R_{\omega q}$ in units of $1/m$.

In the regime $v_0 q \geq \omega$, many authors predict the central peak in the undamped case.[12] In the damped classical case, this regime is studied in a phenomenological way.[13] This yields

$$\delta\chi^R_{\omega q}\cong\frac{\pi^2}{2\eta m}\frac{i\omega}{i\omega-Dq^2} \tag{5.17}$$

around $\omega\sim q\sim 0$, where $D=k_BT/8m\eta$ is the diffusion constant of the soliton. This vanishes for $q\neq 0$ and $\omega=0$ and is finite for $q=0$ and $\omega\to 0$. The second term in the denominator of (5.17) is important for this behavior. Our simplistic low-temperature approximation fails to pick up this term because it vanishes at $T\to 0$. Considering the success of real-time calculations in the classical case, we expect the quantum Langevin equation approach[14] would be powerful in this regime. This problem is under investigation.

It should be also pointed out that our results in the classical limit for $q=0$ do not coincide with the classical result obtained by the transfer-matrix method and decoupling approximations[15,16] in the following points.

First, in the limit $q=0$, $\omega\to 0$, we have $\chi^R_{00}=\pi^2 N_S/2m\eta$ while transfer-matrix calculations gives an additional factor E_S^c/T. This point has been intensively discussed by Guyer and Miller[15] and Büttiker and Landauer[17] who used the dilute soliton-gas approximation in the classical model. Our result support that of Büttiker and Landauer.[17]

Second, the ω dependence of $\delta\chi^R_{\omega 0}$ is still different from the result of Imada,[16] who applied the method of Guyer and Miller[15] to the calculation of $\chi^R_{\omega 0}$. It is not clear to which extent the decoupling approximation is correct, although it is justified in the infinite damping limit by Tsuzuki.[18] On the other hand, our approximation (4.2) might be doubtful for calculating the correlation for $q<N_S$ because the division of the system into subsystems (Appendix B) might not be adequate to calculate such a long-range correlation.

VI. CONCLUDING REMARKS

The finite-temperature properties of the damped quantum sine-Gordon model have been studied using the semiclassical approximation. It is shown that the low-temperature behavior of the renormalized phonon mass and the soliton density are different from the undamped case due to the suppression of quantum fluctuations by damping. This is in contrast to the classical case in which damping has nothing to do with equilibrium properties.

The dynamic response function which is valid in the regime $v_0 q \ll \omega$ is obtained. In the opposite regime, $v_0 q \geq \omega$, our calculation fails. The real-time approach, which is successful in the classical case, is desirable also in the quantum case. The discrepancy found between the classical limit of our theory and the previous classical calculations requires further investigation.

The precise comparison between our theory and the experiment is not possible at the present stage. From this point of view, the extension of the present theory to other quantum systems such as spin systems is desirable.

The concept of a finite-temperature soliton mass introduced by MT is not well defined in the present theory even in the undamped limit. It is not possible to fix the position and velocity of the soliton, because the fluctuations in them are automatically introduced as the translational mode. It should be noted that this concept is not well defined also in the exact Bethe-ansatz scheme.[19]

The inclusion of a breather mode is also in interesting problem. It is clear, however, that the usual breathers become damped modes here and the number of them is no longer conserved. Therefore, it would produce more difficulty than in the undamped case. This is also left for future studies.

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APPENDIX A

The quantity $\langle \phi^2 \rangle_1$ is defined by

$$\langle \phi^2 \rangle_1 = \sum_n \int \frac{dq}{2\pi} \Phi^1_{\omega_n q} , \tag{A1}$$

where $\Phi^1_{\omega_n q}$ is calculated in Sec. V. The correction due to soliton is given by the second and third term of (5.13). By using the identity

$$\int_{-\infty}^{\infty} \frac{x^2 - a^2}{(x^2+a^2)^2} dx = 0 , \tag{A2}$$

the contribution to $\langle \phi^2 \rangle_1$ from these corrections exactly vanishes.

APPENDIX B

In the dilute soliton limit, with soliton (antisoliton) number n_S (n_A), it is possible to divide the system into $N = n_S + n_A$ subsystems with length L_i ($i = 1, \ldots, N$, $\sum_i L_i = L$), each of which contains one soliton or antisoliton. The average value of an intensive physical quantity in the ith subsystem is given by

$$\langle A \rangle_{1,L_i} = \langle A \rangle_{0,L_i} + \delta A / L_i , \tag{B1}$$

where

$$\delta A = L(\langle A \rangle_{1,L} - \langle A \rangle_{0,L}) \tag{B2}$$

and $\langle \ \rangle_{n,L}$ denotes the average in the subsystem with length L and n solitons or antisolitons. $\delta A / L$ is the change of expectation value of A due to the addition of one soliton or antisoliton. It should be noted δA does not depend on L. The average over the total system is given by

$$\begin{aligned}\langle A \rangle_{N,L} &= \sum_{i=1}^{N} L_i \langle A \rangle_{1,L_i} / L \\ &= \left[\sum_{i=1}^{N} L_i \langle A \rangle_{0,L_i} + \sum_{i=1}^{N} \delta A \right] \Big/ L \\ &= \langle A \rangle_{0,l} + N \delta A / L \\ &= \langle A \rangle_0 + N(\langle A \rangle_1 - \langle A \rangle_0) .\end{aligned} \tag{B3}$$

The thermal average over soliton number is given by

$$\begin{aligned}\langle A \rangle &= \frac{1}{Z} \sum_{n_S=0}^{\infty} \sum_{n_A=0}^{\infty} Z_{n_S n_A} [\langle A \rangle_0 + (n_S + n_A) \\ &\qquad \times (\langle A \rangle_1 - \langle A \rangle_0)] \\ &= \langle A \rangle_0 + N_S(\langle A \rangle_1 - \langle A \rangle_0) .\end{aligned} \tag{B4}$$

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