Upper critical field of strongly disordered three-dimensional superconductors: Localization effects

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We calculate the influence of localization on the upper critical field, $H_{c2}(T)$, of strongly disordered superconductors in three dimensions. The present work expands upon our previous paper [L. Coffey, K. A. Muttalib, and K. Levin, Phys. Rev. Lett. 52, 783 (1984)] on this topic. It is our conclusion that studying these field-dependent effects may be the "cleanest" way to analyze the interplay of superconductivity and localization phenomena. Our approach is based on the exact impurity eigenstate method which has been used previously to examine both (zero-field) normal-state and superconducting properties of strongly disordered materials. This approach has the advantage over diagrammatic schemes of being nonperturbative in the disorder although it is fundamentally phenomenological in nature. The most striking qualitative effect of extreme disorder is an enhancement of $H_{c2}(T)$ relative to the standard Werthamer, Helfand, and Hohenberg curve. This occurs at low temperatures corresponding to the high-field suppression of localization. This enhancement has been observed in recent experiments on transition-metal-based alloys. In our approach localization effects in $H_{c2}(T)$ arise mainly from the field dependence of the Coulomb pseudopotential μ^*. The changes with H in the density of states and electron-phonon interaction are found to be relatively less important. The pseudopotential depends on the field-dependent particle-hole function $g_H(\mathbf{r}\mathbf{r}';\omega)$ which is closely related to the density-density correlation function. This same function arises in all exact impurity eigenstate calculations for normal as well as superconducting properties. For weak disorder the Fourier transform $g_H(q;\omega)\approx D_H q^2/(\omega^2+D_H q^2)^2$ where D_H is the diffusion constant at finite H. Localization effects are assumed to enter $g_H(\mathbf{r}\mathbf{r}';\omega)$ and thus μ^* via the scale-dependent diffusion coefficient. Here D_H is modeled by analogy with scaling laws in the zero-field case. With the application of a high magnetic field these localization effects are suppressed and as a result the Coulomb pseudopotential is reduced. The reduction of localization effects in μ^* at high H and the consequent enhancement of $H_{c2}(T)$ at low T are closely related to the behavior of the negative magnetoresistance. This negative magnetoresistance reflects the same field-induced suppression of localization as does the enhancement of $H_{c2}(T)$. Therefore an important test of our theory will be a study of systematic correlations in these two phenomena.

I. INTRODUCTION

Recent investigations of the critical temperatures[1] (T_c) and the upper critical fields[2,3] [$H_{c2}(T)$] of superconductors with high normal-state resistivities (ρ_N) measured just above T_c have revealed interesting behavior. Measurements of $H_{c2}(T)$ in amorphous transition-metal-based alloys and similar materials differ significantly from the results predicted by the standard theory of $H_{c2}(T)$ developed by Werthamer, Helfand, and Hohenberg (WHH) and others.[4,5] For these highly disordered materials with ρ_N ranging from 120 to 200 $\mu\Omega$ cm, the upper critical field $H_{c2}(T)$ is frequently enhanced above the WHH result at low T. In similar systems, such as highly disordered $A15$ compounds and rare-earth ternary borides, it has also been observed[1] that T_c is unusually sensitive to further disorder.

The possibility that spatial inhomogeneities produce an enhancement in H_{c2} has been raised by Carter *et al.*[6] However, it is claimed on the basis of x-ray diffraction and high-resolution transmission microscopy that the samples studied in Ref. 2 are single-phase amorphous. Additional evidence for this are the single very sharp transitions [$\Delta T_c < 0.05$ K] observed for these samples. However it should be pointed out that in the class of transition-metal alloys studied in Ref. 2 there is strong evidence for inhomogeneity effects which are known to perturb the superconducting properties.[7] These inhomogeneities are generally found to lead to broader transitions ($\Delta T_c \geq 0.2$). The situation as regards the degree of homogeneity and its role in the superconductivity in the samples of Ref. 2 is thus somewhat controversial. Tenhover *et al.*[2] have observed a systematic correlation between the observed enhancements of $H_{c2}(T)$ and the values of ρ_N. In view of these correlations, these authors suggested that the deviations of $H_{c2}(T)$ from the standard theory arise from incipient localization effects in these highly disordered materials. It has been argued[8] that localization effects should be evident at resistivities higher than ≈ 150 $\mu\Omega$ cm.

In a previous paper[9] by the present authors, these incipient localization effects on $H_{c2}(T)$ were studied theoretically. The purpose of this paper is to expand upon our earlier rather brief description. We show that the deviations from the standard theory observed in recent experiments can be semiquantitatively explained by considering the interaction of magnetic fields and localization phe-

nomena.

The interaction of localization effects and superconductivity in T_c and $H_{c2}(T)$ has been studied diagrammatically in two dimensions by Takagi and Kuroda[10] and by Fukuyama and co-workers.[11,12] More recently Fukuyama *et al.*[13] have extended this work to three dimensions. These studies however are valid in the weak scattering regime only ($E_F\tau \gg 1$) where E_F is the Fermi level and τ is the elastic scattering time. In this diagrammatic approach the influence of weak localization on superconducting properties manifests itself through density of states effects as well as a renormalization of the Coulomb interaction or pseudopotential. Both of these effects originate in first order Coulomb corrections to the particle-particle propagator which determines the superconducting instability. The important corrections to T_c and $H_{c2}(T)$ arise from the interaction of impurity scattering and the Coulomb interaction and occur in both self-energy and vertex corrections. The former lead to density-of-states corrections and the latter to an enhancement of the Coulomb interaction between Cooper pair electrons.

By contrast our work is based on the exact impurity eigenstate approach which is thus nonperturbative in the disorder. This approach was applied at $H=0$ by Abrahams, Anderson, Lee, and Ramakrishnan[14] (AALR) to examine the one electron self-energy arising from Coulomb interactions in the normal state. Subsequently this method was used by Anderson, Muttalib, and Ramakrishnan[15] (AMR) to study the critical temperatures of strongly disordered superconductors. Using the localization derived enhancement of the Coulomb pseudopotential these authors could explain the unusual sensitivity of the superconducting transition temperatures of $A15$ compounds and the ternary borides to the normal-state resistivity.

The exact eigenstate method has a number of advantages over diagrammatic approaches. Because it is not perturbative it allows one to examine the limit of strong disorder which is of particular interest in three dimensions. In addition it is simple and intuitive and does not involve diagrams more complex than those of Hartree-Fock theory. Where comparisons have been made,[16] both approaches are found to lead to essentially identical results.

In our exact impurity eigenstate method we find an enhancement of $H_{c2}(T)$ above the standard theoretical prediction. This derives from the fact that the application of a magnetic field to the superconductor weakens the effects of localization and thus reduces the Coulomb pseudopotential μ^*. This in turn strengthens the superconducting properties of the material and thus leads to an enhanced $H_{c2}(T)$. The enhancement grows as the temperature decreases and as larger and larger fields can be applied to the superconductor without driving it into the normal phase. Within our framework density of states and electron phonon effects are found to be considerably less sensitive to the magnetic field and thus are neglected in the calculation of H_{c2}.

Within the present theoretical approach it is clear that the magnitude of the localization induced enhancement in $H_{c2}(T)$ is closely correlated with the negative magnetoresistance $\rho_N(H)$.[17] The latter arises in strongly disordered materials as a consequence of the suppression of localization effects by the applied magnetic field. This correlation between $\rho_N(H)$ and H_{c2} should have experimental consequences when more systematic studies are undertaken. While it cannot be claimed that in all disordered systems observed enhancements from the WHH curve are attributable to localization effects, the extent to which localization plays a role in $H_{c2}(T)$ can be ascertained by studying these correlations.

Since our paper[9] on this work was published, there have been a number of theoretical studies[13,18] of disorder effects on H_{c2} in three-dimensional systems. These other approaches reach the same conclusion: that incipient localization effects will lead to enhancements in $H_{c2}(T)$ relative to the results of conventional theory. In the diagrammatic calculation of Ref. (13) the behavior of H_{c2} is dominated by dynamical screening effects. The authors in Ref. (13) also find corrections to the slope at T_c relative to the WHH expression which is in contrast to the results found here. Our results (and a variety of experiments) suggest that this slope at T_c coincides with the WHH result. Our conclusion derives from the weak field dependence in small fields of the magnetoresistance $\rho_N(H)\approx H^2$. For these low H, localization effects are essentially unperturbed by the field. It should be pointed out that Ref. 13 represents the first attempt to deal with dynamical Coulomb effects on superconductivity. These effects, which may lead to significant pair breaking are ignored in the present work.

In Ref. 18 the effects of anomalous diffusion on the particle-particle propagator were considered very close to the mobility edge. These authors also presented predictions for anomalous behavior in H_{c2}. However, it should be noted that the work of Kapitulnik and Kotliar[18] is rather complementary to the present work since Coulombic interactions which we focus on are ignored in Ref. 18. These authors proposed that close to the mobility edge the dominant length scale should depend on frequency as $\omega^{-1/3}$ in three dimensions. We note that this frequency length scale is included naturally in our model. This will be discussed in detail in Sec. IV A, where it is shown that, away from the mobility edge and when Coulomb effects are ignored, the present theory and that of Ref. 18 are essentially equivalent. It should also be observed that in the vicinity of the mobility edge the magnetic length becomes smaller than the scaling length. In this limit there may be difficulties in calculating H_{c2} which arise from uncertainties in characterizing the localization transition in the presence of strong magnetic fields. At present there is no satisfactory theory of these magnetic-field-localization effects.

It should be noted that we and these previous calculations[10–13,18] have not addressed the important contribution of spin orbit coupling to $H_{c2}(T)$. These effects are particularly complex because large spin-orbit scattering will destroy localization phenomena. Because of this complexity and because the main conclusions of this work address a predicted correlation between the negative magnetoresistance and the enhancement in H_{c2}, we will not consider spin-orbit effects. If spin-orbit scattering is suf-

ficiently strong to destroy the negative magnetoresistance then it is clear that any enhancement in $H_{c2}(T)$ cannot be attributed to incipient localization. When, however, a negative magnetoresistance is observed with behavior as expected from localization theory, then our work suggests an enhancement in $H_{c2}(T)$ should also be evident.

II. REVIEW OF EXACT IMPURITY EIGENSTATE APPROACH

The exact eigenstate approach is based on one electron exact eigenstates $\phi_n(\mathbf{r})$ in the absence of electron-electron interactions. In a magnetic field, these field-dependent wave functions satisfy the Schrödinger equation

$$\frac{1}{2m}\left[\mathbf{p}-\frac{e\mathbf{A}}{c}\right]^2\phi_n(\mathbf{r})+V(\mathbf{r})\phi_n(\mathbf{r})=\epsilon_n\phi_n(\mathbf{r}) \tag{2.1}$$

where $V(\mathbf{r})$ represents the potential due to nonmagnetic impurity scattering. This approach has been used previously[14] in characterizing perturbatively the interaction of the Coulomb repulsion and impurity scattering. The Coulomb interaction in second-quantized form,

$$H^c=\int d^3\mathbf{r}d^3\mathbf{r}'\Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r}')V^c(\mathbf{r}-\mathbf{r}')\Psi(\mathbf{r}')\Psi(\mathbf{r}) , \tag{2.2}$$

can be transformed to

$$H^c=\sum_{lmnp}\left[\int d^3\mathbf{r}d^3\mathbf{r}'\phi_l^*(\mathbf{r})\phi_m^*(\mathbf{r}')V^c(\mathbf{r}-\mathbf{r}')\times\phi_n(\mathbf{r}')\phi_p(\mathbf{r})\right]a_l^\dagger a_m^\dagger a_n a_p , \tag{2.3}$$

where the one-electron field operator $\Psi^\dagger(\mathbf{r})=\sum_m a_m^\dagger\phi_m^*(r)$ and $a_m^\dagger$ is the electron creation operator corresponding to the mth exact eigenstate. Thus in the exact impurity eigenstate basis, the Coulomb interaction is

$$V^c_{\text{eff}}=\int d^3\mathbf{r}d^3\mathbf{r}'\phi_l^*(\mathbf{r})\phi_m^*(\mathbf{r}')V^c(\mathbf{r}-\mathbf{r}')\phi_n(\mathbf{r}')\phi_p(\mathbf{r}) . \tag{2.4}$$

In this basis the one-electron Green's function for the nth energy state is defined as

$$G(n,t-t')=-i\langle\mathcal{T}[a_n(t)a_n^\dagger(t')]\rangle , \tag{2.5}$$

where $\mathcal{T}$ is the time-ordering operator. When dynamical screening effects are not included in V, the normal-state self-energy is independent of frequency and is given in Hartree-Fock theory by

$$\Sigma(m)=k_BT\sum_n\sum_{q\Omega}L_{mnnm}(q)V^c(q)G(n,\Omega) , \tag{2.6}$$

where T is the temperature and

$$L_{mnnm}(q)=\int d^3\mathbf{r}\int d^3\mathbf{r}'e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\times\phi_m^*(r)\phi_n(\mathbf{r})\phi_n^*(\mathbf{r}')\phi_m(\mathbf{r}') . \tag{2.7}$$

Equation (2.6) is impurity averaged over all impurity configurations according to

$$\Sigma(E)=(1/N_0)\left\langle\sum_n\delta(\epsilon_m-E)\Sigma(m)\right\rangle . \tag{2.8}$$

In Eq. (2.8), N_0 is the single-particle density of states per spin at the Fermi level and for convenience the volume is set equal to 1.0. The symbol $\langle\ \rangle$ represents impurity averaging. Furthermore, it is assumed in Eq. (2.6) that $L_{mnnm}(q)$ and $G(n,\Omega)$ are impurity averaged separately. Therefore, the normal-state self-energy can be written as

$$\Sigma(E)=k_BT\frac{1}{N_0}\int d\xi\int d^3\mathbf{r}d^3\mathbf{r}'V^c(r-r')\left\langle\sum_{mn}\delta(\epsilon_m-E)\delta(\xi-\epsilon_n)\phi_m^*(\mathbf{r})\phi_n(\mathbf{r})\phi_n^*(\mathbf{r}')\phi_m(\mathbf{r}')\right\rangle . \tag{2.9}$$

This self-energy is seen to depend on the field-dependent correlation function

$$N_0^2 g_H(\mathbf{r},\mathbf{r}',E-\xi)=\left\langle\sum_{mn}\delta(\epsilon_m-E)\delta(\xi-\epsilon_n)\phi_m^*(\mathbf{r})\times\phi_n(\mathbf{r})\phi_n^*(\mathbf{r}')\phi_m(\mathbf{r}')\right\rangle . \tag{2.10}$$

AALR (who considered only the zero-field limit) pointed out[14] that this function is closely connected to the density-density correlation function,

$$A(\mathbf{r}\mathbf{r}',t-t')=\langle[\rho(\mathbf{r},t),\rho(\mathbf{r}',t')]\rangle .$$

In the presence of translational symmetry this may be transformed as follows

$$A(q,\omega)=N_0^2\omega\int g_H(\mathbf{r}\mathbf{r}';\omega)e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}d^2rd^3r' . \tag{2.11}$$

The functional form of g_H is determined by the way in which electronic density fluctuations behave in a disordered medium. The essential physical assumption of AALR was that $g_{H=0}(q,\omega)\approx D_0q^2/[(D_0q^2)^2+\omega^2]$, where D_0 is the weak scattering zero-field diffusion coefficient. Thus density fluctuations are characterized by a diffusive behavior in a highly disordered system.

A similar exact eigenfunction approach was used by AMR (Ref. 15) to derive the superconducting gap or "off-diagonal" self-energy at $H=0$. This can be evaluated in the exact impurity eigenstate basis in analogy with the normal state self-energy. The superconducting order parameter $F(\mathbf{r},\mathbf{r}';t-t')$ is defined by $-i\langle\mathcal{T}[\Psi_\uparrow(\mathbf{r},t)\Psi_\downarrow(\mathbf{r}',t')]\rangle$. In frequency space this parameter is written as

$$F(\mathbf{r},\mathbf{r}';\omega)=-k_BT\sum_{\omega'}\int d^3\mathbf{r}_1d^3\mathbf{r}_2G(\mathbf{r},\mathbf{r}_2;\omega)G(\mathbf{r}',\mathbf{r}_1,-\omega)\times F(\mathbf{r}_1,\mathbf{r}_2;\omega')\lambda(\mathbf{r}_1,\mathbf{r}_2;\omega-\omega') , \tag{2.12}$$

where $\lambda(\mathbf{r}_1,\mathbf{r}_2;\omega-\omega')$ represents the net attractive interaction between Cooper pair electrons. In Eq. (2.12), the Green's function

$$G(\mathbf{r};\mathbf{r}';\omega)=\sum_n \phi_n^*(r')\phi_n(r)G(n,\omega)$$

is the Fourier transform of $-i\langle\mathcal{T}[\Psi_\uparrow(\mathbf{r},t)\Psi_\uparrow^+(\mathbf{r}',t')]\rangle$. The total superconducting gap, which is the sum of the phonon and Coulomb contributions, is given by

$$\Delta(\mathbf{r},\mathbf{r}',\omega)=\sum_{\omega'}\lambda(\mathbf{r},\mathbf{r}';\omega-\omega')F(\mathbf{r},\mathbf{r}';\omega')$$
$$=-k_BT\sum_{\omega'}\int d^3\mathbf{r}_1 d^3\mathbf{r}_2 G(\mathbf{r},\mathbf{r}_1;\omega')G(\mathbf{r}',\mathbf{r}_2;-\omega')$$
$$\times\lambda(\mathbf{r},\mathbf{r}';\omega-\omega')\Delta(\mathbf{r}_1,\mathbf{r}_2;\omega')\ . \tag{2.13}$$

A critical assumption of the T_c calculation of AMR is that localization effects, stemming from the high level of disorder, enter the electron-phonon and Coulomb contributions to the gap in different ways. Schmidt[19] and subsequently Muttalib[20] considered the role of strong disorder in the electron-phonon coupling. It was shown in Ref. 20 that the constraint of charge neutrality results in the elimination of particle-hole diffusion renormalizations [which enter through $g_{H=0}\ (\mathbf{r}\mathbf{r}';t-t')$] in the electron-phonon coupling. By contrast this particle-hole diffusion renormalization dominates the Coulomb contribution to the gap. For this reason we, as do AMR and Refs. 10–13, will ignore localization corrections to the electron-phonon interaction. It should, however, be noted that the scale dependence of the diffusion coefficient[21] and anharmonic effects may alter the electron-phonon interaction in the limit of strong disorder. These effects have not been treated in sufficient detail to determine their importance for the present calculation.

The Coulomb contribution to the gap parameter in coordinate space can be written in terms of exact impurity eigenstates as

$$\Delta^c(\mathbf{r},\mathbf{r}')=\sum_n \phi_n(\mathbf{r})\phi_{\bar{n}}(\mathbf{r}')\Delta_n^c\ , \tag{2.14}$$

where $\phi_n(\mathbf{r})$ and $\phi_{\bar{n}}(\mathbf{r})$ represent the two time-reversed states n and $\bar{n}$, forming the Cooper pair. Equation (2.13) readily yields the exact impurity eigenstate representation of the Coulomb contribution to the gap,

$$\Delta_n^c=-k_BT\sum_{\omega'}\sum_m \prod_{\bar{n}nm\bar{m}} G(m,\omega')G(\bar{m},-\omega')\Delta_m(\omega')\ , \tag{2.15}$$

where the renormalized Coulomb interaction is given by

$$\prod_{\bar{n}nm\bar{m}}=\int d^3r d^3r' V^c(r-r')\phi_{\bar{n}}^*(\mathbf{r})\phi_n^*(\mathbf{r}')\phi_m(\mathbf{r}')\phi_{\bar{m}}(\mathbf{r}). \tag{2.16}$$

In our approach the Coulomb repulsion is represented by a static interaction potential so that the Coulomb contribution to the gap parameter has no frequency ω dependence. The total gap on the right-hand side of Eq. (2.15), in general, has a frequency dependence arising from the electron-phonon interaction, which usually enters as a frequency cutoff.

By exact analogy with Eq. (2.8) the impurity-averaged gap parameter at energy E is defined by

$$\Delta_E^c=\frac{1}{N_0}\left\langle\sum_n \delta(\epsilon_n-E)\Delta_n^c\right\rangle\ . \tag{2.17}$$

In the spirit of the normal-state calculation of AALR it is assumed in Eq. (2.15) that Δ_m and the two Green's functions are averaged separately. The averaged Coulomb gap can be written as

$$\Delta_E^c=N_0k_BT\sum_\omega\int d\zeta d^3r d^3r' V^c(\mathbf{r}-\mathbf{r}')g_H(\mathbf{r}\mathbf{r}';E-\zeta)$$
$$\times G(\zeta;\omega)G(\bar{\zeta};-\omega)\Delta_\zeta(\omega)\ , \tag{2.18}$$

where

$$N_0^2 g_H(\mathbf{r}\mathbf{r}',E-\zeta)=\left\langle\sum_{mn}\phi_m^*(\mathbf{r}')\phi_m(\mathbf{r})\phi_n^*(\mathbf{r})\right.$$
$$\left.\times\phi_n(\mathbf{r}')\delta(\epsilon_n-E)\delta(\zeta-\epsilon_m)\right\rangle\ ,$$

and we have used the time-reversal operator to rewrite $\phi_{\bar{m}}$ in terms of ϕ_m.

In setting up the gap equation, it has been assumed that two time-reversed eigenstates can both be represented by diagonal Green's functions. As will be discussed in more detail below, the assumption that ϕ_n and $\phi_{\bar{n}}$ are *simultaneously* eigenstates of the magnetic-field-dependent Hamiltonian [Eq. (2.1)], is a reasonable approximation only in the semiclassical limit.

It should be noted that the correlation function $g_H(\mathbf{r}\mathbf{r}';E-\zeta)$ which appears in superconducting properties also arises in the expression for the normal-state self-energy given in Eq. (2.9). Because it depends on the diffusion coefficient, $g_H(\mathbf{r}\mathbf{r}';E-\zeta)$ will naturally incorporate localization effects arising from strong disorder. Furthermore, since in general $\phi_n(\mathbf{r})$ depends on the magnetic field, $g_H(\mathbf{r}\mathbf{r}';E-\zeta)$ will also contain information concerning the interplay of localization and magnetic fields.

The calculations, summarized above (for the case $H=0$), of AALR and AMR are closely analogous. In the Nambu matrix formulation of superconductivity these two theories can be viewed as corresponding to different components of the matrix self-energy. It is clear, however, that a more complete and systematic treatment of the interplay of Coulomb interactions and disorder should involve a coupling between the two equations. This coupling yields a density of states correction to the superconducting gap equation which was not included by AMR. A more detailed discussion of the exact eigenstate fully coupled Eliashberg-like equations with dynamical screening is discussed in Ref. 22. We note that for the purposes of the present calculation the field dependence of the density of states is weak for the range of fields appropriate to most experiments. Furthermore in calculating the Coulomb pseudopotential μ^* the density-of-states factors roughly cancel out. However it is not clear that the field dependence of the density of states is entirely negligible for the phonon coupling constant. Because of uncertain-

uncertainties in treating localization effects in the electron phonon interaction we have ignored any field-dependent effects in this term. However, it should be noted that the density-of-states prefactor in the phonon coupling constant will lead to an enhancement in H_{c2} at high H or low T. This will add to the effects we focus on below which arise from the Coulomb pseudopotential.

III. DERIVATION OF $H_{c2}(T)$ EQUATION

In the exact eigenstate basis the equation for the superconducting gap can be seen to depend on $(\epsilon_n - \epsilon_{\bar{n}})$: the difference in energies of time reversed states. In the presence of a time-reversal symmetry-breaking perturbation, such as magnetic field, $(\epsilon_n - \epsilon_{\bar{n}})$ is nonzero. The energy splitting of the time-reversed states is equivalent to the pair breaking parameter generally called ρ. In this exact eigenfunction basis it is easy to see why pair breaking arises whenever there exists a term in the Hamiltonian which is non-time-reversal invariant.

In the presence of a time-reversal symmetry-breaking perturbation H', the energy difference of the two time-reversed states can be calculated in terms of the commutator $[H',K]$, where K is the time-reversal operator.[23] In the case or orbital coupling with the applied field,

$$[H',K]=\frac{-e}{mc}(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p})K \ . \tag{3.1}$$

When strong disorder is present, the quantity multiplying K on the right-hand side in Eq. (3.1) can be treated semiclassically as a c number. This semiclassical approximation also implicitly ignores effects arising from the Landau diamagnetic term $e^2/(2mc^2)A^2$ which can be justified in the presence of strong disorder. Using the usual quantum mechanical evaluation of the electromagnetic interaction one obtains

$$\langle -e/(2mc)(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p})\rangle=(n+1/2)\hbar\omega_c \ ,$$

where ω_c is the cyclotron frequency eH/mc. Diffusion effects can be incorporated into the evaluation of the commutator $[H',K]$ by replacing $\hbar/2m$ by the diffusion coefficient D_H/i. This transcription can be understood as arising from the difference between the Schrödinger equation and the diffusion equation. It is necessary in order for the time dependence of the motion to be exponentially damped as $e^{-t/\tau}$ rather than varying as e^{iEt}. With this substitution into the ground-state electromagnetic energy level one finds the usual result for the pair breaking parameter $\rho=i2D_HeH/e$. Note that ρ which is purely imaginary can be interpreted as a type of lifetime for the Cooper pair.

Because ρ is imaginary, it follows from the general analytic structure for the Green's function that $G_{\bar{\zeta}}(\omega)$ is given in terms of the energy ζ as

$$G_{\bar{\zeta}}(\omega)=(i\omega-\zeta+\rho\,\mathrm{sgn}\omega)^{-1} \ . \tag{3.2}$$

Here it is assumed that $G_{\zeta}(\omega)$ derives from an impurity average of the exact eigenstates $G_{\zeta}(\omega)$ and $G_{\bar{\zeta}}(\omega)$ corresponds to the associated time-reversed state. Inserting this result into Eq. (2.18), it follows that

$$\Delta_E^c=-N_0k_BT \times\sum_{\omega}\int\frac{d\zeta d^3rd^3r'V^c(\mathbf{r}-\mathbf{r}')g_H(\mathbf{r}\mathbf{r}',E-\zeta)\Delta_{\zeta}(\omega)}{(i\omega-\zeta)(-i\omega-\zeta+\rho\,\mathrm{sgn}\omega)} \ . \tag{3.3}$$

We now introduce the Coulomb pseudopotential in order to combine Δ_E^c with the phonon contribution Δ_E^{ph} to the total gap $\Delta_E(\omega)=\Delta_E^{\mathrm{ph}}(\omega)+\Delta_E^c$. In Eq. (3.3) the sum over ω is broken into two parts coming from $|\omega|<\omega_D$ and $\omega_D<|\omega|<E_F$. In the latter, the total gap parameter $\Delta_E(\omega)$ is equivalent to Δ_E^c since at frequencies greater than ω_D the phonon contribution vanishes. Furthermore the pairbreaking parameter which appears in the denominator of Eq. (3.3) can be neglected in this frequency range since it is relatively small compared to ω_D. Letting $E_F\to\infty$ in the second part of the sum over ω, we may then use the identity

$$\sum_{|\omega|=\omega_D}^{E_F}\frac{1}{(i\omega-\zeta)(-i\omega-\zeta)}=\frac{\Theta(\zeta-\omega_D)}{2\zeta}\frac{1}{k_BT} \ . \tag{3.4}$$

The Coulomb contribution to the gap equation can now be written as

$$\Delta_E^c+\int d\zeta\Omega(E-\zeta)\Delta_{\zeta}^c=N_0V^c\int g_H(\mathbf{r}\mathbf{r},E-\zeta)F(\zeta) \ , \tag{3.5}$$

where

$$F(\zeta)=-k_BT\sum_{|\omega|=0}^{\omega_D}\frac{\Delta_{\zeta}(\omega)}{(i\omega-\zeta)(-i\omega-\zeta+\rho\,\mathrm{sgn}\omega)} \ , \tag{3.6}$$

and

$$\Omega(E-\zeta)=\frac{N_0g_H(\mathbf{r}\mathbf{r};E-\zeta)V^c\Theta(\zeta-\omega_D)}{2\zeta} \ . \tag{3.7}$$

In Eqs. (3.5) and (3.7) we have used the fact that $V^c(\mathbf{r}-\mathbf{r}')$ is a screened Coulomb potential. This can be approximated by $V^c\delta(\mathbf{r}-\mathbf{r}')$ in order to set $\mathbf{r}=\mathbf{r}'$ in these equations.

The Coulomb pseudopotential is defined by

$$\Delta_E^c=\int_0^{\omega_D}\mu^*(E-\zeta)F(\zeta)d\zeta \ . \tag{3.8}$$

Therefore, using Eqs. (3.5)—(3.7), the equation for μ^* is written

$$\mu^*(E-\zeta)=N_0V^cg_H(E-\zeta) -\int_{\omega_d}^{E_F}d\zeta'N_0V^c\frac{g_H(E-\zeta')\mu^*(\zeta'-\zeta)}{\zeta'} \ , \tag{3.9}$$

where $g_H(E-\zeta)\equiv g_H(rr,E-\zeta)$ for notational simplicity. In Eq. (3.9) the integral must depend on $(E-\zeta)$ only so that for convenience ζ can be set equal to zero everywhere. Furthermore, the remaining energy dependence of $\mu^*(E)$ for $E<\omega_D$ can be neglected in our weak-coupling approximation. This is justified since μ^* is directly related to $g_H(E)$ which, as will be discussed in detail below, is approximately independent of E in this energy range. Thus we can evaluate Eq. (3.9) at $E=0$. We define $\mu^*\equiv\mu^*(0)$ as follows:

$$\mu^*=N_0V^c g_H(0)\Big/\left[1+N_0V^c\int_{\omega_D}^{E_F}d\zeta g_H(\zeta)/\zeta\right]. \quad (3.10)$$

Using Eq. (3.10) and performing the ζ integral in Eq. (3.8), Δ^c can be written as

$$\Delta^c=-\left[N_0V^c g_H(0)\Big/\left[1+N_0V^c\int_{\omega_D}^{E_F}\frac{d\zeta'}{\zeta'}g_H(\zeta')\right]\right]\times\sum_{\omega=-\omega_D}^{\omega_D}\frac{\Delta}{[\,|2n+1|+(\rho/i/4\pi T)]}. \quad (3.11)$$

Finally, adding the phonon contribution to Δ^c one obtains the final equation for $H_{c2}(T)$,

$$1=(\lambda^{\rm ph}-\mu^*)\left[\ln\frac{1.14\hbar\omega_D}{k_BT}+\Psi\left[\frac{1}{2}\right]-\Psi\left[\frac{1}{2}+\frac{\hbar D_H H_{c2}}{2\phi_0 k_BT}\right]\right], \quad (3.12)$$

where we have written the pair-breaking parameter in terms of the unit of flux $\phi_0=hc/2e$.

In the absence of localization effects Eq. (3.12) is identical to the standard equation[4,5] for $H_{c2}(T)$. In strongly disordered superconductors, localization effects and their interaction with the applied magnetic field are built into μ^* as well as into the general diffusion coefficient D_H, which appears in the pair-breaking parameter. In the limit $H=0$ Eq. (3.12) together with Eq. (3.10) reduce to the AMR result for T_c.

In Ref. 9 we used Eq. (3.12) to calculate the properties of the upper critical field in strongly disordered superconductors. However it should be pointed out that Ref. 9 contained an error so that μ^* appearing in this equation was not defined correctly as in Eq. (3.10). This error did not lead to any qualitative effects, however.

It is important to stress that although, in the absence of localization the exact eigenfunction basis leads to the standard equation for H_{c2}, this is not particularly obvious *a priori*. In this basis we perform the impurity averaging and introduce a factorization scheme in a way which appears to be different from the conventional coordinate space derivation of Eq. (3.12). I ocalization effects can be introduced in this coordinate space approach by including an additional class of vertex corrections. In this way the Coulomb interaction is effectively renormalized by two vertex factors which are equivalent to the function g_H defined earlier. However to make contact with the results of the present theory one cannot use the standard delta function model for this renormalized Coulomb interaction: $\lambda(r_1,r_2)\sim\delta(r_1-r_2)$ since, this removes the possibility of introducing diffusion corrections into H_{c2}. In summary, it should be stressed that all the effects of localization that we consider enter through the important correlation function g_H which has a q (and ω) dependence which is not accurately modeled by the standard delta function assumption.

IV. NUMERICAL RESULTS

A. Localization effects in the correlation function $g_H(\mathbf{r},\mathbf{r}';\omega)$

The function $g_H(\mathbf{r},\mathbf{r}';\omega)$ enters the final expression for $H_{c2}(T)$ [Eq. (3.12)] through the Coulomb pseudopotential μ^*. In zero field, this function can be rather well characterized. For short times $t<\tau$ the correlation function $g_{H=0}(\mathbf{r},\mathbf{r}'t)=g(\mathbf{r}-\mathbf{r}';t)$ has a free-particle form while for $t>\tau$ it is diffusive. In the presence of strong disorder, diffusive behavior is modeled by a scale-dependent diffusion coefficient. From the continuity equation and the constitutive relation the Fourier transform of the particle-hole correlation function has the form

$$N_0g(q,\omega)=\frac{D(q,\omega)q^2}{\omega^2+[D(q,\omega)q^2]^2}. \quad (4.1)$$

Therefore in the limit in which D depends on q and ω, $g(\mathbf{r};t)$ will be modified from its usual form $(4\pi D_0t)^{-3/2}e^{-r^2/4D_0t}$, appropriate for the weak-scattering limit. Here $D_0=2E_F\tau/3m$.

In the scaling theory of localization[21] in zero field the conductivity σ has an anomalous $1/L$ dependence in three dimensions at short length scales (or equivalently short times). From the Einstein relation $\sigma=e^2N_0D$, it follows that this scale dependence will also be present in the diffusion coefficient. These ideas have been incorporated by AMR into a simple model for the zero-field diffusion coefficient:

$$D(L)=D_0l_e/L,\quad l_e<L<L_s \quad (4.2)$$

$$D(L)=D_0l_e/L_s,\quad L_s<L. \quad (4.3)$$

Here l_e is the elastic mean free path. L_s is the localization length. This length in turn is determined by the relation $L_s/l_e=\rho_N/\rho_c$, where ρ_c is a critical resistivity above which localization phenomena should be apparent and ρ_N is the characteristic normal-state resistivity. Using Eqs. (4.2) and (4.3) and the particle-conservation condition,

$$\int g(\mathbf{r};t)d^3r=1, \quad (4.4)$$

this scale dependence can be incorporated phenomenologically into $g(\mathbf{r},t)$. It follows that for $\mathbf{r}=0$,

$$g(0;\omega)=1+\frac{1.75}{(E_F\tau)^2}\int_\tau^{\alpha\tau}\frac{\cos(\omega t)}{t}dt+\frac{1.15(\alpha)^{1/2}}{(E_F\tau)^2}\int_{\alpha\tau}^{\infty}\frac{\cos(\omega t)}{t^{3/2}}dt, \quad (4.5)$$

where α is $(\rho_N/\rho_c)^3=(L_s/l_e)^3$ and τ is the usual elastic scattering time.

We emphasize here that the frequency length scale $l_\omega/l_e=(\omega\tau)^{-1/3}$, discussed in Ref. 18, occurs naturally in this model. To see this we note that an approximate solusolution to the integral in Eq. (4.5) is

$$g(0,\omega)\approx 1+\frac{C}{(E_F\tau)^2}\ln(\alpha^{-1}+\omega\tau), \quad (4.6)$$

for $\omega<\tau^{-1}$ and where C is a constant of order unity. Thus the parameter $(\alpha\tau)^{-1}$ defines a character frequency which is determined by the disorder. The function $g(0,\omega)$ becomes appreciably frequency dependent only for frequencies which are high compared to this characteristic frequency. We can rewrite the logarithm in Eq. (4.6) as

$$\ln(\alpha^{-1}+\omega\tau)=\ln[(l_e/L_s)^3+\omega\tau]\approx\ln(l_e/L)^3 ,$$

where L is the minimum of L_s and the frequency length scale l_ω, and we define

$$l_\omega/l_e\equiv(\omega\tau)^{-1/3} .$$

Note that for the attractive phonon interaction, as long as $(\alpha\tau)^{-1}$ is less than the Debye frequency ω_D, l_ω is an irrelevant length scale. For a reasonable choice of parameters, assuming $E_F\tau\sim O(1)$ it follows that l_ω is irrelevant for the phonon interaction whenever $L_s/l_e=\rho_N/\rho_c$ is less than about five. This is in contrast to the situation for the Coulomb piece in which l_ω plays a key role, and which has no counterpart in Ref. 18 (where Coulomb interactions are not treated). This intermediate-disorder regime $(\rho_N/\rho_c\leq 5)$ appears to be characteristic of the systems studied here. In this limit, and when Coulomb effects are ignored the sole effect of disorder is to renormalize the diffusion coefficient by a frequency-independent parameter and we therefore find that the present theory coincides with that of Ref. 18. These scaling effects in D, however, lead to no changes in the reduced critical field

$$h_{c2}(T)=H_{c2}(T)/[T_c(dH_{c2}/dT)]_{T_c}$$

relative to the universal WHH curve at $T=0$ and $T=T_c$. Therefore the rescaling of the diffusion constant due to localization effects cannot be invoked to explain observed enhancements in h_{c2} which occur, for example, at $T=0$.

In the high-disorder regime $(\rho_N/\rho_c>5)$ considered in Ref. 18, it is clear that the present theory may breakdown for a number of reasons. Among these is our neglect of any scale dependence in the pair-breaking parameter ρ which from Ref. 18 appears to be of some importance.

Yoshika *et al.*[24] and Ting[25] have extended the diagrammatic derivation of scaling of the diffusion coefficient by Vollhardt and Wolfe[26] to the non-zero-field case. The application of a field results in a coupled set of equations which when solved yield the finite field diffusion coefficient. This coupling reflects the fact that at $H\neq 0$ the particle-hole and particle-particle channels are distinct but interrelated. We have numerically solved these equations to gain insight into the scaling properties of the diffusion coefficient at finite H. Based on the behavior at $H=0$ and this numerical work we suggest a simple field dependent scaling model for the diffusion coefficient of the form

$$D_H(L)=D_0 l_e/L,\quad l_e<L<L_s(H) \tag{4.7}$$

$$D_H(L)=D_0 l_e/L_s(H),\quad L_s(H)<L . \tag{4.8}$$

We estimate the scaling length by assuming that

$$\frac{L_s(H)}{l_e}=\frac{\rho_N(H)}{\rho_c} . \tag{4.9}$$

Here $\rho_N(H)$ is the normal-state magnetoresistance. This model is a natural extension of the zero-field model in Eqs. (4.2) and (4.3).

The resistance $\rho_N(H)$ and hence the scaling length $L_s(H)$ decreases upon applying a magnetic field resulting in a reduction of the anomalous $(1/L)$-dependent regime in Eq. (4.7) and an increase in the macroscopic diffusion Eq. (4.8). The magnitude of $\rho_N(H)$ or negative magnetoresistance can be calculated quantitatively by following Kawabata[17] who notes that the conductivity $\sigma(H,\omega)$ is determined dominantly by the particle-particle correlation function $g_H^{pp}(\mathbf{r},\mathbf{r}';\omega)$ as follows:

$$\sigma(H,\omega)-\sigma(H=0,\omega)=\frac{\sigma(H=0)}{\pi}\int g_H^{pp}(\mathbf{r},\mathbf{r};\omega)d^3\mathbf{r} . \tag{4.10}$$

In the absence of localization, an exact solution for $g_H^{pp}(\mathbf{r},\mathbf{r}';\omega)$ can be obtained by noting that it satisfies the Schrödinger equation for a particle of mass $1/2D_H$ and charge $2e$ in a magnetic field for imaginary times. For the magnetic field in the z direction, the Fourier transform $g_H^{pp}(r,r';t)$ is given by

$$g_H^{pp}(r,r';t)=\frac{N_0^{-1}}{Q_H(t)}\exp\left[\frac{-1}{4Dt}\left[\frac{(x-x')^2+(y-y')^2}{\tanh(\omega_H t/2)/(\omega_H t/2)}+(z-z')^2+i\omega_H t(x'y-xy')\right]\right] , \tag{4.11}$$

where

$$Q_H(t)=(4\pi Dt)^{3/2}\sinh(\omega_H t/2)/(\omega_H t/2),$$

and $\omega_H=(4DeH)/hc$. Use of this exact solution improves upon a truncation approximation used by Kawabata[17] and leads to a slightly larger (in magnitude) negative magnetoresistance.

Multiplying Eq. (4.10) by ρ_c and using Eq. (4.9) and its zero-field counterpart gives

$$\frac{l_e}{L_s(H)}=\frac{l_e}{L_s}+\frac{l_e}{L_s}\int\frac{g_H^{pp}(\mathbf{r},\mathbf{r};\omega)d^3\mathbf{r}}{\pi} . \tag{4.12}$$

In a strong magnetic field, Eq. (4.12) yields a $\sqrt{H}$ magnetic field dependence for $L_s(H)^{-1}$ given by

$$\frac{l_e}{L_s(H)}-\frac{l_e}{L_s}\propto\frac{l_e}{l_h} , \tag{4.13}$$

where $l_h=\sqrt{\hbar c/eH}$ and the coefficient of proportionality is approximately unity. It should be noted that in Kawabata's approximate calculation this coefficient would be ~ 0.2. In a weak magnetic field $[L_s(H)]^{-1}$ acquires an H^2 dependence given by

$$\frac{l_e}{L_s(H)}-\frac{l_e}{L_s}\propto\left[\frac{l_e}{l_h}\right]^4 . \tag{4.14}$$

Here the coefficient of proportionality is approximately $(l_i/l_e)^{3/2}$. The inelastic mean free path l_i arises upon replacing ω in Eq. (4.12) by the corresponding inelastic lifetime $1/\tau_i$. The inelastic mean free path is generally assumed[19] to behave as $T^{-3/2}$ in three dimensions.

It should be stressed that while there is some evidence for scaling behavior in D_H there is no satisfactory calculation of $g_H(q,\omega)$ in the presence of both a field and strong disorder. Houghton *et al.*[27] have calculated $g_H(q,\omega)$ in a nonzero field in the weak scattering limit. They find that $g_H(q,\omega)$ is given by Eq. (4.1) with an effective field-dependent diffusion coefficient D_H which is independent of q and ω. Localization effects, arising from the maximally crossed diagrams, were not considered in Ref. 27 and their inclusion would complicate the analysis.

Because in the presence of a field there is a coupling[24,25] between the particle-hole and the particle-particle channels, this suggests that not all field effects are accommodated by a field-dependent diffusion coefficient inserted into Eq. (4.1). This particle-particle effects lead to phase coherence phenomena such as the Aharonov-Bohm effect.[28] In the absence of localization one can compare the particle-particle and particle-hole correlation functions as a function of frequency and field. For reasonable values of H they are found to be essentially identical except for frequencies $\omega<10^{-2}\omega_D$. Since this represents a narrow range of frequencies relative to those important to superconductivity it may be argued that for our purposes we can ignore these phase coherence effects in $g_H(\mathbf{r},\mathbf{r}';\omega)$ in computing H_{c2}.

Therefore, in the absence of any more complete theory, we adopt the simplest possible model for the particle-hole correlation function by substituting an effective field-dependent diffusion coefficient $D_H(q,\omega)$ into the canonical zero-field form for $g_H(q,\omega)$, Eq. (4.1). We thus assume that in the presence of localization effects,

$$N_0 g_H(q;\omega)=\frac{D_H(q,\omega)q^2}{\omega^2+(D_H(q,\omega)q^2)^2} , \tag{4.15}$$

where $D_H(q,\omega)$ is determined by Eqs. (4.7) and (4.8).

The incorporation into $g_H(\mathbf{r},\mathbf{r}';\omega)$ of the interplay of localization effects and the magnetic field causes μ^* to decrease monotonically with the application of a magnetic field. As a consequence $H_{c2}(T)$ is enhanced. The magnitude of the reduction in μ^* is determined by that of the negative magnetoresistance which is reflected in the field-dependent scaling length $L_s(H)$. Therefore a close connection exists between the enhancement of $H_{c2}(T)$ and the normal-state negative magnetoresistance. Finally, it should be noted that we use throughout the *macroscopic* diffusion coefficient defined by Eq. (4.8) in the pair-breaking parameter in Eq. (3.12). This pair-breaking parameter corresponds to a small energy or frequency scale and thus to long time scales.

B. Upper critical field slope $[dH_{c2}(T)/dT]$ near T_c

It is relatively straightforward to obtain an exact expression for the slope of H_{c2} near T_c, corresponding to the limit $H\to 0$. From Eq. (3.12) it follows that

$$\left.\frac{dH_{c2}}{dT}\right|_{T_c}=-\frac{(\lambda^{\rm ph}-\mu^*)}{T_c}\left[\frac{1}{\frac{\partial\mu^*}{\partial H}\ln\left[\frac{1.14\hbar\omega_D}{k_BT_c}\right]+\frac{3D_H\zeta(2)\hbar}{2\phi_0k_BT_c}(\lambda^{\rm ph}-\mu^*)}\right] , \tag{4.16}$$

where $\zeta(2)$ is the Riemann zeta function. The behavior of $\partial\mu^*/\partial H$ in Eq. (4.16) is determined by the field dependence of $L_s(H)$ which was discussed in Sec. III. In a weak magnetic field, $[L_s(H)]^{-1}$ varies as H^2, as shown in Eq. (4.14). As a consequence, the derivative of $g_H(\mathbf{r},\mathbf{r}';\omega)$, with respect to H, vanishes as H approaches zero. Thus $\partial\mu^*/\partial H$ vanishes in this limit also.

Therefore near T_c, $dH_{c2}(T)/dT$ approaches the WHH result for the upper-critical-field slope, which is given by

$$\left.\frac{dH_{c2}(T)}{dT}\right|_{T_c}=\frac{-2\phi_0k_B}{3D_H\zeta(2)\hbar} . \tag{4.17}$$

More generally, whenever $[L_s(H)]^{-1}$ has an $H^{1+\epsilon}$ dependence with ϵ greater than zero in a weak magnetic field, $\partial\mu^*/\partial H$ will approach zero as H approaches zero, so that the slope of the upper-critical-field curve will coincide with that deduced from conventional theory. This result can be verified experimentally, as will be discussed in the next section.

C. Comparison with experiments

The results of Eq. (3.12) can be plotted in the reduced units $h_{c2}(T)=H_{c2}(T)/[T_c(dH_{c2}/dT)]_{T_c}$. In these units the WHH theory reduces to a universal curve which is indicated in Fig. 1. In our theory the same physical mechanism that leads to an enhancement in $H_{c2}(T)$ also causes the upper critical field in reduced units to be enhanced over the universal WHH curve. It should be pointed out that in these reduced units, if μ^* effects are ignored, there can be no deviation from the universal curve at $T=0$ and $T=T_c$ (arising solely from a rescaling of the diffusion coefficient). Hence, plotting h_{c2} represents a particularly convenient way of exhibiting localization effects which arise from contributions other than a rescaling in D_H. The curves labeled A–D shown in Fig. 1 correspond to increasing degrees of disorder. For definiteness we chose $\lambda^{\rm ph}=0.59$, $\omega_D/E_F=10^{-2}$, $E_F\tau=1.0$, and $\Theta_D=300$ K. These are fairly typical numbers for strongly disordered

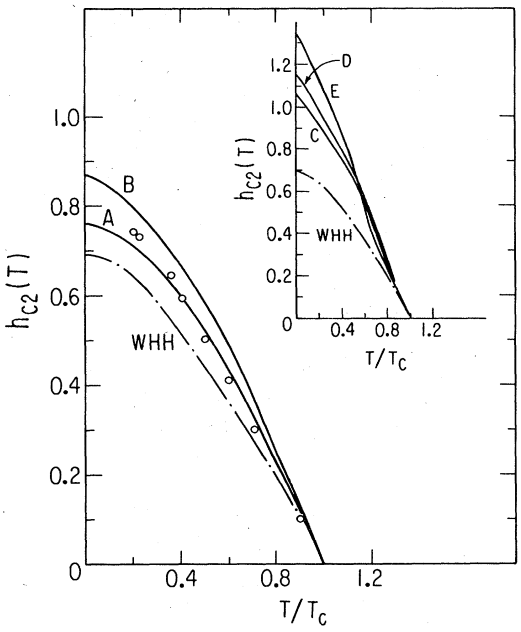

FIG. 1. Calculated temperature dependence of the upper critical field (in reduced units) for various $\rho_N/\rho_c=3.0$ (A) and 4.0 (B), and in the inset $\rho_N/\rho_c=5.0$ (C), 5.5 (D), and 6.0 (E). The standard dirty superconductor result (labeled WHH) and experimental data (circles) from Ref. 2 are shown for comparison.

transition-metal alloys such as amorphous MoRe.[29] With this parametrization we find that the curve labeled A corresponds to a sample with $H_{c2}(0)=185$ kG and $D_H=0.38$ cm^2 sec^{-1} at $H=0$ which appears appropriate for MoRe. The data for MoRe of Ref. 2 are indicated in Fig. 1 by the open circles. While our calculations are not sufficiently accurate to be compared quantitatively with experiment, it is clear from the figure that the theoretical behavior is in reasonable[30] qualitative accord with the results of Ref. 2. Furthermore, effects from inhomogeneities cannot be entirely ruled out.

Although the $T=0$ and $T=T_c$ limits are unaffected, the intermediate-temperature dependence of $H_{c2}(T)$ depends on the inelastic mean free path l_i which enters the definition of $L_s(H)$ in Eq. (4.9). We chose $l_i=5\times10^3 l_e/(T/T_c)^{3/2}$ so that the intermediate points in curve A were reasonably close to the MoRe data of Ref. 2. From numerical studies we have concluded that the magnitude of the inelastic mean free path relative to the other length scales l_e, l_h, and $L_s(H)$ determines the extent of the upward curvature in the H_{c2} curve. The smaller the ratio of l_i/l_e the more pronounced the upward curvature. This upward curvature is most apparent in curve E of Fig. 1. Some evidence for this anomalous curvature has been reported in $Mo_{75}Si_{25}$.[3]

We have found that a nonzero $(l_i)^{-1}$ weakens localization and thus reduces the enhancing effect of the applied magnetic field on the final $H_{c2}(T)$ curve. If l_i is comparable to l_h this reduction in the enhancement will be most noticeable. However as the temperature drops, inelastic scattering freezes out since l_i grows to infinity. Because l_i is then an irrelevant length scale at very low temperatures the magnetic field regains the full effect of weaking localization.

An important feature of the experimental results as well as of our theory (see Sec. IV. B) is that the H_{c2} curve follows the WHH result in a region around T_c. The experimental verification of this predicted behavior serves to reinforce our ansatz that relates $L_s(H)$ to the magnetoresistance.

For completeness it should be noted that there are highly disordered Zr-based alloys[31] with resistivities of the order 170 to 190 $\mu\Omega$ cm which show no evidence for an enhancement of the upper critical field. The Zr-based amorphous alloys have positive magnetoresistances.[32] Spin-orbit effects are important in these systems. These and the presence of magnetic elements, such as Ni and Co, in these compounds may diminish the effect of localization. One cannot claim that the experimental results of Hake and collaborators[30] on ZrNi, ZrCo, etc., are necessarily inconsistent with the predictions of the present theory. Further experiments which conclusively determine the extent to which localization plays a role in these Zr-based alloys are clearly needed.

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