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Novel Lifshitz tricritical point and critical dynamics

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The d-dimensional time-dependent Ginzburg-Landau (TDGL) model is mapped onto a special $(d+1)$-dimensional model which exhibits a Lifshitz tricritical point (LTP). Many of the LTP critical properties follow from those of the TDGL model, and are shown to belong to a novel universality class of LTP's which results from a (previously ignored) relevant, nonlocal, quartic spin operator. These properties are analyzed with the use of scaling, ϵ expansion, and the $n \to \infty$ limit.

There has been much recent interest in the mapping of the critical dynamics of d-dimensional models onto the equilibrium properties of related systems in $D=(d+1)$ dimensions.[1-5] One way to obtain such a mapping is to consider d-dimensional systems with purely dissipative equations of motion described by time-dependent Ginzburg-Landau (TDGL) models.[6] The associated Fokker-Planck equation is then reduced to an imaginary-time Schrödinger equation describing a d-dimensional quantum system. This, in turn, can be mapped onto a classical $(d+1)$-dimensional equilibrium model.[4,5] Alternatively, the mapping can be obtained directly by integrating over the Gaussian noise field.[7] Thus, the dynamics of an n-component spin model is mapped onto the equilibrium statistical mechanics of another (n-component) model in $d+1$ dimensions. The coupling constants of the latter model satisfy specific relations, which represent competing interactions.[1-3,5] It follows that as criticality is approached in the dynamic problem, a *multicritical point* of the $(d+1)$-dimensional problem is approached along a special (Riccati[4]) trajectory.

In this Rapid Communication we elucidate the nature of this multicritical point. For the standard TDGL model, the appropriate $(d+1)$-dimensional multicritical point is a special *Lifshiftz tricritical point* (LTP).[7] LTP's were analyzed previously by Nicoll, Chang, Tuthill, and Stanley.[8] However, the fixed point W^* studied by these authors is unstable. The critical behavior of the dynamic problem is governed by a different fixed point, LTP*. This new fixed point is stable for $n=1$ and $n>n_c=4+O(\epsilon)$ $(\epsilon=5-D=4-d)$. Thus, in this range, both the TDGL problem and the Lifshitz tricritical one are controlled by LTP*. Although, out of this range, LTP* is also unstable with respect to a term $\tilde{V}$ (see below), a special (super)symmetry[7] on the Riccati trajectory guarantees that the TDGL problem is still described by LTP*. The exponents of the TDGL model at criticality (η,z) thus determine those at LTP*$(\eta_\parallel,\eta_\perp)$, independently of the trajectory. Ratios among other exponents also follow. The properties of LTP* are discussed using both the ϵ expansion and the $n\to\infty$ limit. The latter analysis resolves a second problem associated with W^*. As shown by Emery,[9] the usual tricritical points split into critical and critical end points for $n\to\infty$ and $d<3$. The same problem occurs (at $d<4$) for the LTP associated with W^*, but not for our new LTP*, which remains tricritical as $n\to\infty$. Moreover, the Riccati trajectory, approaching LTP*, has interesting analytic properties which yield new information on exponents and scaling functions.

In addition to their theoretical interest, LTP's have been studied experimentally in various systems.[10,11] Our Rapid Communication presents new values for the tricritical exponents, with which these experiments should be critically compared. We note that similar new universality class arise for general order Lifshitz multicritical points as well.[8]

Our analysis is based on a d-dimensional model, with an energy functional $\mathscr{H}\{\mathbf{S_q}\}$, where $\mathbf{S_q}$ is the Fourier transform of the n-component spin. The time evolution is described by the Langevin equation,

$$\frac{\partial \mathbf{S_q}}{\partial t}=-k_BT\frac{\partial H}{\partial \mathbf{S_{-q}}}+\boldsymbol{\eta}_\mathbf{q}(t) \quad , \tag{1}$$

where the Gaussian noise source satisfies

$$\langle \boldsymbol{\eta}_\mathbf{q}(t)\rangle=0 \quad , \tag{2}$$
$$\langle \boldsymbol{\eta}_\mathbf{q}(t)\cdot\boldsymbol{\eta}_{\mathbf{q}'}(t')\rangle=2k_BT\delta(\mathbf{q}+\mathbf{q}')\delta(t-t') \quad .$$

This model has been shown[4,5] to be equivalent to a $(d+1)$-dimensional static classical model, in which the frequencies ω serve as the $d+1$ dimension. Using the standard Ginzburg-Landau Hamiltonian,

$$\mathscr{H}=\frac{1}{2}\int_\mathbf{q}(r_0+q^2)(\mathbf{S_q}\cdot\mathbf{S_{-q}})$$
$$+u_0\int_{\mathbf{q}_1}\int_{\mathbf{q}_2}\int_{\mathbf{q}_3}(\mathbf{S}_{\mathbf{q}_1}\cdot\mathbf{S}_{\mathbf{q}_2})(\mathbf{S}_{\mathbf{q}_3}\cdot\mathbf{S}_{-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3}) \quad , \tag{3}$$

where $\int_q\equiv(2\pi)^{-d}\int d^dq$ (integration over the Brillouin zone), the effective Hamiltonian is

$$\mathscr{H}_{\text{eff}}=\frac{1}{2}\int_{\mathbf{q},\omega}(R_0+\mu_0 q^2+q^4+\omega^2)(\mathbf{S}_{\mathbf{q},\omega}\cdot\mathbf{S}_{-\mathbf{q},-\omega})$$
$$+\int_{\mathbf{q}_1,\omega_1}\int_{\mathbf{q}_2,\omega_2}\int_{\mathbf{q}_3,\omega_3}(U_0+V_0 q_1^2)(\mathbf{S}_{\mathbf{q}_1,\omega_1}\cdot\mathbf{S}_{\mathbf{q}_2,\omega_2})(\mathbf{S}_{\mathbf{q}_3,\omega_3}\cdot\mathbf{S}_{-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3,-\omega_1-\omega_2-\omega_3})+W_0\int\int\int\int\int(\mathbf{S}\cdot\mathbf{S})(\mathbf{S}\cdot\mathbf{S})(\mathbf{S}\cdot\mathbf{S}), \quad (4)$$

with the Riccati constraints

$$R_0=r_0^2-a(n+2)u_0,\quad \mu_0=2r_0,\quad U_0=8u_0r_0,\quad V_0=2u_0,\quad W_0=32u_0^2\ , \quad (5)$$

and with the generalized Brillouin zone $q^4+\omega^2<1$. Introducing a cutoff on ω affects the coefficient $a(n+2)u_0$ in the expression for R_0.

As r_0 approaches its critical value, the scaling fields t, μ, and U (associated with R_0, μ_0, and U_0), all vanish as well. Looking at $\mathscr{H}_{\text{eff}}$ in the general parameter space (t,μ,U), the $\mu=0$ plane corresponds to Lifshitz point behavior, while $U=0$ corresponds to tricritical behavior. The point $t=\mu=U=0$ is thus a LTP. The parameter V, related to the momentum-dependent quartic spin terms, was not included in earlier studies of the LTP.[8] The associated operator has the same upper critical dimensionality as W, i.e., $D=d+1=5$.

For a general Lifshitz problem,[12] $\mathbf{q}$ and ω are replaced by m- and $(D-m)$-component vectors $\mathbf{q}^{\parallel}$ and $\mathbf{q}^{\perp}$. In our special case, $D=d+1$, $m=d$, $\mathbf{q}^{\parallel}=\mathbf{q}$, $q^{\perp}=\omega$. Further, for $n\neq 1$, one should include an additional term,

$$\tilde{V}\int(\mathbf{q}_1^{\parallel}+\mathbf{q}_2^{\parallel})^2(\mathbf{S}_{\mathbf{q}_1}\cdot\mathbf{S}_{\mathbf{q}_2})(\mathbf{S}_{\mathbf{q}_3}\cdot\mathbf{S}_{-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3})\ .$$

Although this term is relevant for LTP* when $n<n_c$ (for $m=d=D-1$), and is important for all $n\neq 1$ for $m\neq d$, it does not arise in the TDGL problem. In what follows we consider only $m=d$ and LTP*, for which $\tilde{V}=0$.

We start from a scaling analysis near LTP*. The spin-correlation function has the form

$$G(q,\omega,t,\mu,U)=\langle\mathbf{S}_{\mathbf{q},\omega}\cdot\mathbf{S}_{-\mathbf{q},-\omega}\rangle$$
$$=t^{-\gamma_{\text{LT}}}g(qt^{-\nu_{\parallel}},\omega t^{-\nu_{\perp}},\mu t^{-\Phi_\mu},Ut^{-\Phi_u})\ , \quad (6)$$

where Φ_μ and Φ_u are crossover exponents. At LTP*, $t=\mu=U=0$, and Eq. (6) reduces to

$$G(q,\omega,0,0,0)=q^{-(4-\eta_{\parallel})}\hat{g}(\omega/q^{(4-\eta_{\parallel})/(2-\eta_{\perp})})\ , \quad (7)$$

with the usual scaling relations[12] $\gamma_{\text{LT}}=\nu_{\parallel}(4-\eta_{\parallel})=\nu_{\perp}\times(2-\eta_{\perp})$. Equation (7) should be identical to the one obtained for the TDGL model[6]

$$\langle\mathbf{S}_{\mathbf{q},\omega}\cdot\mathbf{S}_{-\mathbf{q},-\omega}\rangle=q^{-(2-\eta+z)}\hat{g}(\omega/q^z)\ , \quad (8)$$

where z is the dynamic exponent.[6] Comparing (7) and (8), the LTP* exponents $\eta_{\parallel}$ and $\eta_{\perp}$ are uniquely determined by those known for the TDGL model[13] via

$$4-\eta_{\parallel}=2-\eta+z,\quad 2-\eta_{\perp}=(2-\eta+z)/z\ . \quad (9)$$

Consider now the Riccati trajectory. In the original Hamiltonian (3), there is only one relevant variable, related to the temperature; we denote the appropriate scaling field by $\tilde{t}$. The mapping of the associated TDGL model onto the $(d+1)$-dimensional system of Eq. (4) implies that t, μ, and U all vanish when $\tilde{t}=0$. Since t, μ, U, and $\tilde{t}$ are all analytic in the parameters of the original problem (e.g., r_0,u_0), we expect t,μ, and U to be analytic in $\tilde{t}$. In the simplest case t, μ, and U will be linear in $\tilde{t}$ (to leading order). We can now use two alternative renormalization schemes. In the first, we rescale lengths by a factor b in $\mathscr{H}$, obtain $\mathscr{H}'$ with $\tilde{t}'=b^{1/\nu}\tilde{t}$, and then map onto $\mathscr{H}'_{\text{eff}}$, with t', μ', and U' all linear in $b^{1/\nu}\tilde{t}$. Alternatively, we can first map $\mathscr{H}$ onto $\mathscr{H}_{\text{eff}}$, and then renormalize. The parameter of the resulting $\mathscr{H}''_{\text{eff}}$ will be $t''\propto b^{1/\nu_{\parallel}}t\propto b^{1/\nu_{\parallel}}\tilde{t}$; $\mu''\propto b^{\Phi_\mu/\nu_{\parallel}}\tilde{t}$, $U''\propto b^{\Phi_\mu/\nu_{\parallel}}\tilde{t}$. Since t'', μ'', and U'' are analytic (and therefore, in the simplest case, linear) in t', μ', U', we conclude that at least one of $1/\nu_{\parallel}, \Phi_\mu/\nu_{\parallel}$, and $\Phi_\mu/\nu_{\parallel}$ should be equal to $1/\nu$. (In general, one might also have rational ratios.) Both our $O(\epsilon)$ and $n\to\infty$ results yield $\Phi_\mu/\nu_{\parallel}=1/\nu$, and we expect this equality to hold in general. With this equality, the Riccati trajectory is identified as a straight line in the (t,μ,U) space, approaching LTP*.

The intermediate quantum model in the mapping from the TDGL system to the $(d+1)$-dimensional classical and static model has zero ground-state energy by construction, implying that along the Riccati trajectory the free energy is analytic.[3,5] Writing the singular part of the free energy in the scaling form

$$F_s=t^{2-\alpha_{\text{LT}}}f(\mu t^{-\Phi_\mu},Ut^{-\Phi_u})\ , \quad (10)$$

and noting that $2-\alpha_{\text{LT}}=\nu_{\parallel}d+\nu_{\perp}$ is usually a noninteger, we conclude that, on the trajectory, $F_s=0$. Since, to $O(\epsilon)$, both $\Phi_\mu<1$ and $\Phi_u<1$, it follows that $f(0,0)=0$. This is a rather unusual and strong constraint of $f(x,y)$.

Our ϵ-expansion analysis is a straightforward extension of the renormalization-group study of Lifshitz points[12] to Eq. (4). Rescaling $\mathbf{q}^{\parallel}\to\mathbf{q}^{\parallel}/a$, $\mathbf{q}^{\perp}\to\mathbf{q}^{\perp}/b$, $\mathbf{S}_{a\mathbf{q}^{\parallel},b\mathbf{q}^{\perp}}\to\zeta\mathbf{S}_{\mathbf{q}^{\parallel},\mathbf{q}^{\perp}}$, with $b^{2-\eta_{\perp}}=a^{4-\eta_{\parallel}}$ and $\zeta^2=a^m b^{D-m+2-\eta_{\perp}}$, we find that near the Gaussian fixed point $V'=a^{6+m-2D}V$; $\tilde{V}'=a^{6+m-2D}\tilde{V}$, and $W'=a^{2(6+m-2D)}W$, identifying the upper critical dimensionality as $D_u=3+m/2$. For the LTP problem, nonclassical behavior is expected for $d=m=D-1<4$, the same as for the TDGL model. To order $\epsilon=4-d$, $\tilde{V}$ is not generated from (4). Integrating over spins with $b^{-2}<q^4+\omega^2<1$, it is straightforward to write $O(\epsilon)$ recursion relations for R, μ, U, V, and W. Our new fixed point, LTP*, has $W^*=O(\epsilon^2)$ and $V^*=O(\epsilon)$. It is stable against W, with the crossover exponent $\lambda_w=\Phi_w\nu_{\parallel}^{-1}=-(n+26)\epsilon/(n+8)$. The three relevant variables have exponents

$$\nu_{\parallel}^{-1}=4-(n+2)\epsilon/(n+8)\ ,$$
$$\lambda_\mu=\Phi_\mu\nu_{\parallel}^{-1}=2-(n+2)\epsilon/(n+8)\ , \quad (11)$$
$$\lambda_u=\Phi_u\nu_{\parallel}^{-1}=2-\epsilon\ ,$$

and we confirm our (general) result $\lambda_\mu=1/\nu$. To $O(\epsilon)$, we also find $\lambda_{\tilde{V}}=\Phi_{\tilde{V}}\nu_{\parallel}^{-1}=(4-n)\epsilon/(n+8)$, so that $\tilde{V}$ is irrelevant for $n > n_c = 4 + O(\epsilon)$. The exponents (11) should thus be observed in Lifshitz tricritical problems with $m = D - 1$ and $n > n_c$. For $n < n_c$, a new fixed point, with V^*, $\tilde{V}^* \neq 0$ becomes stable.

For $m \neq d$, $\tilde{V}$ is important for all $n \neq 1$, and the LTP exponents are determined by new fixed points, with both $V, \tilde{V}$ of order $\epsilon_m = 3 + m/2 - D$, and W of order ϵ_m^2. The situation simplifies only for $n = 1$, when V and $\tilde{V}$ represent the same operator. In this case, we find

$$\begin{aligned} \nu_{\parallel}^{-1} &= 4 - 12\epsilon_m/(22 - m) \ , \\ \lambda_\mu &= \Phi_\mu \nu_{\parallel}^{-1} = 2 - 12\epsilon_m/(22 - m) \ , \\ \lambda_u &= \Phi_u \nu_{\parallel}^{-1} = 2 - 2(14 + m)\epsilon_m/(22 - m) \ . \end{aligned} \tag{12}$$

Note that for $m = d = D - 1$ these results coincide with our LTP*, Eq. (11). Experimental results[10,11] should be reanalyzed and compared with these corrected exponents.

It turns out that $\tilde{V}$ does not affect critical exponents to the leading order in the large n limit. Therefore, a single calculation yields results for all D and m. Following the usual procedure,[14] the renormalized parameters R, μ, and U are

$$R = R_0 + UI_0(R, \mu) + VI_2(R, \mu) \ , \tag{13a}$$

$$\mu = \mu_0 + VI_0(R, \mu) \ , \tag{13b}$$

$$U = U_0 + WI_0(R, \mu) \ , \tag{13c}$$

with

$$I_l(R, \mu) = \int d^D q \, |\mathbf{q}^{\parallel}|^l (R + \mu |\mathbf{q}^{\parallel}|^2 + |\mathbf{q}^{\parallel}|^4 + |\mathbf{q}^{\perp}|^2) \ .$$

The LTP surface is identified as $U_{\text{LT}} = -WI_0(0,0)$, $\mu_{0L} = -VI_0(0,0)$. For $D < 3 + m/2$, Eq. (13b) yields $\mu \propto R^{1/2}$, and $I_0(R, \mu) - I_0(0,0) \propto \mu/V \propto R^{1/2}/V$. Substituting into Eq. (13a), we obtain

$$R = t - AUR^{1/2} + BWR - CVR^{(2D - m - 2)/4} \ , \tag{14}$$

where A, B, and C are positive numbers and $t = R_0 - R_{\text{LT}}$. We thus identify

$$\begin{aligned} \gamma_{\text{LT}} &= 4/(2D - m - 2) \ , \\ \Phi_u = \Phi_\mu &= (2D - m - 4)/(2D - m - 2) \ , \\ \Phi_w &= -(6 + m - 2D)/(2D - m - 2) \ , \end{aligned} \tag{15}$$

in agreement with our ϵ-expansion results and the relation $\Phi_\mu \nu_{\parallel}^{-1} = (2D - m - 4) = 1/\nu$. For the fixed point[8] W^* (i.e., $V = 0$), Eq. (14) is replaced by

$$R = t - \tilde{A} UR^{(2D - m - 4)/4} + \tilde{B} WR^{(2D - m - 4)/2} \ . \tag{16}$$

Since $\tilde{B} > 0$, this equation does not have a solution for R which vanishes when $t \to 0$. The result is presumably a first-order transition, as in the $n \to \infty$ limit of the usual tricritical point.[9] Fortunately, the introduction of V renormalizes μ via (13b), destabilizes (16), and yields crossover to (14).

After completion of this work we became aware of the work of Dengler,[15] who has investigated the case $n = 1$ for LTP's and agrees with our results.

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