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Quantum diffusion in thin disordered wires

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Quantum Ohmic residual resistance of a thin disordered wire, approximated as a one-dimensional multichannel conductor, is known to scale exponentially with length. This nonadditivity is shown to imply (i) a low-frequency noise-power spectrum proportional to $-\ln(\Omega)/\Omega$, and (ii) a dispersive capacitative impedance proportional to $\tanh(\sqrt{i\Omega})/\sqrt{i\Omega}$. A deep connection to the quantum Brownian motion with linear dynamical frictional coupling to a harmonic-oscillator bath is pointed out and interpreted in physical terms.

The nonadditive and non-self-averaging nature of the residual resistance of a strictly one-dimensional conductor with static potential disorder is now well understood in terms of quantum diffusion in one space dimension, i.e., the one-dimensional (1D) and one-channel (1C) problem.[1-4] The realistic case of a thin disordered wire corresponds to the 1D and nC problem with n large.[5] In this large-n limit, the statistical fluctuation of resistance is expected to be relatively negligible. The nonadditivity, however, persists and the resistance scales exponentially with length as[5]

$$R(l)=R_s(\exp R_{\rm cl}/R_s-1),\ \ldots\ , \tag{1}$$

where $R_{\rm cl}$ is the classical linearly additive resistance and $R_s=r_s\hbar/e^2$ sets the scale of resistance with r_s of the order of unity. For a strictly one-dimensional wire $R_{cl}=(\pi\hbar/2e^2)l/l_e$ with l_e the mean free path for elastic backscattering. For a thin physical wire we set $R_{\rm cl}=(1/\nu)(\pi\hbar/2e^2)l/l_e$, where ν counts in some sense the parallel channels in the classical limit. The point essential to our treatment is the exponential growth of $R(l)$ with l for $R_{\rm cl}\gg R_s$. In this note we derive some physical consequences following from this nonadditivity which is of purely quantum origin. We believe these to be of significance to experiments on high-resistance ($R_{\rm cl}\gg R_s$) thin disordered wires at low temperatures, i.e., in the quantum regime when the inelastic mean free path $l_{\rm in}\gg l_e$. We will demonstrate in particular the appearance of a $(1/f)$-type noise, and of dispersive transport. We will also point out and interpret in physical terms a rather subtle connection between this and the problem of quantum diffusion in the presence of dynamical frictional coupling to a harmonic-oscillator bath. The latter has been and continues to be a subject of intense discussion in the contest of dissipative quantum motion.[6]

First, let us consider the question of $(1/f)$-type noise in such a resistive wire at absolute zero of temperature. For this we consider the quantum diffusion of an electron in the one-dimensional wire of length l whose resistance is given by Eq. (1). We assume $R_{\rm cl}\gg R_s$, so that $R(l)\simeq R_s\exp(R_{\rm cl}/R_s)$. This defines a length-scale-dependent diffusion constant $D(l)$ via the Einsetin relation for a degenerate electron system equating conductivity to $\frac{1}{2}e^2D(l)n_F$. Thus, we get

$$D(l)=(v_Fl)e^{-\beta l/l_e}\ , \tag{2}$$

where $\beta=l/2r_s\nu$, and $n_F=2/\pi\hbar v_F$ is the density of states at the Fermi level per unit length, and v_F the Fermi velocity. Now, following Thouless,[7] we can identify the mean-squared displacement with l^2 and the diffusion time with the time $\tau(l)$ to traverse the sample length l, and write for the length-scale-dependent diffusion constant

$$2D(l)=\frac{l^2}{\tau(l)}\ . \tag{3}$$

From Eqs. (2) and (3) we get for large τ

$$l\sim\frac{l_e}{\beta}\ln\tau+O(\ln\ln\tau)\ , \tag{4}$$

indicating slow diffusion.

We can now view the scaling eqation (4) as the long-time relation between the root-mean-square displacement $\langle X^2(t)\rangle^{1/2}$ and the diffusion time t for an electron within the sample of length l. The associated low-frequency noise-power spectrum $S_x(\omega)$ is given by[8]

$$S_x(\omega)=\lim_{T\to\infty}\left\{\frac{1}{2T}\left|\int_{-T}^{+T}X(t)e^{i\omega t}dt\right|^2\right\}\simeq-2\eta\frac{l_e^2}{\beta^2}\frac{\ln\omega}{\omega}\ , \tag{5a}$$

with

$$\eta\simeq\int_0^{\theta}\ln x\cos x\,dx\simeq O(l)\ \text{ for }\omega\to0\ , \tag{5b}$$

where

$$\theta=\frac{\omega l}{2v_F}\exp\frac{\beta l}{l_e}\ .$$

The upper limit θ in Eq. (5b) reflects the fact that $X(t)$ is bounded l. Equation (5a) is essentially the $1/f$ noise with a slow logarithmic correction factor. It is purely of quantum origin inasmuch as it results from the nonadditive resistance $R(l)$ growing exponentially with the sample length l—a cumulative effect of quantum interference. The latter makes dissusion slower with time. The closest classical analog of this is the $1/f$ noise discussed by Marinari, Parisi, Ruelle, and Windey,[8] where the particle moves classically in a spatially white-noise random static potential and is acted upon by a stochastic force. Again the classical motion gets slowed down with time as it is limited by progressively higher potential barriers to escape over. We should note that the present mechanism, depending on the nonaddivity of resistance in Eq. (1), is dominant only for the condition $R_{\rm cl}(l)\gg R_s$. Now at nonzero temperatures the sample length l in Eq. (1) must be replaced by $l_{\rm in}$ for $l_{\rm in}<l$ because of loss of coherence, and the condition then reads $R_{\rm cl}(l_{\rm in})\gg R_s$. This may get violated at higher temperatures leading to suppression of the $1/f$ noise originating

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from the present mechanism. This provides an experimental test for this mechanism.

We now turn to the question of dispersive transport in such a disordered system. In particular, we shall derive an expression for the ac impedance $Z_l(\omega)$ of such a disordered wire of length l, with $Z_l(0)=R(l)$. Let the origin $x=0$ coincide with the midpoint of the wire such that the wire is bounded by $x=\pm l/2$. Let $V(x)$ be the electrochemical potential at the point x when a dc current I_0 is established as a causative source such that $V(\pm l/2)=\mp V_0/2$. The electrochemical-potential drop across the wire is related to the current I_0 by $V_0=I_0R(l)$. Now the crucial point is that the electrochemical-potential difference $V(x)$ as defined operationally by an ideal potentiometer in a four-probe configuration is *not* given by $V(x]=V_0/2-I_0R[(l/2)+x)$ because of interference from the segment of the wire to the right of the point x. The correct expression for $V(x)$ is given by[9]

$$V(x)=-\frac{V_0}{2}\tanh\left[\beta\left(\frac{x}{l_e}\right)\right] . \tag{6}$$

We should remark here that Eq. (6) is an approximation of the exact expression given in Eq. (17) of Ref. 9 in that the reflection coefficients have been reexpressed in terms of the resistance $R(l)$ via the simple Landauer expression[5,7] and $\tanh(\beta l/l_e)\simeq 1$. We verify that $V(x)=0$ at the midpoint $x=0$, indeed as in the exact expression (17) of Ref. 9. Also, $V(x)\rightarrow \pm V_0/2$ as $x\rightarrow \mp\frac{1}{2}$, recalling that $\tanh(\beta l/(2l_e)\simeq 1$. For the current carrying resistor the change in the electrochemical potential is predominently Coulombic in nature. Thus, $V(x)$ in Eq. (6) can be taken as electrostatic potential leading to accumulation of charge given by the approximately 1D Poisson equation.

$$\frac{\partial^2 V}{\partial x^2}=\frac{-\rho(x)}{\epsilon_0 A} , \tag{7}$$

where A is the effective cross-sectional area of the 1D wire. We should note that for the usual case of the classical additive resistor $V(x)$ is linear in x and $\rho(x)$ is identically zero as is well known. Thus, again it is the nonadditivity of the quantum Ohmic resistance that leads to nonzero $\rho(x)$. Indeed, such a charging effect is to be expected from the "residual resistance dipole" concept of Landauer.[10] The present case is a continuum limit of the same. Thus, the charge per unit length is given by

$$\rho(x)=-\frac{A\beta^2}{l_e^2}V_0\,\mathrm{sech}^2\left[\beta\left(\frac{x}{l_e}\right)\right]\tanh\left[\beta\left(\frac{x}{l_e}\right)\right] . \tag{8}$$

Equations (7) and (8) together, define an effective capacitance per unit length

$$C(x)=\frac{\rho(x)}{V(x)}=\frac{A\beta^2}{l_e^2}\,\mathrm{sech}^2\left[\beta\left(\frac{x}{l_e}\right)\right] . \tag{9}$$

This leads to an equivalent circuit in which the resistor is shunted by distributed capacitance per unit length $C(x)$ connecting the point x to the point $x=0$ (midpoint). The distributed resistance $r(x)$ per unit length at point x is by definition

$$r(x)=-\left[\frac{dV}{dx}\right]/I_0=\frac{1}{2}\frac{\beta R(l)}{l_e^2}\,\mathrm{sech}^2\left[\beta\left(\frac{x}{l_e}\right)\right] . \tag{10}$$

We can now write down at once the time-dependent equations for the propagation of current $i(x,t)$, etc. on the above distributed resistance-capacitance line as

$$\frac{\partial i(x,t)}{\partial x}=-C(x)\frac{\partial V(x,t)}{\partial t} , \tag{11a}$$

$$\frac{\partial V(x,t)}{\partial x}=-i(x,t)r(x) . \tag{11b}$$

Substituting for $i(x,t)$ from Eq. (11b) into Eq. (11a) we get

$$D'\frac{\partial^2 V}{\partial x'^2}=\frac{\partial V}{\partial t} , \tag{12}$$

where

$$D'=\frac{2l_e^3}{\beta^3 A\epsilon_0 R(l)}$$

and

$$x'=\frac{l_e}{\beta}\tanh\left(\frac{\beta x}{l_e}\right) .$$

Equation (12) clearly displays a dispersive transport with *non-Gaussian* spread in the real x space. We will, however, be interested in the ac impedence $Z_l(\omega)$ for the above dispersive line. Straightforward network analysis gives

$$Z_l(\omega)=\frac{\tanh(\sqrt{i\,\Omega})}{\sqrt{i\,\Omega}}R(l) , \tag{13}$$

where $\Omega=(\omega/D')(l_e^2/\beta)$, the dimensionless circular frequency. The frequency scale is set by $\omega_c=D'\beta^2/l_e^2$, which is experimentally the most important single quantity in that $Z_l(\omega)$ shows structure around $\omega\sim\omega_c$. Expressing ω_c in terms of physically transparent quantities, we have

$$\omega_c\equiv 2\pi f_c=\frac{2l}{\epsilon_0 A_0}\frac{1}{R_{\mathrm{cl}}}e^{-R_{\mathrm{cl}}/R_s} . \tag{14}$$

For $R_{\mathrm{cl}}\sim 50$ kΩ, $l=0.01$ m, $A_0\sim 10^{-6}$ m^2 corresponding to a radius of 10^3 μm, and knowing $R_s\sim 10$ kΩ, and $\epsilon_0=1/36\pi\times 10^{-9}$ F m^{-1} we get $f_c\sim 10^7$ Hz. This would correspond to a material with extremely high residual resistivity. But, if we recall that the effective area A_0 can be much larger than the physical cross section of the wire because of fringe fields [i.e., strictly the 1D Poisson equation (7) is not valid], the estimates move within the range of realizability.

Finally, we turn to point out and interpret in physical terms a rather subtle connection between the quantum diffusion in a statically disordered conductor as discussed above and the diffusion in a dynamically disordered medium. Dynamical disorder has been treated by several workers as a parametric stochastic modulation of the potential, which is now known to be inconsistent with the physics of diffusion.[11] The microscopically consistent way to treat dynamical disorder is via a frictional coupling to a dynamical bath which can be taken, without loss of generality, as a set of harmonic oscillators coupled linearly to the electron. For this model one has for the mean-squared displacement

$$X^2(t)\simeq\left[\frac{2\hbar\gamma}{m\pi}\right]\ln(\sqrt{\omega_c}\gamma t) \tag{15}$$

in the quantum regime, $\gamma^{-1}\ll t\ll\hbar/k_BT$, where γ is the "Langevin" frictional coefficient and ω_c is the oscillator

bath cutoff frequency. Thus, for $T=0$ K, we have a slow logarithmic diffusion as $t \rightarrow \infty$, which is qualitatively similar to the quantum diffusion in our Eq. (4). This can be understood in the following physical terms. Static disorder scatters elastically and coherently in time while dynamic disorder scatters inelastically and incoherently. However, at $T=0$ K the oscillators are in the ground state and any inelastic real scattering involving energy transfer from the oscillators to the electron is energetically forbidden. Inelastic scattering involving excitation of oscillators, and concomitant slowing down of the electron, is still allowed, but such a process has eventually a high probability of being followed by a deexcitation of the same oscillator by the slow electron, as in the case of "phonon drag." This virtual excitation-deexcitation process is equivalent to an elastic scattering in which only the momentum is transferred to the bath (which is a dense set of spatially phase-uncorrelated *local* oscillators). In essence, as $T \rightarrow 0$ K, the elastic scattering dominates over the inelastic processes and eventually at $T=0$ K the ground-static oscillators simply act as static scatterers. This raises the interesting possibility of solving the original problem of quantum diffusion for $d>1$ in terms of the dynamical problem which appears tractable in all dimensions for the oscillator-bath model with linear frictional coupling.

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