Simple renormalization-group method for calculating geometrical equations of state

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We describe a real-space renormalization-group procedure to calculate geometrical equations of state for the entire range of values of the external parameters. Its use is as simple as a mean-field approximation; however, it yields nontrivial results and can be systematically improved. We illustrate it by calculating, for all bond concentrations, the site mass density for the complete and the backbone percolating infinite clusters in square lattice: the results are quite satisfactory.

Renormalization-group (RG) techniques have been employed mainly to evaluate critical points and exponents. However, a RG theory has been developed[1] to evaluate the free energy for the *whole range* of the thermodynamic parameters. From the free energy all other quantities of interest (such as magnetization, specific heat, susceptibility, and so on) can be evaluated. This theory and similar ones[2] apply to systems which can be described by a Hamiltonian formalism.

In this paper we want to develop a *simple* real-space RG formalism which may be useful for calculating, for the *whole* range of variation of external parameters such as concentration, *geometrical* quantities which cannot be described by a Hamiltonian formalism. These include, for example, the density of sites in a full percolating cluster and in its backbone for different types of percolation problems (random, correlated, bootstrap, valence, chromatic), and the conductivity for random resistor and random superconducting networks, etc. Great success has been achieved in the use of large cell RG techniques[3] for calculating critical points and exponents. The present method illustrates how this method can be easily adapted (and systematically improved) to extract, with precision, the interesting quantities for the *whole range of variation* of the external parameters. Last, but not least, let us compare the present procedure with the standard mean-field approximation: (i) they share the fact that the entire equation of states is obtained; (ii) for small cells it is, operationally speaking, even simpler; (iii) it yields nontrivial results which, as said before, can be systematically improved.

To illustrate this method we consider first the particular case of random-bond percolation. In this problem one supposes that each bond of a d-dimensional hypercubic lattice of linear size L has a probability p of being active. In the $L \to \infty$ limit, the order parameter P_∞ can be defined as $P_\infty = N_L(p)/L^d$, where $N_L(p)$ is the average number of sites in the infinite cluster. Furthermore, we associate with each site of the lattice an adimensional "mass" m_0; we could in principle choose $m_0 = 1$, but we will instead leave it here as a variable since it will change under renormalization. Following the original idea of scaling as introduced by Kadanoff, we divide the system of L^d sites into a system of L'^d cells of linear size $b = L/L' > 1$. Then we associate with each cell new renormalized variables $p' = f_1(p, m_0)$ and $m_0' = f_2(p, m_0)$. The renormalized variables depend on the particular RG, but they all satisfy the condition that the total mass of the infinite cluster in the system of cells be identical to the mass of the infinite cluster in the original system, namely,

$$N_{L'}(p')m_0' = N_L(p)m_0 \ . \tag{1}$$

Dividing both terms by L^d we obtain

$$P_\infty(p')m_0' = b^d P_\infty(p)m_0 \ , \tag{2}$$

where $P_\infty(p') = N_{L'}(p')/L'^d$.

After n iterations from (2) we have

$$P_\infty(p^{(n)})m_0^{(n)} = b^{nd}P_\infty(p)m_0 \ ; \tag{3}$$

hence,

$$P_\infty(p) = \lim_{n\to\infty} P_\infty(p^{(n)})\frac{m_0^{(n)}}{b^{nd}m_0} \ . \tag{4}$$

For p less than the percolation threshold p_c, $\lim_{n\to\infty} p^{(n)} = 0$. Since $P_\infty(0) = 0$ and $\lim_{n\to\infty} m_0^{(n)}/b^{nd} < \infty$ (this is related to the fact that whatever the cluster we choose to renormalize is, it will have at least one possible percolating path between terminals), we have the expected result $P_\infty(p) = 0$, *for all* $p < p_c$. For $p > p_c$, $\lim_{n\to\infty} p^{(n)} = 1$, and since $P_\infty(1) = 1$, we find

$$P_\infty(p) = \lim_{n\to\infty} \frac{m_0^{(n)}(p)}{b^{nd}m_0} \quad (p > p_c) \ . \tag{5}$$

This gives the desired expression for P_∞ *for all values of* p.

Let us illustrate the procedure on the square lattice by constructing a RG (denoted as RG_{12}) which maps the cell of Fig. 1(b) into that of Fig. 1(a) (hence, $b = 2$). The re-

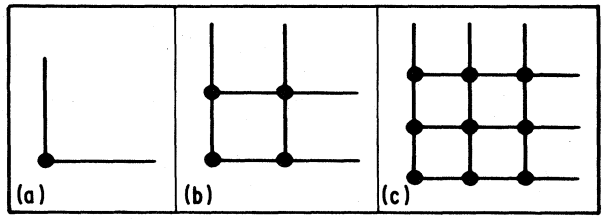

FIG. 1. RG cells used for the calculation of $P_\infty(p)$ and $P_\infty^B(p)$ (see Fig. 2) for square lattice. RG_{12}: cell (b) is renormalized into cell (a); hence, $b = 2$. RG_{13}: cell (c) is renormalized into cell (a); hence, $b = 3$. RG_{23}: cell (c) is renormalized into cell (b); hence, $b = \frac{3}{2}$.

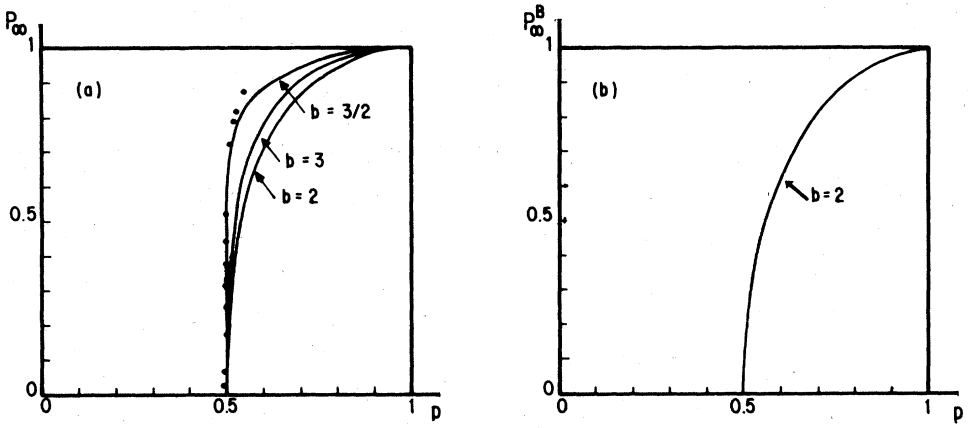

FIG. 2. (a) RG percolation probability P_∞ and (b) RG backbone mass (reduced) density P_∞^B as functions of the bond concentration p for the square lattice. b is the RG linear expansion (see Fig. 1); $p_c = \frac{1}{2}$ for all values of b; the dots represent Monte Carlo data (Ref. 5).

normalized bond probability p' is given by the probability of connecting the bottom to the top of the cells, namely,[4]

$$p' = p^5 + 5p^4(1-p) + 8p^3(1-p)^2 + 2p^2(1-p)^3 \equiv R_2(p) . \quad (6)$$

To find the equation for m_0 we consider all possible cell configurations and the corresponding total mass of the spanning cluster. Then we impose that the average mass is preserved, and obtain

$$\begin{aligned} p'm_0' &= [p^5 + 4p^5(1-p) + p^4(1-p) + \tfrac{21}{2}p^4(1-p)^2 \\ &\quad + \tfrac{13}{2}p^3(1-p)^3 + p^2(1-p)^4]4m_0 \\ &\equiv Q_2(p)m_0 , \end{aligned} \quad (7)$$

which, together with Eqs. (5) and (6), provides $P_\infty(p)$ for all values of p [see Fig. 2(a)]. Note that Eq. (7) is invariant through change of "mass" unit (i.e., $m_0 \to \lambda m_0$ and $m_0' \to \lambda m_0'$): This property is a general one, and enables us to choose $m_0 = 1$ in Eq. (5), which makes the calculation of $P_\infty(p)$ even simpler. The resulting critical exponents ν [for the connectedness length $\xi(p)$] and β [for $P_\infty(p)$] are given in Table I.

We also performed the calculation renormalizing the cell of Fig. 1(c) [6] into that of Fig. 1(a) ($RG_{13}; b = 3$). The role played by Eqs. (6) and (7) is now played, respectively, by

$$p' = 18p^{13} - 117p^{12} + 298p^{11} - 352p^{10} + 149p^9 + 39p^8 - 10p^7 - 37p^6 + 2p^5 + 8p^4 + 3p^3 \equiv R_3(p) \quad (8)$$

and

$$\begin{aligned} p'm_0' &= [9p^{15} + 135p^{14}(1-p) + 941p^{13}(1-p)^2 + 3996p^{12}(1-p)^3 + 11\,336p^{11}(1-p)^4 + 22\,031p^{10}(1-p)^5 + 28\,964p^9(1-p)^6 \\ &\quad + 24\,865p^8(1-p)^7 + 13\,970p^7(1-p)^8 + 5075p^6(1-p)^9 + 1158p^5(1-p)^{10} + 152p^4(1-p)^{11} + 9p^3(1-p)^{12}]m_0 \\ &\equiv Q_3(p)m_0 . \end{aligned} \quad (9)$$

We finally performed the calculation renormalizing the cell of Fig. 1(c) into that of Fig. 1(b) ($RG_{23}; b = \frac{3}{2}$). The corresponding recursive relations are given by

$$R_2(p') = R_3(p) \quad (10)$$

and

$$Q_2(p')m_0' = Q_3(p)m_0 . \quad (11)$$

The results obtained within RG_{13} and RG_{23} improve those obtained within RG_{12} as shown in Fig. 2(a) and Table I.

TABLE I. RG critical exponents $\nu(\xi \propto |p - p_c|^{-\nu})$ and $\beta[P_\infty \propto (p - p_c)^\beta]$ for several values of b (see Fig. 1).

	$b = 2$	$b = 3$	$b = \frac{3}{2}$	Exact
ν	$\frac{\ln 2}{\ln \frac{13}{8}} \simeq 1.428$	$\frac{\ln 3}{\ln \frac{4541}{2048}} \simeq 1.380$	$\frac{\ln \frac{3}{2}}{\ln \frac{4541}{3328}} \simeq 1.305$	$\frac{4}{3} \simeq 1.333$
β	$\frac{\ln \frac{16}{13}}{\ln \frac{13}{8}} \simeq 0.428$	$\frac{\ln \frac{49\,152}{37\,547}}{\ln \frac{4541}{2048}} \simeq 0.338$	$\frac{\ln \frac{39\,936}{37\,547}}{\ln \frac{4541}{3328}} \simeq 0.198$	$\frac{5}{36} \simeq 0.139$

Backbone percolation. We have also calculated the percolating probability $P_\infty^B(p)$ in the backbone of the spanning cluster. The calculation is analogous to the percolation probability $P_\infty(p)$. The RG_{12} equation for p' still is Eq. (6), and Eq. (7) is replaced by

$$p'm'_{0B} = [p^5 + 4p^5(1-p) + p^4(1-p) + \tfrac{19}{2}p^4(1-p)^2 + 6p^3(1-p)^3 + p^2(1-p)^4]4m_{0B} \quad . \tag{12}$$

The full curve for $P_\infty^B(p)$ (never obtained before, as far as we know) is given in Fig. 2(b); the corresponding critical exponent β_B equals $\ln\frac{64}{49}/\ln\frac{13}{8} \simeq 0.550$, to be compared with the Monte Carlo result[7] $\beta_B \simeq 0.53$.

Random resistor network. Although for the random resistor problem treatments related to the present one are available in the literature,[8] we shall discuss here this case for completeness. In this problem one supposes that each bond of a d-dimensional hypercubic lattice of linear size L carries a conductance g_0 with probability p, and 0 with probability $(1-p)$. Following the same procedure as before we find for the average conductivity $\Sigma(p)$ the relation

$$\Sigma(p^{(n)})g_0^{(n)} = b^{n(d-2)}\Sigma(p)g_0 \quad , \tag{13}$$

where $p^{(n)}$ and $g_0^{(n)}$, respectively, are the renormalized probability and conductance after n iterations. From (13) we have, for $p < p_c$, $\Sigma(p) = 0$ [see remarks below Eq. (4)], and, since $\Sigma(1) = g_0$, we have, for $p > p_c$,

$$\Sigma(p) = \lim_{n\to\infty} \frac{g_0^{(n)}(p)}{b^{n(d-2)}} \quad . \tag{14}$$

In conclusion, we have described a method based on a real-space renormalization-group procedure which allows us to obtain the variation of various interesting quantities *for all values of the external parameters.* This method applies to geometrical quantities for which large cell renormalization groups have successfully been employed to get accurate critical points and exponents; it uses essentially the *same information,* provides the *full variation* of the quantities, and is normally expected to recover the *correct asymptotic behaviors* in the vicinity of the trivial fixed points, i.e., when $\xi \to 0$ ($p = 0, 1$ for percolation problems). This method is, for small cells, operationally even simpler than the mean-field approximation procedures; it gives, however, nontrivial results; it constitutes, in principle, a reliable approximation in *both* critical and noncritical regions, and can be systematically improved.

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