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Breakdown of replica analyticity in the one-dimensional axis model

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We consider a random-bond one-dimensional m-component axis model, which has a transition for $m < 1$. We solve the model exactly with and without using the replica trick. Below the transition, the exact zeros of the partition function $\langle Z_N^n \rangle$ move towards the positive real n axis in the thermodynamic limit indicating the breakdown of analytic continuation: We *cannot* approach the $n = 0$ point once the thermodynamic limit is taken. We also show that we cannot interchange the thermodynamic and the replica limits below the transition.

In recent years, we have seen important progress in our theoretical understanding of the properties of random systems, notably spin-glass systems.[1–3] A great deal of this progress is due to Parisi,[4] who provided a stable scheme for replica symmetry breaking (RSB).[5–9] Sompolinsky[10] provided a time-dependent RSB. In order to break RS, one *must* take the thermodynamic limit ($N \to \infty$) before the replica limit ($n \to 0$), even though the correct "replica trick"[11–13] demands taking $n \to 0$ before $N \to \infty$. There are no rigorous results available that show the validity of this interchange of limits in the general case, even though it is either implicitly or explicitly *assumed* to hold in *all* schemes of RSB. Moreover, taking the $n \to 0$ limit implies invoking analytic continuation in the replica variable n. It has been shown rigorously by van Hemmen and Palmer[13] that for the Sherrington-Kirkpatrick model,[12] the interchange of the limit is valid for the analytic continuation of the (wrong) solution adopted by Sherrington and Kirkpatrick. (This continuation leads to a negative entropy at low temperatures.) However, nothing is known about the correct (but yet unknown) analytic continuation in the replica variable, even though similar attempts have been made in other cases,[14,15] where similar formal limits are taken. The major impediment to progress in the present case has been the absence of any results regarding the distribution of zeros of $\langle Z_N^n \rangle$ as the thermodynamic limit $N \to \infty$ is taken.

In our current pursuit to understand the physical significance of RSB and its implications for the physical system[4–9,16] we have elevated this assumption of the validity of the interchange of limits to the level of an identity. However, in the absence of any knowledge about the analyticity of the continuation and, in particular, about the distribution of zeros of $\langle Z_N^n \rangle$, it is imperative that we have some exact results about the validity of the interchange of limits. It is this question that we address here. Our *exact* calculation shows that the interchange is not allowed. One must take the $n \to 0$ limit before $N \to \infty$. This means that there should be no RSB, since this requires reversing the order of limits. This is quite disturbing, in view of the apparent success of the RSB scheme of Parisi in explaining some of the observed effects. Thus, our work, which is the first of its kind in providing exact results regarding the distribution of zeros and the interchange of limits, is important in that it suggests that we ask why RSB is so successful in our understanding of spin-glass systems, even though the main assumption behind it cannot be justified.

In order to obtain exact results, we consider a random-bond one-dimensional axis model with spins with m components. This model gives rise to a phase transition at finite nonzero temperature for $m < 1$. It should be emphasized that the extension of the axis model to unphysical values of m should not worry us, as we are interested only in comparing the results obtained with and without the replica trick (using both orders of limits). Throughout our analysis, we consider $m \geqslant 0$. Thus, negative m's are not allowed. It is known that the $m = 0$ pure system corresponds to a polymer problem.[17] We evaluate the partition function exactly both with and without the replica trick. This enables us to look at the distribution of zeros of $\langle Z_N^n \rangle$ explicitly. The interesting situation occurs below the transition temperature for $m < 1$. We find here that as $N \to \infty$ the zeros begin to move towards the positive real n axis at rational values of n, given by the ratios of two odd integers, indicating phase transitions at these points. This means that the "whole" n axis is *singular*. Thus, there is no way to continue the model to $n = 0$. We also show explicitly that the limits cannot be interchanged. Thus, our results go beyond those of van Hemmen and Palmer[13] by providing all the missing links in their work. There is no problem with the analytic continuation and the interchange of the limits above the transition temperature.

Let $\mathbf{S}_i$ denote an m-component axis spin located at the site i, $i = 1, 2, \ldots, N$ in one dimension. For the sake of clarity, we take periodic boundary condition ($\mathbf{S}_{N+1} = \mathbf{S}_1$). The spin is allowed to take the following $2m$ orientations with equal probability: $(\pm 10, 0, \ldots, 0)$, $(0, \pm 1, 0, 0, \ldots, 0)$, $(0, 0, \pm 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 0, \pm 1)$. The effective Hamiltonian[18] of the system is given by

$$H = \sum_i K_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} , \tag{1}$$

where the variable K_i is a random variable taking the values $\pm K$ with equal probability. It is easily seen that the transfer matrices $T(K) = \exp(K \mathbf{S} \cdot \mathbf{S}')$ and $T(-K) = \exp(-K \mathbf{S} \cdot \mathbf{S}')$ commute. Therefore, they can be diagonalized simultaneously. The eigenvalues of $T(K)$ and their degeneracies are given by

$$\begin{aligned} \alpha &= x + 1/x + 2\overline{m};\ \text{1-fold} , \\ \beta &= x - 1/x;\ m\text{-fold}, \\ \gamma &= x + 1/x - 2;\ \overline{m}\text{-fold} , \end{aligned} \tag{2}$$

where $x=\exp(K)$ and $\bar{m}=m-1$. In the following, we will consider only $m \geqslant 0$. For $m>0$, $\alpha>\gamma$: The two levels never cross, except when $m=0$. However, when $0<m<1$, α and β cross at a critical point

$$1/x_c = e^{-K_c} = 1-m \ , \tag{3}$$

indicating a phase transition at x_c. For $x<x_c$, $\alpha>\beta$, and for $x>x_c$, $\beta>\alpha$. Let us now evaluate the free energy per particle. In a given configuration Γ_L with L antiferromagnetic bonds and the rest ferromagnetic bonds, the free energy is given by

$$f_N(\Gamma_L) = (1/N)\ln Z_N(\Gamma_L) = (1/N)\ln[\alpha^N + m(-)^L\beta^N + \bar{m}\gamma^N] \ .$$

At this point, we wish to make a simple but very important point regarding the evaluation of f_N: Since $\ln Z_N$ is a multivalued function whose value depends on the choice of the Riemann sheet, it is *customary* to choose the sheet that gives a real value for f_N. For physical systems, we certainly want a real free energy. However, when analytic continuation to noninteger or complex m is desired, there is no reason to expect a real f_N. Thus, we may choose any particular Riemann sheet that we wish in order to evaluate f_N; it is given by

$$f_N = (1/N)(ln|Z_N| + C) \ ,$$

where C is a constant, independent of N, that depends on the choice of the Riemann sheet. For example, if Z_N is real and positive (or negative), C is given by $2k\pi i$ [or $(2k+1)\pi i$], where $k=0,1,2,3$, etc. However, this ambiguity in f_N disappears as $N \to \infty$; the choice of the particular Riemann sheet is immaterial in the thermodynamic limit. (In the following, we will always take $N\to\infty$.) Thus, our definition of f_N gives a real free energy in the thermodynamic limit for integer values of m and agrees with its customary definition there. Let us now evaluate the average $\langle f_N \rangle$ given by

$$\langle f_N \rangle = (1/2^N N) \sum_{L=0}^{N} \binom{N}{L} f_N(\Gamma_L) = (1/2N)[\ln(\alpha^N + m\beta^N + \bar{m}\gamma^N) + \ln(\alpha^N - m\beta^N + \bar{m}\gamma^N)] \ . \tag{4}$$

The free energy per site in the thermodynamic limit is given by

$$\langle f \rangle = \lim_{N\to\infty} \langle f_N \rangle = \begin{cases} \ln\alpha, & x<x_c \\ \ln\beta, & x>x_c \end{cases} . \tag{5}$$

Equation (5) is valid only for $0<m<1$. For $m \geqslant 1$, $x_c \to \infty (T=0)$ and the phase above $T=0$ is described by the eigenvalue α. For $m=0$, $T_c \to \infty$ $(x_c=1)$.

The only nonanalyticity in $\langle f \rangle$ appears at x_c. Therefore, our direct evaluation of $\langle f \rangle$ gives a perfectly sensible result which exhibits a critical point at $x=x_c$, even though we have an unphysical choice of m: $m<1$.

Let us now use the replica trick and replicate the system n times, where $n=1,2,3,\ldots$. This will enable us to evaluate $\langle Z_N^n \rangle$. It is easy to see that

$$\langle Z_N^n \rangle = \tfrac{1}{2}[(\alpha^N + m\beta^N + \bar{m}\gamma^N)^n + (\alpha^N - m\beta^N + \bar{m}\gamma^N)^n] = \sum_{k\ \mathrm{even}} \binom{n}{k} a^{n-k} b^k > 0 \ ,$$

where $a=\alpha^N+\bar{m}\gamma^N>0$, and $b=m\beta^N>0$. Following van Hemmen and Palmer,[13] we introduce $\phi_N(n)=(1/N)\times\ln\langle Z_N^n\rangle$,

$$\phi_N(n) = (1/N)\ln\{\tfrac{1}{2}[(a+b)^n + (a-b)^n]\} \ . \tag{6}$$

This function can be immediately continued to any value n by treating n as a real or complex variable in (6). To distinguish this analytic continuation from $\phi_N(n)$ derived only for integer n, we will denote it by $\tilde{\phi}_N(n)$: $\tilde{\phi}_N(n)=\phi_N(n)$ for all n, as long as $N<\infty$.

Let us now calculate its derivative with respect to n at $n=0$:

$$\tilde{\phi}_N'(0) = \frac{1}{N} \left. \frac{(a+b)^n \ln(a+b) + (a-b)^n \ln(a-b)}{(a+b)^n + (a-b)^n} \right|_{n=0} = \frac{\ln(a^2-b^2)}{2N}$$

Therefore,

$$\lim_{N\to\infty} \tilde{\phi}_N'(0) = \begin{cases} \ln\alpha, & x<x_c \\ \ln\beta, & x>x_c, \end{cases} \tag{7}$$

and we find that $\langle f \rangle = \lim_{N\to\infty}\tilde{\phi}_N'(0)$ as expected.[13]

Let us now evaluate

$$\phi(n) = \lim_{N\to\infty}\phi_N(n) = \begin{cases} n\ln\alpha, & x<x_c, \\ n\ln\beta, & x>x_c \text{ with } n \text{ even}, \\ \ln\alpha + \bar{n}\ln\beta, & x>x_c \text{ with } n \text{ odd}, \end{cases} \tag{8}$$

where $\bar{n}=n-1$. Equation (8) is valid for all integers $n=1,2,3,\ldots$. For $x<x_c$, it is evident that the analytic continuation $\tilde{\phi}(n)$ of $\phi(n)$ for all n is $\tilde{\phi}(n)=n\ln\alpha$, so that $\tilde{\phi}(0)=0$ as expected. However, for $x>x_c$, there are *many* different continuations $\tilde{\phi}(n)$. The following is one such possibility:

$$\tilde{\phi}(n) = (n/2)[1+(-)^n]\ln\beta + \tfrac{1}{2}[1+(-)^{\bar{n}}](\ln\alpha + \bar{n}\ln\beta) \ , \tag{9}$$

so that $\tilde{\phi}(0)=0$. However,

$$\tilde{\phi}'(0) = \ln\beta - \tfrac{1}{2}i\pi(\ln\alpha - \ln\beta) \neq \langle f \rangle \ . \tag{10}$$

Thus, the interchange of the two limits, $n\to 0$ and $N\to\infty$, does *not* give identical results, contrary to the assumption made in *all* RSB schemes.

Let us try to obtain $\tilde{\phi}(n)$, not from $\phi(n)$, given in (8), but from $\tilde{\phi}_N(n)$. Let us first consider $n>0$. It is easily seen that for $x<x_c$, $\tilde{\phi}(n)=\phi(n)=n\ln\alpha$ for all n. However, for $x>x_c$, $\tilde{\phi}(n)=\phi(n)=n\ln\beta$ for all n, except when n is odd. For n odd, $\tilde{\phi}(n)=\phi(n)=\ln\alpha+\bar{n}\ln\beta$. This means that our new $\tilde{\phi}(n)$ is identical to $\phi(n)$ given in (8) except that $\tilde{\phi}(n)=n\ln\beta$ not only for even n, but all n, except when it is odd. This continuation is different from the one in (9). However, our most important observation is that $\tilde{\phi}(n)$ has *singularities* at odd n's for all $x>x_c$. In the following we will see that the singularities are at all n's given by the ratio of two odd integers. This is evident from (6), and is shown in Fig. 1.

Let us look at this point more carefully. Let us locate the zeros of $\langle Z_N^n \rangle$, when $x>x_c$. A simple algebra gives the

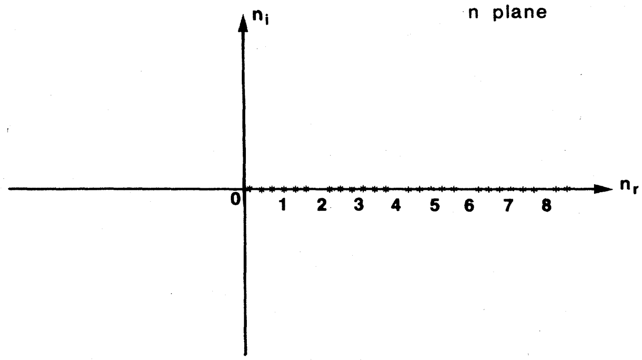

FIG. 1. Location of the zeros of $\langle Z_N^n \rangle$ as $N \to \infty$. The zeros are on the positive real n axis at $(2k+1)/(2l+1)$, where k and l are positive integers. There are *no* zeros when n is even. The above picture is true only below the transition temperature.

zeros at $n = n_r + in_i$, where

$$n_r = \frac{2(k+1)\pi^2}{\pi^2 + (\ln\xi)^2}\ ,\qquad n_i = \frac{n_r \ln\xi}{\pi}\ , \tag{11}$$

where $k = 0, 1, 2, 3, \ldots$ and $\xi = (1-a/b)/(1+a/b)$. It is now clear that as $N \to \infty$ $(x > x_c)$, $a/b \to 0$ and the zeros of the partition function begin to move to the real axis at odd n's. As a matter of fact, it can readily be seen that the situation is even worse. By writing $(-)^n$ as $\exp[(2l+1)in\pi]$, we find that

$$n_r = \frac{(2k+1)(2l+1)\pi^2}{[(2l+1)\pi]^2 + (\ln\xi)^2}\ ,\qquad n_i = \frac{n_r \ln\xi}{(2l+1)\pi}\ , \tag{12}$$

where $k, l = 0, 1, 2, \ldots$. Therefore, as $N \to \infty$, i.e., as $\xi \to 1$, we find that $n_i \to 0$, and $n_r = (2k+1)/(2l+1)$. By a proper choice of k and l, we can get any rational point given by the ratio of two odd integers on the positive real axis. The function $\tilde{\phi}(n)$ is singular over the whole real n axis. Therefore, there is no hope for analytic continuation in our model once $N \to \infty$ has been taken.[19]

This means that the main assumption behind RSB *cannot* be substantiated. The hope for analytic continuation to $n = 0$ cannot materialize. This does not mean that RSB has nothing to do with the physics of the system. All we have shown is that the assumption of the interchange of limits cannot be justified in general.

One can also compute $\tilde{\phi}(n)$ for $n < 0$. Again, we obtain $\tilde{\phi}(n)$ from $\phi_N(n)$. It is evident that because $n < 0$,

$$\tilde{\phi}(n) = n \ln\gamma \quad \text{for all } x \text{ and } n < 0\ .$$

This means that $\phi'(n)$ is *not* continuous at $n = 0$. Moreover, there are *no* singularities when $n < 0$.

Let us briefly summarize our exact results. By solving a one-dimensional axis model for $m < 1$, we have been able to obtain a few exact results. We have shown that the zeros of $\langle Z_N^n \rangle$ begin to pinch on the whole positive real n axis at rational points given by the ratios of two odd integers, as the thermodynamic limit $N \to \infty$ is taken. Thus, there is no hope for an analytic continuation of $\phi(n)$ to $n = 0$. We have also shown that one cannot interchange $N \to \infty$ and $n \to 0$. Thus, the main assumption behind the physical relevance of RSB for spin glasses is not valid in our case. It should be emphasized that RSB may indeed be important for the physics of real spin-glass systems. All we have shown here is that for the model we have considered, the basic assumption behind it cannot be justified. Therefore, we must ask why RSB is so successful in light of this result.

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[18] A factor of $-\beta = -1/kT$ is absorbed in the definition of the effective Hamiltonian.

[19] It may be thought that our pathology has something to do with the informal $m < 1$ feature of the theory. However, this is not the case. It is clear from (6) that the zeros of $\langle Z_N^n \rangle$ are due to the interplay between two terms $(a+b)^n$ and $(a-b)^n$, and are determined by the relative magnitudes of α and β. The constraint $m < 1$ merely allows us to look at the region "below" the transition point, which could not have been possible if m was restricted to be a positive integer. Moreover, as shown above, our direct evaluation of $\langle f \rangle$ does not suffer from any problem. The same result is also obtained if one takes $n \to 0$ before $N \to \infty$. Therefore, the pathology is certainly due to $N \to \infty$ limit before $n \to 0$ limit.