Nonuniversal critical exponents for transport in percolating systems with a distribution of bond strengths

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Numerical simulations are used to examine the dependences of the percolation transport exponents on the distribution of bond strengths g in two-dimensional models. We use the probability density function $p(g)=g^{-\alpha}$, a case that arises naturally in percolation of continuum systems. Our results are consistent with earlier predictions that for $0<\alpha<1$ the exponent $\bar{t}$ differs from its counterpart t in the standard discrete lattice percolation networks by $(\bar{t}-t)\approx\alpha/(1-\alpha)$, while for $\alpha<0$, the exponents $\bar{t}$ and t are equal.

We present in this note numerical results confirming that the transport critical exponent can be nonuniversal for percolating networks with a distribution of bond strengths. These network models, with distributions of bond strengths, arise naturally in continuum models, such as the random-void model, where circular or spherical holes are randomly placed in a uniform transport medium.[1-3] Our numerical results support the argument that the exponents governing the behavior of elasticity, electrical conductivity, and fluid permeability in such media can be substantially larger than the corresponding ones in the conventional lattice percolation models.[1,4] In particular, the present calculations support previous results[1] for discrete systems with a distribution of bond strengths obtained using a nodes-links-blobs analysis, as well as the results obtained much earlier by Kogut and Straley,[5] and by Ben-Mizrahi and Bergman[6] using various other approximation methods. These nonuniversal transport exponents may be contrasted with the exponents for *geometrical* percolation properties, such as the correlation length exponent ν, which are clearly independent of bond strengths.

In this paper we consider the conductivity of a two-dimensional (2D) square network where the bond conductance has a value of g, which is an uncorrelated random variable with a power-law probability density function

$$p(g)=\begin{cases}(1-\alpha)g^{-\alpha}, & 0<g\leq 1\ ,\\ 0, & g>1\ .\end{cases} \tag{1}$$

We remove bonds randomly, and compute the conductivity as a function of the fraction p of bonds remaining. The value of α is chosen according to considerations described in the next few paragraphs. These considerations indicate that $\alpha=0.6$ applies to fluid permeability in the 2D continuum with flow in between the obstacles, and that $\alpha=-1$ applies to the case of electrical conductivity in the same model.

Reference 1 discusses the mapping of the random-void model onto a discrete random network, as described earlier by Elam, Kerstein, and Rehr.[2] In the 2D case of random circular holes of radius a punched in a conducting sheet, a bond is present if the two neighboring holes do not overlap. The "strength" of the ith bond depends crucially on the channel width δ_i. If we approximate the ith neck by a thin rectangle of width δ_i and length $l_i\approx(\delta_i a)^{1/2}$, the 2D electrical conductance is given by $g_i\approx\delta_i^{1/2}/a^{1/2}$, while the permeability of a viscous fluid flowing through the neck varies as $\delta_i^{5/2}/a^{1/2}$.

In the 3D random-void model, the smallest cross section of a bond has roughly the shape of a triangle, and we find that the bond has an electrical conductance $g_i\approx\delta_i^{3/2}/a^{1/2}$, and the flow of a viscous fluid through a narrow channel is proportional to $\delta_i^4/l_i=\delta_i^{7/2}/a^{1/2}$. The force constant γ_i for bond bending or for torsion is proportional to $\delta_i^{7/2}$ in 3D, and is proportional to $\gamma_i\delta_i^{5/2}$ in 2D.

Thus, in all these cases, the bond strength varies as a power law,

$$g\approx\delta^m\ . \tag{2}$$

In all these cases we also find that the probability distribution $P(\delta)$ goes to a finite constant for $\delta\to 0$. Since the probability distribution of δ is related to that of g by $P(\delta)\,d\delta=p(g)\,dg$, it follows that $p(g)$ has the form of Eq. (1), for $\delta\to 0$, with

$$\alpha=1-1/m\ . \tag{3}$$

In order to establish the connection between the transport exponent and α, the simplest and most direct route is to consider a 2D electrical network with a distribution given by Eq. (1). That is what we proceed to do in the next section.

As the fraction of present bonds p approaches p_c, the coherence length $\xi\sim(p-p_c)^{-\nu}$ diverges, so there will be finite sample size errors, regardless of how large a value we choose for the sample size L. We employ the finite-size scaling calculational scheme[4,7,8] to overcome this problem. In this approach, we fix p at the exact value of p_c, and vary L. Because of the self-duality of the square lattice, p_c is $\frac{1}{2}$ exactly. The dependence of the properly averaged transport coefficients on the sample size gives the percolation critical exponent(s). This dependence can be expressed as

$$\bar{C}\sim L^{-\bar{t}/\nu}[a_1+a_2 g_{\bar{t}}(L)] \tag{4}$$

at $p=p_c$. Here $\bar{C}$ is the conductivity of the network with bond strengths given by Eq. (1) for the present bonds and zero conductance for the missing ones; a_1, a_2 are constants. The function $g_{\bar{t}}(L)$ goes to zero for large L.

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Now we will proceed to describe the methods and details of our numerical calculation.

First we generate a square lattice of size $L \times L$ with nearest-neighbor bonds which are randomly occupied with probabitity $p = p_c = \frac{1}{2}$. Then a search routine is used to test whether the two opposite ends of the network are connected or not. If they are, we give each occupied bond a strength according to Eq. (1). The nodes of the top row are tied to a bus bar of voltage 1, and the nodes of the bottom row are tied to a bus bar of voltage 0. Then a sparse matrix inversion solving routine (kindly provided to us by Lasseter[9]) is used to solve for all the node voltages. The conductance $\overline{C}_i$ is calculated for each of $n = 100$ different realizations. Four different statistics of the data are then calculated: the arithmetic mean $M_A = (1/n) \sum_i^n \overline{C}_i$, the geometric mean $M_G = (\prod_i^n \overline{C}_i)^{1/n}$, the harmonic mean $M_H = [(1/n) \sum_i^n \overline{C}_i^{-1}]^{-1}$, and standard deviation $\sigma = [(1/n) \sum_i^n (\overline{C}_i - M_A)^2]^{1/2}$. The entire procedure is repeated for ten different values of L. Equation (4) (with a_2 set equal to zero) is then used to extract the exponents. The results are shown in Fig. 1 for $m = 0$, 1, and 2.5, respectively. The results are summarized in Table I.

We used ten values of L starting with $L = 4$ and going up to $L = 49$. We employ the least-square-fitting technique to obtain the slope, and estimate the errors in slope by dividing the sum of error squared, χ^2, by the standard deviation to get the R factor.[10] The closer the coefficient R of linear regression is to unity, the better is the least-squares fit. The final error for $\overline{t}/\nu$ is estimated by taking the largest among the errors of the individual slopes and the differences in slopes between the different types of average. Note that our estimate completely ignores systematic errors and presumes that the error is purely statistical.

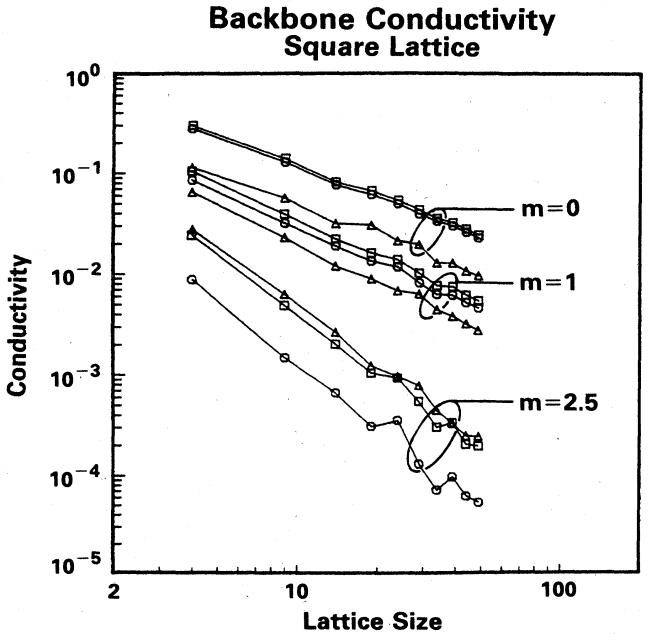

FIG. 1. The conductivity of a sample as a function of size for the standard case of constant bond strength, $m = 0$, as well as the nonstandard cases of $m = 1$ and 2.5 in Eq. (2). The arithmetic mean M_A (square), the geometric mean M_G (circle), and the standard deviation σ (triangle) are shown.

We will now look at the results for specific exponent values. First consider the case of $m = 0$, i.e., the standard percolation problem. We note that M_A (square), M_G (circle), and σ (triangle) all converge to narrowly separated parallel lines. For this case of ordinary conductivity, our estimate of $t/\nu = 0.987 \pm 0.03$ agrees well with the most accepted value[8] $t/\nu = 0.973(+0.005, -0.003)$. In this case the harmonic mean (not shown) also follows a line close to the other three statistics.

Next consider the cases of $m = 1$ and 2.5. Although M_A (square), M_G (circles), σ (triangle), give parallel lines, the M_H (not shown in Fig. 1), i.e., the average of the macroscopic resistances, gives a different slope (see Table I). It may be seen from Eqs. (1) and (3) that for $m > 1$ in our finite-size sample, the harmonic-mean conductance will actually be infinite, if one averages over an infinite number of realizations. In the present case, we discard the M_H. The differences between the various statistics are much larger in the case of $m = 2.5$, where there is wide distribution of bond resistances than in the case $m = 0$, where all the bonds are the same, or $m = 1$, where there is less likelihood of a large resistance. Fortunately, all the quantities (except M_H) show the same scaling dependence, within the error bars, on L.

As was discussed in Ref. 1, the nodes-links-blobs analysis gives an estimate $\overline{t} = m$, for $m = 1/(1-\alpha) > 1$, which we believe to be a lower bound to the true value of $\overline{t}$. On the other hand, the variational analysis of Kogut and Straley[5] obtained an estimate $\overline{t} = t + m - 1$, for $m > 1$, which is a rigorous upper bound to $\overline{t}$. Since $t \approx 1.29$, the difference between the (nonrigorous) lower bound and the rigorous bound is not very large. The numerical results of Table I are certainly consistent with the nonrigorous lower bound. In fact, the observed values of the exponent $\overline{t}/\nu$ are slightly larger than the rigorous upper bound for the case $m = 1$. If we used the value $\nu = \frac{4}{3}$, which is believed to be exact, the

TABLE I. Least squares fit of $\log_{10}\overline{C}$ vs $(\overline{t}/\nu)\log_{10}(L) + \log_{10}(a_1)$ for M_A, M_G, M_H, and σ (see text for definitions).

	$\frac{\overline{t}}{\nu}$	$\log_{10}(a_1)$	$\Delta\frac{\overline{t}}{\nu}$	R
		$m = 2.5$		
M_A	1.934	0.458	0.059	0.996
M_G	2.044	0.840	0.102	0.990
σ	1.929	0.378	0.049	0.997
M_H	3.813	0.967	0.649	0.901
		$m = 1.0$		
M_A	1.171	0.281	0.023	0.998
M_G	1.161	0.375	0.026	0.998
σ	1.232	0.459	0.030	0.998
M_H[a]	1.234	0.444	0.055	0.992
		$m = 0.0$		
M_A	0.987	0.078	0.016	0.999
M_G	0.987	0.047	0.013	0.999
σ	0.990	0.316	0.044	0.992
M_H[a]	0.988	0.017	0.011	1.000

[a] Not used.

upper bound requires that $\bar{t}/\nu$ should not exceed 0.97 for $m=1$ and 2.09 for $m=2.5$, respectively. We have studied several other cases of m and found the results to be consistent with the bounds. For example, the arithmetic mean gives $\bar{t}=1.37\pm0.02$ for $m=0.5$, $\bar{t}=1.84\pm0.05$ for $m=1.5$, and $\bar{t}=2.20\pm0.06$ for $m=2.0$.

The discrepancy in the case of $m=1$, is presumably due to a slow transient in the approach to the asymptotic critical exponent for infinite L, or possibly due to logarithmic correction ($g_{\bar{t}}\sim\log_{10}L$) to the critical behavior. This correction leads to larger effective exponents in the observed range of lattice sizes. Evaluation of a very precise value of the exponent $\bar{t}$ in the transition region of $m=1$ would require a great deal of computation and is beyond the scope of this paper. Nevertheless, the discrepancy in the case of $m=1$ is fairly small, and the cases of $m=1.5$, 2.0, and 2.5 provide strong numerical evidence in support of previous theoretical findings that there should be new transport exponents in cases where the bond strengths have a distribution of values. Our results show that the value of the transport exponent depends very strongly on the details of the bond-strength-distribution density function.

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