Critical behavior of the quantum double-sine-Gordon model

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We investigate the critical behavior of systems modeled by the quantum double-sine-Gordon theory. We find the existence of a Kosterilitz-Thouless transition in a region of the phase space. For different ranges of the physical parameters, the model is shown to be equivalent to a free massive Dirac field.

Two-dimensional models have played a crucial role in statistical mechanics. Much of our understanding of phase transitions in our three-dimensional world has come from insights drawn from models such as the Ising model, the X-Y model, and other two-dimensional models. Recently, two dimensions became an experimental world in its own right and revealed a world of rich and intriguing phenomena. With these discoveries has come a resurgence in the theory of two-dimensional systems. This has been stimulated also from high-energy physics, since the study of phase transitions in two dimensions could shed some light on the problem of quark confinement.

Our aim is to report on the study of the phase transition occurring in the double-sine-Gordon (DSG) model. This model is a widely studied[1] nonintegrable theory of a single scalar field in two dimensions, whose dynamics is described by the classical Lagrangian density:[2]

$$\mathscr{L}(\tilde{\phi},\eta)=\tfrac{1}{2}\,\partial_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi}-\frac{1}{1+4|\eta|}\times\frac{\alpha_0}{\tilde{\beta}^2}[-\cos(\tilde{\beta}\tilde{\phi})+\eta\cos(2\tilde{\beta}\tilde{\phi})]+\gamma_0 \quad (1)$$

In Eq. (1) the sum over repeated indices is implied and α_0, $\tilde{\beta}$, γ_0, and η are real parameters. Without loss of generality, we can take $\tilde{\beta}$ to be positive and fix γ_0 by adjusting our zero of energy density so that the minimum energy is zero. As η is varied over the range $-\infty<\eta<+\infty$, the model describes a variety of physical systems such as magnetic chains,[3,4] He^3 (Ref. 5), or polymers.[6] Of particular interest to us is the range $\eta\geq 1$, where Eq. (1) has been used to model antiferromagnetic Heisenberg chains[6,7] and—in high-energy physics—the bosonic sector of a confining model.[8] In the following we will restrict ourselves to this range of η, where for computational convenience[9,10] we take $\eta=\frac{1}{4}\sinh^2 R$.

Most of the studies on the DSG model have focused on the properties and dynamics[9] of the solitary wave solutions of Eq. (1). However, for many applications to realistic systems, the classical thermodynamics has also been investigated.[3,6,10,11] In this Brief Report we make a step forward in the analysis of the statistical mechanics of this model by reporting the result of an attempt to analyze its critical behavior. For this purpose we investigate the quantum[12] DSG model showing that, in a suitable approximation, this theory can be mapped in the quantum-sine-Gordon model. As a consequence we have the following: (i) the existence of a critical line of Kosterlitz-Thouless type[13] separating two physically different zones in the parameter space $R-\tilde{\beta}$; (ii) the existence in the plane $R-\tilde{\beta}$ of a line on which the DSG equation describes the charge zero sector of a free massive Dirac field theory.

As a by-product, we provide, in the context of quantum field theory, a version of the conjecture[13] about the irrelevance of higher than unit charges in the phase transition of the X-Y model. In fact, one can show[13] that those charges correspond to adding higher harmonics [i.e., terms such as $\cos(\eta\tilde{\beta}\tilde{\phi})$] to the sine-Gordon Lagrangian.

Our analysis starts from the consideration that, for any theory of a single scalar field in two dimensions with nonderivative interactions, all divergences that occur in any order of the perturbation theory can be removed by normal ordering of the Hamiltonian.[14,15] For the DSG model normal ordering is easily performed if one recalls[16] that, for $\tilde{\beta}^2\simeq 4\pi$,

$$:\!\cos\tilde{\beta}\tilde{\phi}\!:^2\equiv\lim_{\Lambda\to\infty}\lim_{x\to y}:\!\cos[\tilde{\beta}\tilde{\phi}(x)]\!::\!\cos[\tilde{\beta}\tilde{\phi}(y)]\!:=-\frac{8}{\tilde{\beta}^2}(\partial_\mu\phi)^2 \ . \quad (2)$$

Here Λ is a suitable cutoff and enclosure by a pair of colons (: :) denotes normal ordering in respect to an arbitrary mass m, which is set equal to one in the following.

Use of Eq. (2) leads to the normal ordered version of the Lagrangian density [Eq. (1)]:

$$\mathscr{L}=\tfrac{1}{2}\left[1+\frac{16\eta\alpha^0}{(1+4\eta)\tilde{\beta}^4}\right](\partial_\mu\tilde{\phi})^2+\frac{\alpha_0}{(1+4\eta)\tilde{\beta}^2}:\!\cos(\tilde{\beta}\tilde{\phi})\!: \ . \quad (3)$$

Upon rescaling the field $\tilde{\phi}$ according to

$$Z\tilde{\phi}=\phi\ , \quad (4a)$$

$$Z=1+\frac{16\eta\alpha_0}{(1+4\eta)\tilde{\beta}^4}\ , \quad (4b)$$

one gets the well-known quantum-sine-Gordon theory

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+\frac{\alpha}{\beta^2}:\!\cos(\beta\phi)\!:\ , \quad (5)$$

with

$$\alpha = \frac{\alpha_0 \tilde{\beta}^4}{(1+4\eta)\tilde{\beta}^4 + 16\eta\alpha_0} , \tag{6a}$$

$$\beta^2 = \tilde{\beta}^2 \left[\frac{(1+4\eta)\tilde{\beta}^4}{(1+4\eta)\tilde{\beta}^4 + 16\eta\alpha_0} \right] . \tag{6b}$$

As shown by Coleman,[14] the quantum-sine-Gordon model is equivalent to a massive Thirring model,[17] whose mass m and coupling constant g are given by

$$m = \frac{\alpha}{\beta^2} , \tag{7a}$$

$$\frac{\beta^2}{4\pi} = \frac{\pi}{\pi + g} . \tag{7b}$$

As $\beta^2 = 8\pi$ the system undergoes[14,15] a transition of the Kosterlitz-Thouless type. In terms of the DSG parameters η and $\tilde{\beta}$ (α_0 is fixed and negative) this condition reads as

$$\frac{\tilde{\beta}^4(\tilde{\beta}^2 - 8\pi)}{2^7\pi\alpha_0} = \frac{\eta}{1+4\eta} = \frac{1}{4}\tanh^2 R , \tag{8}$$

and leads to a critical line separating two different phases. As $\beta^2 = 4\pi$ the model is equivalent to a free massive Dirac field theory. The line corresponding in the R-$\tilde{\beta}$ plane is obtained as

$$\frac{\tilde{\beta}^4(\tilde{\beta}^2 - 4\pi)}{2^7\pi\alpha_0} = \frac{1}{4}\tanh^2 R . \tag{9}$$

Equations (8) and (9) explain (i) and (ii) in the neighborhood of $\tilde{\beta}^2 \simeq 4\pi$. We expect that our results are qualitatively true for all values of $\tilde{\beta}^2$; however, for $\tilde{\beta}^2 > 4\pi$, it requires an infinite wave-function renormalization.

We stress that the correspondence of the quantum DSG model with the quantum-sine-Gordon theory relies upon Eq. (2), which is only an approximation to the full quantum theory, since in Eq. (2) we neglected contributions coming from the contraction terms appearing in the Wick reordering of the DSG interaction. In fact,

$$:\cos^2(\tilde{\beta}\tilde{\phi}): = :\cos(\tilde{\beta}\tilde{\phi}):^2 + \cdots , \tag{10}$$

where the ellipsis represents unspecified contraction terms. To find a compact form for the contributions of the contraction terms in Eq. (10) is a major computational task. Conventional wisdom,[13,18,15] however, indicates that those terms are irrelevant for the phase transition of this model.

Equation (10) shows that the introduction of higher harmonics in the classical-sine-Gordon Lagrangian—apart from a renormalization of the kinetic term—produces quantum effects through the contraction terms. This observation may bring to light novel quantal phenomena in the kink sector of the DSG model. A semiclassical analysis, in fact, showed[9,19] that the introduction of higher harmonics in the classical-sine-Gordon Lagrangian leads to the existence of a meson-kink bound state whose frequency is parametrized by R. Our study here indicates that such a state contributes to the quantum theory only through the field renormalization and the contraction terms. This peculiar aspect of the model needs further investigation.

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