Effects of kinematical interactions on magnon line broadening in a ferromagnet

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Magnon line broadening in a Heisenberg ferromagnet is analyzed by a semiclassical method that includes the effects of both dynamical and kinematical interactions. At temperatures where kinematical interactions become important, appreciable broadening is generated by the same long-range three-magnon effective interaction that gives rise to a gap in the magnon spectrum. In particular, whenever two magnon wave packets collide, there is a transient energy modulation of other magnons throughout the crystal. This source of broadening is maximized at long wavelengths.

I. INTRODUCTION

In a recent article,[1] the writer analyzed the Heisenberg model of a three-dimensional ferromagnet by a semiclassical method that attempted to include the effects of both dynamical and kinematical interactions. At low temperatures, the results of Dyson[2] were reproduced, thus indicating the accuracy of the semiclassical approximation. At temperatures where the kinematical interactions become important, it was shown that a gap appears in the magnon excitation spectrum. This results, in part, from a long-range three-magnon effective interaction. The most serious deficiency of the calculation was the failure to calculate the magnon line broadening. We will attempt to correct this deficiency in the present paper.

The long-range three-magnon effective interaction results from the renormalization of the energy of a magnon wave packet when an appreciable fraction of all the lattice sites of the crystal are occupied by two or more magnons. A more dynamical way of describing this interaction is as follows. Whenever two magnon wave packets collide, there is a transient modulation of the energy of other magnons throughout the crystal. This modulation leads to a line broadening of the magnon excitation spectrum. At temperatures where the kinematical interactions are important, it appears that this source of line broadening is far more important than that due to two-magnon scattering, especially at long wavelengths.

We briefly recapitulate the semiclassical procedure used in I. The Heisenberg Hamiltonian is

$$H=-\tfrac{1}{2}\sum_{i,j} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j\ , \tag{1}$$

$$J_{ij}=J(\,|\mathbf{R}_i-\mathbf{R}_j|\,),\quad J_{ii}=0\ , \tag{2}$$

where the double sum is over the N lattice sites of the crystal. The quantum-mechanical equations of motion are

$$\frac{d\mathbf{S}_i}{dt}=\mathbf{S}_i\times\mathbf{H}_i,\quad \mathbf{H}_i\equiv\hbar^{-1}\sum_j J_{ij}\mathbf{S}_j\ . \tag{3}$$

The semiclassical approximation consists of solving these equations classically. We restrict ourselves to the spin-$\frac{1}{2}$ case. In the absence of magnons, we choose all spins to be pointing along the x axis, i.e., an orientation given by the spherical coordinates $\theta_0=\frac{1}{2}\pi$, $\phi_0=0$. In the presence of an assembly of magnons, the spin on the ith site points in the direction $\theta_0+\theta_i$, $\phi_0+\phi_i$, where

$$\begin{aligned}\phi_i&=\sum_k C_k\sin(\mathbf{k}\cdot\mathbf{R}_i-\omega_k t+\Phi_k)\ ,\\ \theta_i&=\sum_k C_k\cos(\mathbf{k}\cdot\mathbf{R}_i-\omega_k t+\Phi_k)\ .\end{aligned} \tag{4}$$

C_k, the amplitude of mode $\mathbf{k}$, is related to f_k, the number of magnons of wave vector $\mathbf{k}$, by the expression

$$C_k^2=4N^{-1}f_k\ . \tag{5}$$

In order to solve the equations of motion, a series of transformations of coordinates are introduced, different sets for each lattice site, in order to make the precessing spins appear static. In carrying the calculation to completion, it is necessary to introduce the essential approximation of *linearizing* certain quantities with respect to their arguments. Specifically, we linearize a set of quantities A_i, functions of $\phi_i,\dot{\phi}_i,\theta_i,\dot{\theta}_i$, and a set of quantities B_{ij}, functions of $\phi_i,\phi_j,\theta_i,\theta_j$. The linearization of $f(\theta)$ with respect to θ is

$$\begin{aligned}\mathit{lin}f(\theta)=&\tfrac{1}{2}\langle f(\theta)+f(-\theta)\rangle\\ &+\tfrac{1}{2}\theta\langle (d/d\theta)[f(\theta)-f(-\theta)]\rangle\ .\end{aligned} \tag{6}$$

The angular brackets denote averaging with respect to time and with respect to the random phases Φ_k appearing in θ_i and ϕ_i. The solution to the "static" problem in the transformed coordinates leads to the expression for the magnon excitation energy,

$$\hbar\omega_k=\tfrac{1}{2}\sum_j J_{ij}(\alpha_{0ij}+\alpha_{1ij}\{1-\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\})\ , \tag{7}$$

where α_{0ij} and α_{1ij} are functions of the three quantities:

$$\begin{aligned}b&\equiv N^{-1}\sum_k 2f_k\ ,\\ d&\equiv N^{-1}\sum_k 2\omega_k f_k\ ,\\ g_{ij}&\equiv N^{-1}\sum_k 2f_k\{1-\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\}\ .\end{aligned} \tag{8}$$

Specifically,

$$\alpha_{1ij} \equiv \tfrac{1}{2}(1+e^{-(1/2)b})^{-1}[e^{-g_{ij}}(1+e^{-g_{ij}}+2e^{-(1/2)b}) + e^{-2b}(e^{+g_{ij}}-1)] \,, \quad (9)$$

$$\tfrac{1}{2}\sum_j J_{ij}\alpha_{0ij} = (e^{+(1/2)b}+1)^{-1}\hbar d - \tfrac{1}{2}(1+e^{-(1/2)b})^{-1}e^{-2b} \times \sum_j J_{ij}(e^{+g_{ij}}-1) \,. \quad (10)$$

From the definition of d, we have

$$\hbar d = \tfrac{1}{2}\sum_j J_{ij}(\alpha_{0ij}b + \alpha_{1ij}g_{ij}) \,. \quad (11)$$

Thus we can choose

$$\alpha_{0ij} \equiv (1-b+e^{+(1/2)b})^{-1} \times [\alpha_{1ij}g_{ij} - e^{-(3/2)b}(e^{+g_{ij}}-1)] \,. \quad (12)$$

Thus both α_{0ij} and α_{1ij} can be expressed as functions of b and g_{ij} alone.

A separate calculation leads to an alternative form of $\hbar\omega_k$, namely

$$\hbar\omega_k = \tfrac{1}{2}\sum_j J_{ij}(\alpha_{0ij} + \alpha_{2ij}\{1-\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\}) \,. \quad (13)$$

This calculation suffers from the fact that $\hbar\omega_k$ contains an unknown additive constant, independent of $\mathbf{k}$. The constant is chosen such that the two forms of $\hbar\omega_k$ agree in the long-wavelength limit $\mathbf{k}=0$. We have

$$\alpha_{2ij} = \tfrac{1}{4}[(3+e^{-g_{ij}})e^{-g_{ij}} + (e^{+g_{ij}}-1)e^{-2b}] \,. \quad (14)$$

It is instructive to expand the α's in powers of b and g_{ij}. Terms proportional to $b^{n_1}g_{ij}^{n_2}$ result from n-magnon effective interactions, where $n=(n_1+n_2+1)$:

$$\alpha_{0ij} = \tfrac{3}{4}g_{ij}(b-g_{ij}) - \tfrac{1}{48}g_{ij}(18b^2+3bg_{ij}-20g_{ij}^2) + \cdots \,, \quad (15)$$

$$\alpha_{1ij} = 1-g_{ij}+\tfrac{1}{2}g_{ij}(2g_{ij}-b) + \tfrac{1}{24}g_{ij}(9b^2+3bg_{ij}-10g_{ij}^2) + \cdots \,, \quad (16)$$

$$\alpha_{2ij} = 1-g_{ij}+\tfrac{1}{2}g_{ij}(2g_{ij}-b) + \tfrac{1}{12}g_{ij}(6b^2-3bg_{ij}-5g_{ij}^2) + \cdots \,. \quad (17)$$

To the accuracy of three-magnon interactions, $\alpha_{1ij}=\alpha_{2ij}$. The leading terms of α_{0ij} result from three-magnon interactions.

II. LINE BROADENING

The second form of $\hbar\omega_k$ is actually an average. Specifically, Eq. (13) is

$$\hbar\omega_k = \langle E_k \rangle \,, \quad (18)$$

where

$$E_k \equiv \hbar(A_{ix}-\Omega) + \tfrac{1}{2}\sum_j J_{ij}\{B_{ijxx} - B_{ijyy}\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\} \,. \quad (19)$$

Here $\hbar\Omega$ is the already mentioned additive constant, while

$$A_{ix} = \tfrac{1}{2}(\dot{\theta}_i\sin\phi_i - \dot{\phi}_i\sin\theta_i) \,, \quad (20)$$

$$B_{ijxx} = \tfrac{1}{2}[\cos(\phi_j-\phi_i)\cos\theta_i\cos\theta_j + \sin\theta_i\sin\theta_j + \cos(\theta_j-\theta_i)\cos\phi_i\cos\phi_j + \sin\phi_i\sin\phi_j] \,, \quad (21)$$

$$B_{ijyy} = \tfrac{1}{2}[\cos(\phi_j-\phi_i) + \cos(\theta_j-\theta_i)\sin\phi_i\sin\phi_j + \cos\phi_i\cos\phi_j] \,. \quad (22)$$

Making use of the first form of $\hbar\omega_k$, we can write

$$\hbar A_{ix} = \tfrac{1}{2}\sum_j J_{ij}A_{ijxx} \,, \quad (23)$$

where we define

$$A_{ijxx} \equiv \tfrac{1}{2}\sum_k C_k\{(\alpha_{0ij}+\alpha_{1ij}) - \alpha_{1ij}\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\} \times [\sin\phi_i\sin(\mathbf{k}\cdot\mathbf{R}_i - \omega_k t + \Phi_k) + \sin\theta_i\cos(\mathbf{k}\cdot\mathbf{R}_i - \omega_k t + \Phi_k)] \,. \quad (24)$$

Thus

$$E_k + \hbar\Omega = \tfrac{1}{2}\sum_j J_{ij}((A_{ijxx}+B_{ijxx}-B_{ijyy}) + B_{ijyy}\{1-\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\}) \,. \quad (25)$$

Note that

$$\alpha_{2ij} = \langle B_{ijyy} \rangle \,. \quad (26)$$

We now define

$$(\hbar\tilde{\omega}_k)^2 \equiv \langle E_k^2 \rangle - \langle E_k \rangle^2 \,, \quad (27)$$

the mean-square deviation in the magnon excitation energy (the second moment of the line-shape function). We can write

$$(\hbar\tilde{\omega}_k)^2 = \tfrac{1}{4}\sum_{j,p} J_{ij}J_{ip}(\tilde{\alpha}_{0ijp} + 2\tilde{\alpha}_{1ijp}\{1-\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\} + \tilde{\alpha}_{2ijp}\{1-\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\}\{1-\cos[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)]\}) \,, \quad (28)$$

where

$$\tilde{\alpha}_{0ijp} \equiv \langle (A_{ijxx}+B_{ijxx}-B_{ijyy})(A_{ipxx}+B_{ipxx}-B_{ipyy})\rangle - \langle (A_{ijxx}+B_{ijxx}-B_{ijyy})\rangle \times \langle (A_{ipxx}+B_{ipxx}-B_{ipyy})\rangle , \quad (29)$$

$$\tilde{\alpha}_{1ijp} \equiv \langle B_{ijyy}(A_{ipxx}+B_{ipxx}-B_{ipyy})\rangle - \langle B_{ijyy}\rangle \langle (A_{ipxx}+B_{ipxx}-B_{ipyy})\rangle , \quad (30)$$

$$\tilde{\alpha}_{2ijp} \equiv \langle B_{ijyy}B_{ipyy}\rangle - \langle B_{ijyy}\rangle \langle B_{ipyy}\rangle . \quad (31)$$

In order to determine $(\hbar\tilde{\omega}_k)^2$, we need to calculate the $\tilde{\alpha}$'s.
As in I, for the angles we are considering, we have

$$\langle \theta^{2n}\rangle = [(2n)!/n!2^n]\langle \theta^2\rangle^n, \quad \langle \theta^{2n+1}\rangle = 0 , \quad (32)$$

$$\langle \cos\theta\rangle = \exp(-\tfrac{1}{2}\langle \theta^2\rangle), \quad \langle \sin\theta\rangle = 0 , \quad (33)$$

$$\langle \phi_i^2\rangle = \langle \theta_i^2\rangle = b , \quad (34)$$

$$\langle \dot{\theta}_i\phi_i\rangle = -\langle \dot{\phi}_i\theta_i\rangle = d , \quad (35)$$

$$\langle \dot{\theta}_i\sin\phi_i\rangle = -\langle \dot{\phi}_i\sin\theta_i\rangle = de^{-(1/2)b} , \quad (36)$$

$$\langle \dot{\theta}_i\sin\theta\rangle = \langle \dot{\theta}_i\theta\rangle\langle \cos\theta\rangle , \quad (37)$$

$$\langle \sin^2\phi_i\rangle = \langle \sin^2\theta_i\rangle = e^{-b}\sinh b , \quad (38)$$

$$\langle \dot{\phi}_i^2\rangle = \langle \dot{\theta}_i^2\rangle = \tilde{d} . \quad (39)$$

Here we are defining

$$\tilde{d} \equiv N^{-1}\sum_k 2\omega_k^2 f_k . \quad (40)$$

We can write

$$\hbar^2\tilde{d} = \tfrac{1}{4}\sum_{j,p} J_{ij}J_{ip}[\alpha_{0ij}\alpha_{0ip}b + \alpha_{0ij}\alpha_{1ip}g_{ip} + \alpha_{0ip}\alpha_{1ij}g_{ij} + \alpha_{1ij}\alpha_{1ip}(g_{ij}+g_{ip}-g_{jp})] . \quad (41)$$

We have

$$\langle A_{ix}^2\rangle - \langle A_{ix}\rangle^2 = \tfrac{1}{4}(\langle \dot{\theta}_i\sin\phi_i\rangle^2 + \langle \dot{\phi}_i\sin\theta_i\rangle^2 + \langle \dot{\theta}_i^2\rangle\langle \sin^2\phi_i\rangle + \langle \dot{\phi}_i^2\rangle\langle \sin^2\theta_i\rangle) = \tfrac{1}{2}e^{-b}(d^2+\tilde{d}\sinh b) . \quad (42)$$

It follows that

$$\langle A_{ijxx}A_{ipxx}\rangle - \langle A_{ijxx}\rangle\langle A_{ipxx}\rangle = \tfrac{1}{2}e^{-b}\{(\alpha_{0ij}\alpha_{0ip}b + \alpha_{0ij}\alpha_{1ip}g_{ip} + \alpha_{0ip}\alpha_{1ij}g_{ij})(b+\sinh b) + \alpha_{1ij}\alpha_{1ip}[g_{ij}g_{ip} + (g_{ij}+g_{ip}-g_{jp})\sinh b]\} . \quad (43)$$

Consider the angle

$$\begin{aligned}\theta &= (\nu_1\phi_i + \nu_2\phi_j + \nu_3\phi_p + \nu_4\theta_i + \nu_5\theta_j + \nu_6\theta_p) \\ &= \sum_k C_k\{\nu_1 + \nu_2\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] - \nu_5\sin[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] + \nu_3\cos[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)] \\ &\qquad - \nu_6\sin[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)]\}\sin(\mathbf{k}\cdot\mathbf{R}_i - \omega_k t + \Phi_k) \\ &\quad + \sum_k C_k\{\nu_4 + \nu_2\sin[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] + \nu_5\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] + \nu_3\sin[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)] \\ &\qquad + \nu_6\cos[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)]\}\cos(\mathbf{k}\cdot\mathbf{R}_i - \omega_k t + \Phi_k) . \end{aligned} \quad (44)$$

$$\begin{aligned}\tfrac{1}{2}\langle \theta^2\rangle &= \tfrac{1}{4}\sum_k C_k^2(\{\nu_1 + \nu_2\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] - \nu_5\sin[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] + \nu_3\cos[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)] \\ &\qquad - \nu_6\sin[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)]\}^2 + \{\nu_4 + \nu_2\sin[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] + \nu_5\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] \\ &\qquad + \nu_3\sin[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)] + \nu_6\cos[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)]\}^2) \\ &= N^{-1}\sum_k f_k\{(\nu_1^2+\nu_2^2+\nu_3^2+\nu_4^2+\nu_5^2+\nu_6^2) + 2(\nu_1\nu_2+\nu_4\nu_5)\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)] \\ &\qquad + 2(\nu_1\nu_3+\nu_4\nu_6)\cos[\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_i)] + 2(\nu_2\nu_3+\nu_5\nu_6)\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_p)]\} , \end{aligned} \quad (45)$$

or

$$\tfrac{1}{2}\langle \theta^2\rangle = \tfrac{1}{2}[(\nu_1+\nu_2+\nu_3)^2 + (\nu_4+\nu_5+\nu_6)^2]b - (\nu_1\nu_2+\nu_4\nu_5)g_{ij} - (\nu_1\nu_3+\nu_4\nu_6)g_{ip} - (\nu_2\nu_3+\nu_5\nu_6)g_{jp} . \quad (46)$$

We can write

$$B_{ijuu} = \tfrac{1}{8}\sum_m c_m^u\cos(\nu_{m1}\phi_i + \nu_{m2}\phi_j + \nu_{m3}\theta_i + \nu_{m4}\theta_j) , \quad (47)$$

where $u=x,y$, and

$$(\nu_{mm'})\equiv\begin{pmatrix} +1 & -1 & 0 & 0 \\ +1 & +1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 \\ +1 & -1 & +1 & +1 \\ +1 & -1 & -1 & -1 \\ +1 & +1 & +1 & -1 \\ +1 & +1 & -1 & +1 \end{pmatrix} \quad \text{with } m=1,2,\ldots,10,\ m'=1,2,3,4, \tag{48}$$

$$(c_m^x)=\begin{pmatrix} +2 \\ -2 \\ +2 \\ -2 \\ +2 \\ +2 \\ +1 \\ +1 \\ +1 \\ +1 \end{pmatrix},\quad (c_m^y)=\begin{pmatrix} +6 \\ +2 \\ 0 \\ 0 \\ +1 \\ +1 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}. \tag{49}$$

We now have $(u,v=x,y)$

$$\langle B_{ijuu}B_{ipvv}\rangle-\langle B_{ijuu}\rangle\langle B_{ipvv}\rangle=\tfrac{1}{64}\sum_{m,n}c_m^u c_n^v e^{-\alpha_{mn}}(\cosh\beta_{mn}-1)\,, \tag{50}$$

where we define

$$\alpha_{mn}\equiv\tfrac{1}{2}[(\nu_{m1}+\nu_{m2})^2+(\nu_{m3}+\nu_{m4})^2+(\nu_{n1}+\nu_{n2})^2+(\nu_{n3}+\nu_{n4})^2]b-(\nu_{m1}\nu_{m2}+\nu_{m3}\nu_{m4})g_{ij}-(\nu_{n1}\nu_{n2}+\nu_{n3}\nu_{n4})g_{ip}\,, \tag{51}$$

$$\beta_{mn}\equiv[(\nu_{m1}+\nu_{m2})(\nu_{n1}+\nu_{n2})+(\nu_{m3}+\nu_{m4})(\nu_{n3}+\nu_{n4})]b-(\nu_{n1}\nu_{m2}+\nu_{n3}\nu_{m4})g_{ij}-(\nu_{m1}\nu_{n2}+\nu_{m3}\nu_{n4})g_{ip}$$
$$-(\nu_{m2}\nu_{n2}+\nu_{m4}\nu_{n4})g_{jp}\,. \tag{52}$$

For the cases $uv=xx$, xy, yy, there are 100, 60, 36 terms, respectively, in the above double sum over m and n. This makes the above expression tedious to evaluate.

Consider the angle

$$\theta=(\nu_1\phi_i+\nu_2\phi_j+\nu_3\theta_i+\nu_4\theta_j)\,. \tag{53}$$

We have

$$\langle\dot\theta_i\theta\rangle=\tfrac{1}{2}\sum_k C_k^2\omega_k\{\nu_1+\nu_2\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\}$$
$$=(\nu_1+\nu_2)d-\nu_2\tilde g_{ij}\,, \tag{54}$$

$$-\langle\dot\phi_i\theta\rangle=\tfrac{1}{2}\sum_k C_k^2\omega_k\{\nu_3+\nu_4\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\}$$
$$=(\nu_3+\nu_4)d-\nu_4\tilde g_{ij}\,. \tag{55}$$

Here we are defining

$$\tilde g_{ij}\equiv N^{-1}\sum_k 2\omega_k f_k\{1-\cos[\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_i)]\}\,. \tag{56}$$

We can write

$$\hbar\tilde g_{ij}=\tfrac{1}{2}\sum_p J_{ip}[\alpha_{0ip}g_{ij}+\alpha_{1ip}(g_{ij}+g_{ip}-g_{jp})]\,. \tag{57}$$

We have

$$A_{ix}B_{ijuu}=\tfrac{1}{16}\sum_m c_m^u(\dot\theta_i\sin\phi_i-\dot\phi_i\sin\theta_i)\cos(\nu_{m1}\phi_i+\nu_{m2}\phi_j+\nu_{m3}\theta_i+\nu_{m4}\theta_j)$$
$$=\tfrac{1}{32}\sum_m c_m^u\dot\theta_i\{\sin[(\nu_{m1}+1)\phi_i+\nu_{m2}\phi_j+\nu_{m3}\theta_i+\nu_{m4}\theta_j]-\sin[(\nu_{m1}-1)\phi_i+\nu_{m2}\phi_j+\nu_{m3}\theta_i+\nu_{m4}\theta_j]\}$$
$$-\tfrac{1}{32}\sum_m c_m^u\dot\phi_i\{\sin[\nu_{m1}\phi_i+\nu_{m2}\phi_j+(\nu_{m3}+1)\theta_i+\nu_{m4}\theta_j]-\sin[\nu_{m1}\phi_i+\nu_{m2}\phi_j+(\nu_{m3}-1)\theta_i+\nu_{m4}\theta_j]\}\,. \tag{58}$$

Thus

$$\langle A_{ix}B_{ijuu}\rangle-\langle A_{ix}\rangle\langle B_{ijuu}\rangle=\frac{1}{16}\sum_m c_m^u e^{-(1/2)b}e^{-\alpha_m}\{d(\cosh\beta_m+\cosh\gamma_m-2)-[(\nu_{m1}+\nu_{m2})d-\nu_{m2}\tilde{g}_{ij}]\sinh\beta_m$$
$$-[(\nu_{m3}+\nu_{m4})d-\nu_{m4}\tilde{g}_{ij}]\sinh\gamma_m\}\ , \quad (59)$$

where we define

$$\alpha_m\equiv\frac{1}{2}[(\nu_{m1}+\nu_{m2})^2+(\nu_{m3}+\nu_{m4})^2]b-(\nu_{m1}\nu_{m2}+\nu_{m3}\nu_{m4})g_{ij}\ , \quad (60)$$

$$\beta_m\equiv(\nu_{m1}+\nu_{m2})b-\nu_{m2}g_{ij}\ , \quad (61)$$

$$\gamma_m\equiv(\nu_{m3}+\nu_{m4})b-\nu_{m4}g_{ij}\ . \quad (62)$$

It follows that

$$\langle A_{ipxx}B_{ijuu}\rangle-\langle A_{ipxx}\rangle\langle B_{ijuu}\rangle=\frac{1}{16}\sum_m c_m^u e^{-(1/2)b}e^{-\alpha_m}\{(\alpha_{0ip}b+\alpha_{1ip}g_{ip})(\cosh\beta_m+\cosh\gamma_m-2)$$
$$-[(\nu_{m1}+\nu_{m2})(\alpha_{0ip}b+\alpha_{1ip}g_{ip})-\nu_{m2}\alpha_{0ip}g_{ip}$$
$$-\nu_{m2}\alpha_{1ip}(g_{ij}+g_{ip}-g_{jp})]\sinh\beta_m$$
$$-[(\nu_{m3}+\nu_{m4})(\alpha_{0ip}b+\alpha_{1ip}g_{ip})-\nu_{m4}\alpha_{0ip}g_{ip}$$
$$-\nu_{m4}\alpha_{1ip}(g_{ij}+g_{ip}-g_{jp})]\sinh\gamma_m\}\ . \quad (63)$$

In order to understand the significance of the results we have obtained, it is necessary to expand in powers of b and the g's. We will keep only the lowest-order nonvanishing terms, which in all cases are bilinear in b and the g's. To this order, Eqs. (43), (50), and (63) become, respectively,

$$\langle A_{ijxx}A_{ipxx}\rangle-\langle A_{ijxx}\rangle\langle A_{ipxx}\rangle=\frac{1}{2}[b(g_{ij}+g_{ip}-g_{jp})+g_{ij}g_{ip}]\ , \quad (64)$$

$$\langle B_{ijuu}B_{ipuu}\rangle-\langle B_{ijuu}\rangle\langle B_{ipuu}\rangle=\frac{1}{128}\sum_{m,n}c_m^u c_n^v\beta_{mn}^2\ , \quad (65)$$

$$\langle A_{ipxx}B_{ijuu}\rangle-\langle A_{ipxx}\rangle\langle B_{ijuu}\rangle=-\frac{1}{16}\sum_m c_m^u\{[(\nu_{m1}+\nu_{m2})^2+(\nu_{m3}+\nu_{m4})^2]bg_{ip}+(\nu_{m2}^2+\nu_{m4}^2)g_{ij}(g_{ij}+g_{ip}-g_{jp})$$
$$-[(\nu_{m1}+\nu_{m2})\nu_{m2}+(\nu_{m3}+\nu_{m4})\nu_{m4}][b(g_{ij}+g_{ip}-g_{jp})+g_{ij}g_{ip}]\}\ . \quad (66)$$

Specifically, we have

$$\langle(B_{ijxx}-B_{ijyy})(B_{ipxx}-B_{ipyy})\rangle-\langle(B_{ijxx}-B_{ijyy})\rangle\langle(B_{ipxx}-B_{ipyy})\rangle=\langle B_{ijyy}B_{ipyy}\rangle-\langle B_{ijyy}\rangle\langle B_{ipyy}\rangle$$
$$=\frac{1}{2}(g_{ij}+g_{ip}-g_{jp})^2\ , \quad (67)$$

$$\langle B_{ijyy}(B_{ipxx}-B_{ipyy})\rangle-\langle B_{ijyy}\rangle\langle(B_{ipxx}-B_{ipyy})\rangle=0\ , \quad (68)$$

$$(A_{ipxx}(B_{ijxx}-B_{ijyy})\rangle-\langle A_{ipxx}\rangle\langle(B_{ijxx}-B_{ijyy})\rangle=\langle A_{ipxx}B_{ijyy}\rangle-\langle A_{ipxx}\rangle\langle B_{ijyy}\rangle$$
$$=-\frac{1}{2}g_{ij}(g_{ij}+g_{ip}-g_{jp})\ . \quad (69)$$

Thus, to this order, we have

$$\tilde{\alpha}_{0ijp}=\frac{1}{2}[(b-g_{ip})(g_{ij}+g_{ip}-g_{jp})+g_{ij}g_{ip}]\ , \quad (70)$$

$$\tilde{\alpha}_{1ijp}=-\frac{1}{2}g_{ij}(g_{ij}+g_{ip}-g_{jp})\ , \quad (71)$$

$$\tilde{\alpha}_{2ijp}=\frac{1}{2}(g_{ij}+g_{ip}-g_{jp})^2\ . \quad (72)$$

Inserting these results into Eq. (28), we have an expression for $(\hbar\tilde{\omega}_k)^2$ bilinear in b and the g_{ij}. Thus we have included the effects of three-magnon interactions. However, also included are effects of two-magnon interactions. Since b and the g_{ij} are linear in the statistical factors $f_{k'}$, our expression for $(\hbar\tilde{\omega}_k)^2$ can be written as a double sum of $\mathbf{k}'$ and $\mathbf{k}''$ of terms proportional to $f_{k'}f_{k''}$. The terms where $\mathbf{k}'\neq\mathbf{k}''$ are due to three-magnon interactions; those where $\mathbf{k}'=\mathbf{k}''$ are due to two-magnon interactions. For example, consider the hypothetical case where only one statistical factor, $f_{k'}$, is nonvanishing. Then $\hbar\tilde{\omega}_k$ is directly proportional to $f_{k'}$, as should be the case for a two-particle interaction between magnons of wave vectors $\mathbf{k}$ and $\mathbf{k}'$.

Note that $\tilde{\alpha}_{1ijp}$ is negative. This means that $\tilde{\omega}_k$ has a maximum at $\mathbf{k}=0$. Compare this with results obtained in the literature[3,4] for the inverse lifetime τ_k^{-1} for a magnon of wave vector $\mathbf{k}$. Rather than having a maximum, τ_k^{-1} vanishes in the long-wavelength limit. There are two reasons for this difference. First, both decay and modulation processes[5] contribute to $\tilde{\omega}_k$; only decay processes contribute to τ_k^{-1}. Secondly, effects of kinematical interactions on τ_k^{-1} are almost certainly not included in any calculations in the literature, for exactly the same reason that kinematical interactions are not included in the results of Dyson.[2] (Effects of kinematical interactions do not appear in calculations based on consideration of orders of interaction, e.g., diagrammatic perturbation theory.) The writer believes that the finite value of $\tilde{\omega}_k$ at $\mathbf{k}=\mathbf{0}$ results predominately, if not entirely, from modulation effects rather than lifetime effects, specifically the modulation effects described in the second paragraph of this paper. This is related to the long range of the three-body effective kinematical interaction.[1]

[1]R. H. Parmenter, Phys. Rev. B **30**, 2745 (1984). We shall refer to this paper as I.

[2]F. J. Dyson, Phys. Rev. **102**, 1217 (1956); **102**, 1230 (1956).

[3]J. F. Cooke and H. A. Gersch, Phys. Rev. **153**, 641 (1967).

[4]A. B. Harris, Phys. Rev. **175**, 674 (1968).

[5]An analogous situation occurs in the calculation of linewidths in nuclear magnetic resonance, where inhomogeneities contribute to $\tilde{\omega}_k$ but not to τ_k^{-1}.