Two-dimensional XY model in a random uniaxial field

Brian J. Minchau* and Robert A. Pelcovits
Department of Physics, Brown University, Providence, Rhode Island 02912
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We present a low-temperature analysis of the two-dimensional XY model in the presence of a random uniaxial field, which points in the $\pm x$ directions with equal probability. We first construct a domain argument, valid at zero temperature in all dimensions, which indicates that in all dimensions the ground state is ferromagnetic, for small field, and the magnetization points in the y direction. We then construct Kosterlitz-style recursion relations for this model in two dimensions. Analyzing these equations, we find an ordered phase with Ising symmetry at nonzero temperature and small values of the random-field strength. Thus, the random field induces long-range order in a system which exhibits only quasiorder in its pure state. We also consider the p-state generalization of the model, and find an ordered phase with $2p$-fold symmetry at nonzero temperature. Our results are in disagreement with earlier work by Dotsenko and Feigelman, who found a paramagnetic phase at low temperatures.

I. INTRODUCTION

Since the development of theoretical tools to study the two-dimensional XY model, several studies have focused on the properties of the model with quenched random disorder. The types of disorder considered were bond disorder,[1] random Dzyaloshinski-Moriya interactions,[2] and random symmetry-breaking fields.[3] Here we study the XY model in the presence of a *uniaxial* random field. This model was previously studied by Dotsenko and Feigelman,[4] but our results disagree with theirs, even on a qualitative basis.

Our model is defined by the Hamiltonian

$$H=-J\sum_{\langle rr'\rangle}\cos[\theta(r)-\theta(r')]-h\sum_{r}\cos[\theta(r)-\phi(r)]\ , \tag{1.1}$$

where $\theta(r)$ is the angle of the spin at site r on a square lattice, and J is a uniform, ferromagnetic coupling. The field strength h is assumed to be nonrandom. The quenched angle variable $\phi(r)$ takes on values 0 and π with equal probability. If $\theta(r)$ and $\phi(r)$ are measured from zero on the x axis, then for $h>0$ the random field attempts to align the spin with the $+x$ direction for $\phi=0$ and the $-x$ direction for $\phi=\pi$. In the isotropic random-field models studied in Ref. 3, ϕ assumed *all* values between 0 and 2π with equal probability.

For $h\gg J$ the system will be paramagnetic, with the spins following the local random fields. Of interest is the behavior in the nontrivial regime, $h<J$. This regime was considered in $4-\epsilon$ dimensions by Aharony,[5] who found that the spins order in the direction *perpendicular* to the fields (i.e., the y direction in our model) at low and zero temperatures. At each site the spin points primarily in the y direction, with a small component along the local random field. This ordered phase, which has Ising symmetry, is bounded by a phase boundary with Ising exponents (see Fig. 1). In Sec. II we construct a domain argument, valid at zero temperature, which demonstrates that in all dimensionalities the ground state is ferromagnetic.

Dotsenko and Feigelman[4] have studied (1.1) in two dimensions using the renormalization group, and find a paramagnetic phase at all temperatures including zero temperature. As this result conflicts with our domain argument, we have undertaken an independent renormalization-group analysis and have found that long-range order in the y direction does exist at least at intermediate temperatures. At very low temperatures we are unable to determine the phase diagram, though we expect

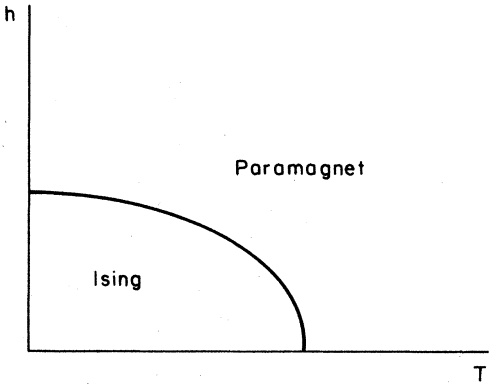

FIG. 1. Phase diagram for (1.1) in all dimensionalities greater than 1. This phase diagram was obtained near four dimensions in Ref. 5, where the Ising fixed point controlling the phase boundary is accessible in the renormalization-group analysis. In our two-dimensional renormalization-group analysis (Sec. III) we can only establish the Ising nature of the phase at intermediate temperatures but we speculate on the basis of our domain argument (Sec. II) that the full phase diagram is as shown. The Ising fixed point is not accessible in our analysis. For $p=2$ [see (1.2)] the phase diagram is qualitatively similar, though the ordered phase now has the symmetry of a four-state clock model and the phase transition is presumably in the universality class of that model.

that it has the same qualitative form as Fig. 1. Thus, a model, which in its pure state exhibits only quasiorder at nonzero temperatures, displays long-range order when placed in a uniaxial random field. This order is brought about by a fluctuation-induced nonrandom Ising symmetry-breaking field, as is the case near four dimensions. Dotsenko and Feigelman have apparently not considered this possibility.

We also study the p-fold generalization of (1.1). This generalization is obtained by replacing the last term in (1.1) by

$$-h \sum_r \cos[p\theta(r)-\phi(r)] , \tag{1.2}$$

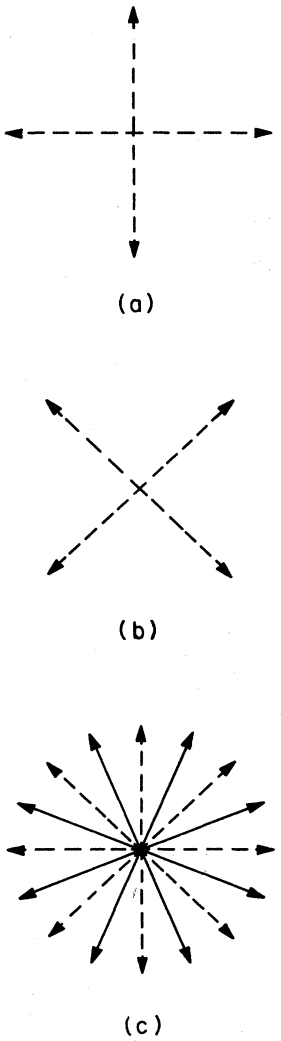

FIG. 2. Illustration of the special directions in the model (1.2) when $p=4$. Panels (a) and (b) denote the two sets of random axes which occur with equal probability in the system. In panel (c) we superpose these random-field directions, and show the directions (using solid vectors) of the eightfold nonrandom symmetry-breaking field generated under renormalization (see Sec. III).

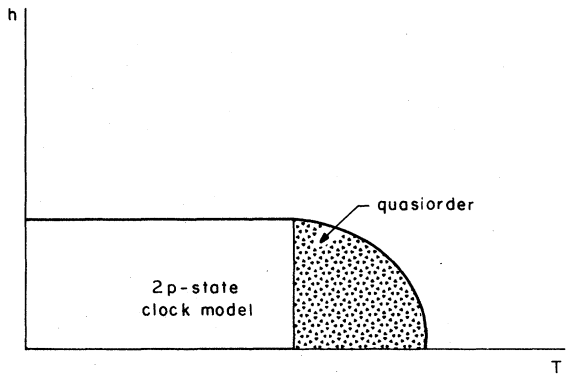

FIG. 3. Phase diagrams for (1.2) in two dimensions with $p \geq 3$. The phase indicated by dots exhibits quasi-long-range order. The low-temperature phase is ordered with the symmetry of a $2p$-state clock model. As in Fig. 1, the phase boundary terminating the ordered phases is inaccessible to our theory.

where again $\phi=0$ or π with equal probability. For $p=2$, e.g., the last term in (1.2) corresponds to a local easy axis, parallel to either the x or y axis with equal probability. By contrast, in the isotropic random-axis model,[3] the local easy axis points in any direction in the x-y plane with equal probability. Our renormalization-group analysis of Sec. III indicates that under renormalization a nonrandom $2p$-fold symmetry-breaking field is generated, provided $k_B T/J < 4\pi/p^2$. This $2p$-fold field induces long-range order with the symmetry of a $2p$-state clock model. The long-range order is along one of the $2p$ bisectors of the angles between adjacent directions picked out by the random field (see Fig. 2). For $p=2$ the phase diagram is qualitatively similar to that of $p=1$ (Fig. 1). For $p \geq 3$, however, as first noted by Dotsenko and Feigelman, a phase of quasi-long-range order exists for a range of temperatures below the Kosterlitz-Thouless point. This phase then meets the low-temperature ordered phase as shown in Fig. 3.

Our low-temperature results differ substantially from those of Dotsenko and Feigelman, who predict a paramagnetic phase at all temperatures for $p=1$ and 2, and at low temperatures for $p \geq 3$. These authors have apparently not considered the possibility of generating a nonrandom $2p$-fold symmetry-breaking field under renormalization.

II. DOMAIN ARGUMENT

We analyze (1.1) in d dimensions at zero temperature by means of a domain argument in the spirit of Imry and Ma.[6] Because of the uniaxial nature of the field, the system can break up into domains, yet still exhibit ferromagnetic order. We assume that the system breaks up into domains of size L and in each domain the average magnetization makes an angle Θ with the average field direction (see Fig. 4). In a system of volume V, the energy of our collection of domains is given approximately by

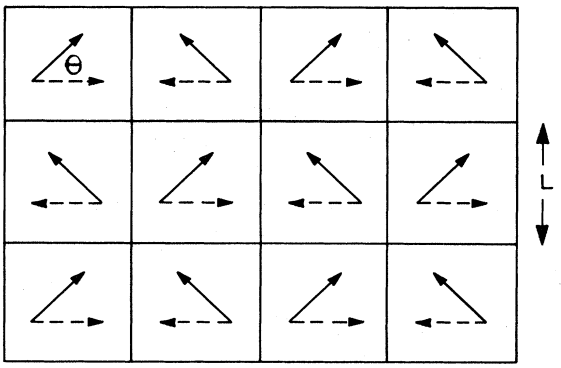

FIG. 4. Domains in the XY model with a random uniaxial field. The dashed vectors denote the average field direction in a domain, while the solid vectors denote the average magnetization near the center of the domain.

$$E=\frac{V}{L^d}[JL^{d-2}(\pi-2\Theta)^2-hL^{d/2}(\cos\Theta)] . \tag{2.1}$$

The first term in (2.1) accounts for the cost in exchange energy. It assumes that every domain is surrounded by domains with fields of opposite directions and the spin rotates smoothly from one domain to the next. Improving on this assumption, however, would only alter the prefactor of this term and not affect the critical dimensionality. Note that for $\Theta=0$, (2.1) reduces to the energy considered by Imry and Ma in their analysis of the isotropic random-field model.

We now minimize E with respect to L and Θ to see if the system prefers $\Theta=0$ (paramagnet) or $\Theta\neq 0$ (unsaturated ferromagnet). We obtain two equations:

$$\frac{\partial E}{\partial L}=V\left[-2JL^{-3}(\pi-2\Theta)^2+\frac{hd}{2}L^{-d/2-1}\cos\theta\right]=0 , \tag{2.2a}$$

$$\frac{\partial E}{\partial \Theta}=V[-4JL^{-2}(\pi-2\Theta)+hL^{-d/2}\sin\Theta]=0 . \tag{2.2b}$$

Rather than solving (2.2) exactly, we simply examine the stability of the solution for $\Theta=0$ (paramagnetic behavior). For $\Theta=0$, (2.2a) is solved by

$$L=L_c\equiv\left[\frac{4\pi^2J}{hd}\right]^{1/(2-d/2)} . \tag{2.3}$$

In the case of the isotropic random field (i.e., $\Theta=0$), if we insert (2.3) into (2.1) we find that below $d=4$ the energy with $L=L_c$ is lower than the corresponding energy with $L\to\infty$. Hence, one obtains the Imry-Ma result, i.e., the ground state is paramagnetic for $d<4$.

In the case of the uniaxial field, though the system prefers to have $L\neq\infty$ below $d=4$, the ground state can still be ferromagnetic, though unsaturated. We assess this possibility by evaluating $\partial E/\partial\Theta$ at $\theta=0, L=L_c$ and find

$$\left.\frac{\partial E}{\partial\Theta}\right|_{\Theta=0,\,L=L_c}=-V\left[\frac{4\pi^2J}{hd}\right]^{4/(d-4)}(4\pi J) . \tag{2.4}$$

Thus, for all dimensionalities the system can lower its energy by having $\Theta\neq 0$, and thereby assume an unsaturated ferromagnetic state with order in the y direction.

III. RENORMALIZATION-GROUP ANALYSIS

We now analyze (1.1) in two dimensions (2D) by constructing renormalization-group recursion relations using the Kosterlitz method.[7] We use the replica method and have obtained identical results without the use of replicas. We first study (1.1) in the spin-wave approximation. It is simple, to the order that we work, to include the effects of the vortices in the spin field at the final stage. We consider the replicated partition function,

$$Z^n=\prod_{\alpha=1}^{n}\prod_{r}\left[\int_{-\infty}^{+\infty}d\theta_\alpha(r)\exp\left[-\sum_{\alpha=1}^{n}\bar{H}_\alpha\right]\right] , \tag{3.1}$$

where the reduced Hamiltonian $\bar{H}_\alpha$ is given by

$$\bar{H}_\alpha=\frac{K}{2}\sum_{\langle rr'\rangle}[\theta_\alpha(r)-\theta_\alpha(r')]^2-H\sum_{r}\cos[\theta_\alpha(r)+\phi(r)] , \tag{3.2}$$

with $-\infty<\theta_\alpha<\infty$, $K=J/k_BT$, and $H=h/k_BT$. We follow the usual procedure[3,8] for studying symmetry-breaking fields in the 2D XY model, and represent the field portion of the action as follows:

$$\exp\left[H\sum_{\alpha}\sum_{r}\cos[\theta_\alpha(r)+\phi(r)]\right]=\prod_{r}\prod_{\alpha}\sum_{m_\alpha(r)=-\infty}^{\infty}I_{m_\alpha(r)}(H)\exp\left[i\sum_{\alpha}m_\alpha(r)[\theta_\alpha(r)+\phi(r)]\right] , \tag{3.3}$$

where the $I_m(H)$ are modified Bessel functions. We can then readily average (3.3) over the quenched variable $\phi(r)$ to obtain

$$\tfrac{1}{2}\int_0^{2\pi}d\phi(r)[\delta(\phi(r))+\delta(\phi(r)-\pi)]\exp\left[H\sum_{\alpha}\sum_{r}\cos[\theta_\alpha(r)+\phi(r)]\right]=\prod_{\alpha}\sum_{m_\alpha(r)=-\infty}^{\infty}{}'I_{m_\alpha(r)}(H)\exp\left[i\sum_{\alpha}m_\alpha(r)\theta_\alpha(r)\right] . \tag{3.4}$$

The prime on the summation sign on the right-hand side of (3.4) signifies the constraint induced by integrating over the randomness, namely

$$\sum_{\alpha=1}^{n}m_\alpha(r)=\mathscr{E} , \tag{3.5}$$

where $\mathscr{E}$ denotes an even number, at every site r. By contrast, for isotropic random fields,[3(b)] the right-hand side of (3.5) is replaced by zero.

Our effective partition function is then

$$Z_{\text{eff}}=\prod_r \left[\int_0^{2\pi} \frac{d\phi(r)}{2} [\delta(\phi(r))+\delta(\phi(r)-\pi)] Z^n \right]$$
$$= \prod_\alpha \prod_r \left[\int_{-\infty}^{\infty} d\theta_\alpha(r) \sum_{m_\alpha(r)}{}' I_{m_\alpha}(r)(H) \exp\left[-\tfrac{1}{2}K \sum_{\langle r,r'\rangle} \sum_\alpha [\theta_\alpha(r)-\theta_\alpha(r')]^2 + i \sum_\alpha m_\alpha(r)\theta_\alpha(r) \right] \right] . \tag{3.6}$$

The general forms of the charges satisfying (3.5) are either $m_\alpha^{(s)}$, s even, or $m_\alpha^{(s_1,s_2)}$ with s_1 and s_2 both odd. The superscripts denote the absolute value of the entries in the charge in replica space. Writing the charges as n-dimensional vectors we have

$$\mathbf{m}^{(1,1)}=(0,0,\pm 1,0,\ldots,\pm 1,0,0) , \tag{3.7a}$$
$$\mathbf{m}^{(2)}=(0,\pm 2,0,0,\ldots,0,0,0) , \tag{3.7b}$$
$$\mathbf{m}^{(1,3)}=(0,0,\pm 1,0,\ldots,\pm 3,0,0) , \tag{3.7c}$$
$$\mathbf{m}^{(4)}=(0,0,\pm 4,0,\ldots,0,0,0) . \tag{3.7d}$$

The ellipses in (3.7) are all zero. More complicated charges of the forms, say $m^{(1,1,1,1)}$ or $m^{(2,4)}$, are considered to be superpositions of the most elemental charges, i.e., (3.7a), (3.7b), and (3.7d) at the same lattice site, and thus are not independent entities.

The fugacities associated with the charges (3.7) take on the initial values

$$y_{1,1}=\left[\frac{I_1(H)}{I_0(H)}\right]^2 , \tag{3.8a}$$
$$y_2=\frac{I_2(H)}{I_0(H)} \tag{3.8b}$$
$$y_{1,3}=\frac{I_1(H)I_3(H)}{I_0^2(H)} , \tag{3.8c}$$
$$y_4=\frac{I_4(H)}{I_0(H)} , \tag{3.8d}$$

where we have factored out from (3.6) $[I_0(H)]^n$, the fugacity of a zero charge.

Integrating (3.6) over $\theta_\alpha(r)$ we write the result as an expansion in the numbers of the various types of charges: for simplicity we display only $m^{(1,1)}$ and $m^{(2)}$. We find

$$Z_{\text{eff}}=(Z_{\text{sw}})^n[I_0(H)]^{nM} \sum_{\{N\}} f(N)Z(N) , \tag{3.9}$$

where M is the number of lattice sites, and

$$Z_{\text{sw}}=\prod_r \left[\int_{-\infty}^{\infty} d\theta(r) \exp\left[-\tfrac{1}{2}K \sum_{\langle rr'\rangle} [\theta(r)-\theta(r')]^2 \right] \right] , \tag{3.10a}$$

$$\sum_{\{N\}} = \prod_{\mathbf{m}^{(1,1)}} \sum_{N(\mathbf{m}^{(1,1)})=0}^{\infty} \prod_{\mathbf{m}^{(2)}} \sum_{N(\mathbf{m}^{(2)})=0}^{\infty} , \tag{3.10b}$$

$$f(N)=\prod_{\mathbf{m}^{(1,1)}} \frac{y_{1,1}^{N(\mathbf{m}^{(1,1)})}}{[N(m^{(1,1)})]!} \prod_{\mathbf{m}^{(2)}} \frac{y_2^{N(\mathbf{m}^{(2)})}}{[N(\mathbf{m}^{(2)})]!} \tag{3.10c}$$

$$Z(N)=\prod_{\mathbf{m}^{(1,1)}} \prod_{i(\mathbf{m}^{(1,1)})=1}^{N(\mathbf{m}^{(1,1)})} \int_{a_0} \frac{d^2 r_{i(\mathbf{m}^{(1,1)})}}{a_0^2} \prod_{\mathbf{m}^{(2)}} \prod_{j(\mathbf{m}^{(2)})=1}^{N(\mathbf{m}^{(2)})} \left[\int_{a_0} \frac{d^2 r_{j(\mathbf{m}^{(2)})}}{a_0^2} e^{\overline{H}\{m\}} \right] , \tag{3.10d}$$

and

$$\overline{H}=\frac{1}{4\pi K} \sum_{\mathbf{m}} \sum_{i(\mathbf{m})=1}^{N(\mathbf{m})} \sum_{\mathbf{m}'} \sum_{j(\mathbf{m}')=1}^{N(\mathbf{m}')} \mathbf{m}\cdot\mathbf{m}' G(r_{i(\mathbf{m})}-r_{j(\mathbf{m}')}) . \tag{3.10e}$$

In (3.10b), $N(\mathbf{m}^{(1,1)})$ represents the number of charges of a particular permutation of (3.7a); a similar statement holds for $N(\mathbf{m}^{(2)})$. In (3.10d), $\prod_{\mathbf{m}^{(1,1)}}$ represents a product over all configurations in replica space of the charge form (3.7a). In (3.10e) the summations over $\mathbf{m}$ and $\mathbf{m}'$ are each separation sums over all permutations of the charges $\mathbf{m}^{(1,1)}$ and $\mathbf{m}^{(2)}$. The integrations in (3.10d) require that no two charges both with a nonzero entry in the same replica component come closer together than the lattice spacing a_0. The Green's function $G(r-r')$ is the usual

logarithmic interaction between charges:

$$G(r-r')=\ln\left[\left|\frac{\mathbf{r}-\mathbf{r}'}{a_0}\right|\right]+\frac{\pi}{2}\ . \tag{3.11}$$

The slashes in the summation signs in (3.10b) denote that all charge configurations contributing to (3.10) must obey spatial charge neutrality, i.e.,

$$\sum_{m^{(1,1)}} m^{(1,1)}N(m^{(1,1)})+\sum_{m^{(2)}} m^{(2)}N(m^{(2)})=0\ . \tag{3.12}$$

The details of the renormalization-group calculation are relegated to the Appendix. We find the following recursion relations for K, $y_{1,1}$, y_2, $y_{1,3}$, and y_4, and the vortex fugacity y, correct to second order in these fugacities:

$$\frac{dK}{dl}=\pi[2(n-1)y_{1,1}^2+4y_2^2+20(n-1)y_{1,3}^2+16y_4^2]$$
$$-4\pi^3K^2y^2\ , \tag{3.13a}$$

$$\frac{dy_{1,1}}{dl}=\left[2-\frac{1}{2\pi K}\right]y_{1,1}+4\pi(n-2)(y_{1,1}^2+y_{1,3}^2)$$
$$+4\pi(y_{1,1}y_2+y_{1,3}y_2+y_{1,3}y_4)\ , \tag{3.13b}$$

$$\frac{dy_2}{dl}=\left[2-\frac{1}{\pi K}\right]y_2+2\pi(n-1)(y_{1,1}^2+2y_{1,3}y_{1,1}+y_{1,3}^2)$$
$$+2\pi y_2y_4\ , \tag{3.13c}$$

$$\frac{dy_{1,3}}{dl}=\left[2-\frac{5}{2\pi K}\right]y_{1,3}+2\pi(y_{1,1}y_2+y_{1,1}y_4+y_{1,3}y_2)$$
$$+4\pi(n-2)y_{1,1}y_{1,3}\ , \tag{3.13d}$$

$$\frac{dy_4}{dl}=\left[2-\frac{4}{\pi K}\right]y_4+2\pi y_2^2+4\pi(n-1)y_{1,1}y_{1,3}\ , \tag{3.13e}$$

$$\frac{dy}{dl}=(2-\pi K)y\ . \tag{3.13f}$$

The most relevant charge fugacity in (3.13) is $y_{1,1}$, which is relevant for temperatures less than $K^{-1}=4\pi$. The eigenvalue of the fugacity associated with a general charge m is $2-\mathbf{m}^2/4\pi K$. Thus, the charges considered in (3.13) are the only ones relevant for temperatures greater than $K^{-1}=4\pi/9$ (corresponding to the temperature where $\mathbf{m}^{(3,3)}$ becomes relevant). Since y becomes relevant at $K^{-1}=\pi/2$, we can use (3.13) in the temperature range $4\pi/9<K^{-1}<\pi/2$. We also note that unlike the situation in the isotropic random-field model,[3(b),3(c)] there is only one coupling K here, and no off-diagonal (in replica space) coupling is developed. In the isotropic case, the off-diagonal coupling represents a random Dzyaloshinski-Moriya interaction,[2] which is not generated when the field is uniaxial.

For the general p model defined in (1.2), (3.13) are replaced by similar equations, except that in (3.13b)–(3.13e), $K\to K/p^2$. We see then that for $p\geq 3$ there is a range of temperatures below the Kosterlitz-Thouless point where all of the charge fugacities are irrelevant.[4]

We now analyze (3.13) and their p-state counterparts with $n=0$. As discussed in the last paragraph, our analysis is restricted to $4\pi/13p^2<K^{-1}<\pi/2$. We have considered, however, the extension of (3.13) to all charges with $\mathbf{m}^2\leq 100$, allowing us to probe temperatures down to $K^{-1}=4\pi/53p^2$ (Figs. 5 and 6 are based on the latter analysis). The generalization of (3.13) to all charges is shown at the end of the Appendix.

Analyzing (3.13) numerically with $dl=0.01$, we find that $y_{1,1}$ initially grows and reaches a maximum of about 0.1. Subsequently $y_{1,1}$ drives y_2 negative. The negative y_2 drives all fugacities of the form y_{1,s_2}, s_2 odd, to zero. Similarly, y_6 drives all fugacities of the form y_{3,s_2} to zero, etc. This behavior is illustrated in Fig. 5 for initial values: $H=0.1$, $K^{-1}=0.35\pi$.

Since all of the double-subscripted fugacities have been driven to zero, or are negligibly small, the replica coupling in the system has vanished and only a *nonrandom* field remains. At each site in the system, this field yields a factor in the partition function given by [cf. (3.7)]

$$e^{V(\theta)}=1+2y_2(l^*)\cos[2p\theta(r)]$$
$$+2y_4(l^*)\cos[4p\theta(r)]+\cdots\ , \tag{3.14}$$

where only fugacities with a single even subscript appear, and l^* corresponds to our last iteration of (3.13) (see Fig. 5). We have plotted (3.14) in Fig. 6 for $p=1$. We see that $V(\theta)$ is a potential, that induces order along the directions $\theta=\pi/2$ and $3\pi/2$, i.e., along the y axis. Thus there is an ordered phase in this temperature range with Ising symmetry. For general p, (3.14) is a function with $2p$ peaks and periodicity $2\pi/2p$. We also note that when the fugacities $y_{1,1}$ and $y_{1,3}$ are set to zero in (3.13), we obtain the recursion relations derived in Ref. 8 for an XY model in the presence of a nonrandom $2p$-fold symmetry-breaking field.

It is difficult to ascertain the behavior of this model at very low temperatures where more and more fugacities become relevant. Given our domain-argument result of Sec.

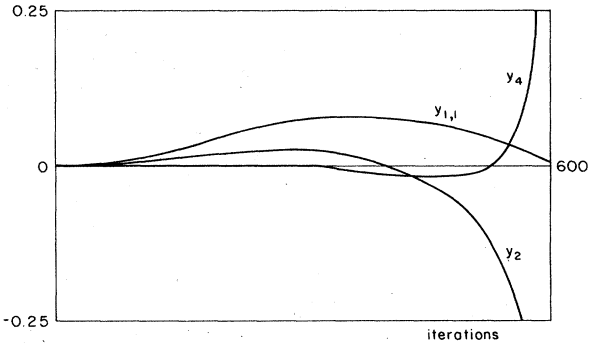

FIG. 5. Numerical solution of (A8) for charges with $\mathbf{m}^2\leq 100$, for initial values $H=0.1$, $K^{-1}=0.35\pi$. For simplicity, we plot only $y_{1,1}$, y_2, and y_4.

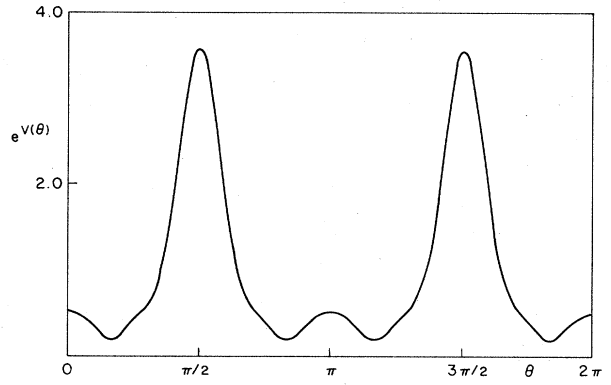

FIG. 6. Plot of $e^{V(\theta)}$, (3.14), after 588 iterations with initial values $H=0.1$ and $K^{-1}=0.35\pi$. The nonzero charge fugacities at this iteration point are $y_{1,1}=1.28\times10^{-2}$, $y_2=-0.562$, $y_4=0.411$, $y_6=-0.163$, $y_8=0.100$, and $y_{10}=-3.8\times10^{-2}$. We have neglected $y_{1,1}$ in making this plot, as Fig. 6 indicates that it is rapidly tending to zero. The subsidiary peaks in $\exp[V(\theta)]$ occurring at $\theta=0,\pi,2\pi,\ldots$ have no obvious physical interpretations and may be artifacts of the second-order equations.

II, we expect that the phase diagrams will be as indicated in Fig. 1 for $p=1$ and 2, and Fig. 3 for $p\geq3$, with the long-range order persisting down to zero temperature.

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APPENDIX: DERIVATION OF THE RECURSION RELATIONS (3.13)

In this appendix we sketch the derivation of (3.13) using the methods of Refs. 3(b), 7, and 9. We perform a Kosterlitz-style renormalization-group analysis of the effective partition $Z_{\rm eff}$ defined in (3.10) and (3.11). We integrate out the short-wavelength degrees of freedom, i.e., we integrate over the positions of all charges which come within a distance $a_0+\delta a$ of each other. This procedure is accomplished by breaking up the integrations in (3.10d) as follows:

$$\int_{a_0}=\int_{a_0}^{a_0+\delta a_0}+\int_{a_0+\delta a_0}\,. \tag{A1}$$

We keep terms only to order δa_0, i.e., we integrate out only *pairs* of charges that come within $a_0+\delta a_0$ of each other. We will thereby obtain recursion relations to second order in the fugacities. Consideration of triplets and quartets of charges would, in principle, yield higher-order terms in (3.13). If the two charges in the pair under consideration are equal and opposite, they will form a dipole, screening the interaction between the remaining charges. This effect is the source of the terms on the right-hand side of (3.13a). The last term in that equation is the screening due to spin vortices and agrees with the original Kosterlitz result.[7] The remaining two terms in that equation arise from the screening due to the random-field charges. The first term proportional to $y_{(1,1)}^2$ involves a pair of charges of the $\mathbf{m}^{(1,1)}$ variety, while the term proportional to y_2^2 involves a pair of $\mathbf{m}^{(2)}$ charges. The specific form of these terms can be derived by following Appendix A of Ref. 3(b). For example, considering the $m^{(1,1)}$ charges, we expand the part of $\overline{H}$, (3.10e), involving this pair: denoting this part of $\overline{H}$ as $\overline{H}_{ij}$, we have

$$\overline{H}_{ij}=\frac{1}{2\pi K}\sum_r \mathbf{m}(r)\cdot\mathbf{m}^{(1,1)}(r_i)[G(r-r_i)-G(r-r_j)]\,, \tag{A2}$$

where the sum is over the lattice sites indicated in (3.10d) excluding $r=r_i$ and $r=r_j$.

As usual, we consider $|r-r_i|, |r-r_j|\gg a_0$, so that we can expand the difference of the Green's functions. Thus,

$$\overline{H}_{ij}=\frac{1}{2\pi K}\sum_r \mathbf{m}^{(1,1)}(r)\cdot\mathbf{m}^{(1,1)}(r_i)[\mathbf{a}_{ij}\cdot\nabla G(r-r_i)+\tfrac{1}{2}(\mathbf{a}_{ij}\cdot\nabla)^2G(r-r_i)+\cdots]\,, \tag{A3}$$

where $\mathbf{a}_{ij}=\mathbf{r}_i-\mathbf{r}_j$, $a_0<|\mathbf{a}|<a_0+\delta a_0$. Upon expanding the exponential of $\overline{H}_{ij}$ to second order in a, one finds [after dropping terms that contribute zero after the trace over $\mathbf{m}^{(1,1)}(r_i)$]

$$e^{H_{ij}}=1+\frac{1}{8\pi^2K^2}\sum_{r,r'}\{[\mathbf{a}_{ij}\cdot\nabla G(r-r_i)][\mathbf{a}_{ij}\cdot\nabla G(r-r_i)][\mathbf{m}(r)\cdot\mathbf{m}^{(1,1)}(r_i)][\mathbf{m}(r')\cdot\mathbf{m}^{(1,1)}(r_i)]\}\,. \tag{A4}$$

To trace over $m^{(1,1)}(r_i)$, we use

$$\tfrac{1}{2}\sum_{\mathbf{m}^{(1,1)}}(\mathbf{A}\cdot\mathbf{m}^{(1,1)})(\mathbf{m}^{(1,1)}\cdot\mathbf{B})=\frac{1}{2n}(\mathbf{m}^{(1,1)})^2\left[\sum_{\mathbf{m}^{(1,1)}}(1)\right]\mathbf{A}\cdot\mathbf{B} \tag{A5}$$

for any vectors $\mathbf{A}$ and $\mathbf{B}$. The right-hand side of (A5) can be simplified by noting that $(\mathbf{m}^{(1,1)})^2=2$, and

$\sum_{m^{(1,1)}}(1)=2^2\binom{n}{2}=2n(n-1)$, which is just the number of distinct charges of the $m^{(1,1)}$ variety. We then integrate over the shell, sum over a, and integrate over r_i [see, e.g., Appendix A of Ref. 3(b)]. The contribution to (3.13a) proportional to $y_{1,1}^2$ follows immediately. Similarly, to derive the contribution proportional to y_2^2, we follow the same procedure, replacing (A5) by

$$\frac{1}{2}\sum_{\mathbf{m}^{(2)}}(\mathbf{A}\cdot\mathbf{m}^{(2)})(\mathbf{m}^{(2)}\cdot\mathbf{B})=\frac{1}{2n}(\mathbf{m}^{(2)})^2\left[\sum_{\mathbf{m}^{(2)}}(1)\right]\mathbf{A}\cdot\mathbf{B}=4\mathbf{A}\cdot\mathbf{B}\ . \tag{A6}$$

The second-order terms in (3.13b)—(3.13f) arise from counting the number of ways two charges can form a composite charge of the same form as the one whose fugacity we are renormalizing [see, e.g., Refs. 3(b) and 9]. For example, in (3.13c) the term proportional to $y_{1,1}^2$ arises from the following vector addition:

$$(2,0,0,\ldots)=(1,0,0,\ldots,0,+1,0,\ldots,0)+(1,0,0,\ldots,0,-1,0,\ldots,0)\ . \tag{A7}$$

There are $(n-1)$ positions for the $+1,-1$ entries which will satisfy (A7). The term proportional to $y_{1,1}^2$ then has a coefficient $2\pi(n-1)$, the factor of 2π arising from a geometrical factor of 2π occurring in the integration over **a**. Using these techniques it is straightforward to generalize (3.13) to second order in all charge fugacities:

$$\frac{dK}{dl}=\pi\left[(n-1)\sum_{i=1,3,5,\ldots,}\ \sum_{j=1,3,5,\ldots,}(i^2+j^2)y_{i,j}y_{i,j}+\sum_{i=2,4,6,\ldots,}i^2y_i^2\right]-4\pi^3K^2y^2\ , \tag{A8a}$$

$$\begin{aligned}\frac{dy_{s_1,s_2}}{dl}=&\left[2-p^2\frac{(s_1^2+s_2^2)}{4\pi k}\right]y_{s_1,s_2}\\&+2\pi\sum_{i=1,3,5,\ldots,}[2(n-2)y_{s_1,i}y_{s_2,i}+y_{s_1+i}y_{s_2,i}+y_{s_2+i}y_{s_1,i}+y_{s_1+i+1,s_2}y_{i+1}+y_{s_2+i+1,s_1}y_{i+1}]\\&+2\pi\sum_{i=1,3,5,\ldots,}^{s_1-2}(y_{i,s_2}y_{s_1-i})+2\pi\sum_{j=1,3,5,\ldots,}^{s_2-2}(y_{j,s_1}y_{s_2-j})\ ,\end{aligned} \tag{A8b}$$

where s_1 and s_2 are both odd, and

$$\begin{aligned}\frac{dy_s}{dl}=&\left[2-\frac{p^2s^2}{4\pi K}\right]y_s+2\pi(n-1)\sum_{j=1,3,5,\ldots,}\left[2\sum_{i=1,3,5,\ldots,}y_{i,j}y_{s+i,j}+\sum_{i=1,3,5,\ldots,}^{s-1}y_{i,j}y_{s-i,j}\right]\\&+2\pi\sum_{i=2,4,6,\ldots,}y_iy_{s+i}+2\pi\sum_{i=2,4,6,\ldots,}^{s-2}y_iy_{s-i}\ ,\end{aligned} \tag{A8c}$$

where s is even. If the upper limit in any summation is less than the first value to be summed on, then there is no contribution from that sum.

The generalizations of (A5) and (A6) are, respectively,

$$\begin{aligned}&\frac{1}{2}\sum_{\mathbf{m}^{(s_1,s_2)}}(\mathbf{A}\cdot\mathbf{m}^{(s_1,s_2)})(\mathbf{m}^{(s_1,s_2)}\cdot\mathbf{B})\\&\quad=\frac{1}{2n}(\mathbf{m}^{(s_1,s_2)})^2\left[\sum_{\mathbf{m}^{(s_1,s_2)}}(1)\right]\mathbf{A}\cdot\mathbf{B}\ ,\end{aligned} \tag{A9}$$

and

$$\frac{1}{2}\sum_{\mathbf{m}^{(s)}}(\mathbf{A}\cdot\mathbf{m}^{(s)})(\mathbf{m}^{(s)}\cdot\mathbf{B})=\frac{1}{2n}(\mathbf{m}^{(s)})^2\left[\sum_{\mathbf{m}^{(s)}}(1)\right]\mathbf{A}\cdot\mathbf{B}\ . \tag{A10}$$

*Present address: Institut für Festkörperforschung der Kernförshungsanlage Jülich GmbH, D-5170 Jülich, West Germany.

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