Finite-size scaling study of the two-dimensional Ising spin glass

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We have calculated finite-size correlation lengths for strips of the square-lattice Ising spin glass using the transfer-matrix method. In order to minimize sampling errors we study strip lengths up to 5×10^6 lattice spacings. A phenomenological renormalization-group analysis indicates that there are strong corrections to simple power-law scaling near the zero-temperature critical point, as is to be expected near a lower critical dimension. We examine models with Gaussian, $\exp(-J^2/2)$, and exponential, $\exp(-|J|)$, distributions of couplings; the Gaussian distribution shows stronger finite-size corrections. The correlation-length exponent is estimated to be $\nu=4.2\pm0.5$, although we do not want to rule out the possibility that ν is significantly larger than this.

The question of the lower critical dimensions (LCD) of spin glasses has been strongly debated recently.[1] For Ising spin glasses a LCD of 4 has been suggested by various authors.[2–4] However, recent massive Monte Carlo simulations on a special-purpose computer[5] and finite-size scaling studies[6,7] indicate that spin-glass ordering occurs at a finite temperature in the three-dimensional Ising spin glass. Thus it appears that the LCD is less than or equal to 3.

In the two-dimensional Ising spin glass, ordering apparently does not occur at any finite temperature, T. Many studies have looked only at binary ($\pm J$) distributions of bond strengths which result in a large ground-state degeneracy. As this degeneracy is peculiar to that particular distribution of bond strengths and strongly alters the low-temperature behavior, we will here look at continuous distributions of bond strengths that are presumably more "generic" and do not exhibit ground-state degeneracy in finite systems. Initial studies[8] on small systems were interpreted[9] as indicating that the correlation length diverges as $\xi\sim T^{-\nu}$ with $\nu=2$. Cheung and McMillan[10] then looked at the system with bond strengths distributed uniformly between $-J$ and $+J$ using the transfer-matrix method. They obtained finite-size correlation lengths for $0.15J/k_B\leq T\leq0.75J/k_B$ and strips of widths and lengths up to 7 by 10^5 and 11 by 10^4. They fitted their data to a finite-size scaling ansatz with a bulk correlation length of $\xi(T)=aT^{-\nu}+\xi_0$, with $\nu=2.96\pm0.22$ and $\xi_0=2.97$. If we define an effective exponent by

$$\nu_{\rm eff}(T)=-\frac{\partial\ln\xi(T)}{\partial\ln T}\ , \tag{1}$$

then Cheung and McMillan[10] have a fairly strong variation of $\nu_{\rm eff}$ with T due to the large value of ξ_0 they found. The parameter ξ_0 they have allowed is a correction to the pure $\xi\sim T^{-\nu}$ scaling; we argue below that stronger corrections to scaling are present near the $T=0$ critical point.

Bray and Moore[11] and McMillan[12] have estimated ν using finite-size scaling for "domain-wall" energies[13] at $T=0$, obtaining $\nu=3.4\pm0.1$ from systems of sizes up to 12×12 and $\nu=3.27\pm0.16$ for sizes up to 8×8, respectively. There is a tendency in the results we have quoted for the estimate of ν to increase as one looks at either lower temperatures or larger systems, as should occur near or at the LCD of a system. In view of this, we felt it was worth doing a more careful transfer matrix study of the two-dimensional Ising spin glass, going to low temperatures, and using very long strips in order to reduce statistical errors. Allowing for the strong corrections to the $\xi\sim T^{-\nu}$ scaling that are present near a lower critical dimension, and can be seen in a phenomenological renormalization-group[14] (RG) analysis of our data, we find evidence that the exponent ν is significantly larger than estimated by previous authors.[8–12] We estimate $\nu=4.2\pm0.5$ on the basis of our data for strip widths 2, 4, and 8 and, since we are limited to such narrow strips, do not wish to rule out the possibility that ν is actually significantly larger than this estimate.

We consider the Edwards-Anderson model on a square lattice with only nearest-neighbor interactions. The Hamiltonian is

$$H=\sum_{\langle ij\rangle}J_{ij}s_is_j\ , \tag{2}$$

where the $s_i=\pm1$ are Ising spins and the J_{ij} for each bond are independently distributed according to a probability distribution $P(J)$. The distributions we have used are the Gaussian, $P(J)\propto\exp(-J^2/2)$, and the exponential, $P(J)\propto\exp(-|J|)$. Monte Carlo simulation studies of this model at very low temperatures are prevented by huge equilibrium times. The transfer-matrix method avoids this difficulty by actually summing over all states of the system. We consider long strips of width W lattice spacings and periodic boundary conditions across the strips. We obtain the finite-size correlation length $\xi_W(T)$ for correlations along the length of the strip using transfer matrices. How this is done for spin glasses has been described by Cheung and McMillan.[15] In the present study we found it necessary to run very long strips (up to 5×10^6 lattice spacings) in order to get very accurate estimates of $\xi_W(T)$. For a strip of width $W=8$ our program will go a length of 1.8×10^5 per minute of Cray Research Cray-1 computer CPU (central processing unit) time.

Our results for $\xi_W(T)$ for widths $W=6$ and $W=12$ and the Gaussian distribution are shown in Fig. 1. At low temperatures, where $\xi_W(T)\gg W$, the systems behave like a one-dimensional spin glass with $\xi_W(T)\sim T^{-1}$. This asymptotic low-temperature behavior is indicated by the dashed lines in Fig. 1. As temperature is increased and $\xi_W(T)$ becomes of order W, the behavior crosses over to that of a two-dimensional spin glass which has ξ changing more rapidly with T.

As $T\to0$ each finite strip of spin glass goes into one of its two ground states that are related by a global spin reversal. The low-energy excitations that cause a finite correlation length at low temperatures where $\xi_W(T)\gg W$ are domain walls ("kinks") passing across the strip. These domain walls separate domains of the two ground states and may have arbitrarily low energies, depending on the local bond strengths $\{J_{ij}\}$. The correlation length measures the average spacing along the strip between such excited domain walls in this low-temperature one-dimensional regime.

We have calculated phenomenological renormalization-group[14] (RG) recursion relations based on our results for $\xi_W(T)$. The renormalized temperature $T'_W(T)$ under a rescaling by a factor of $b=2$ obtained from comparing strips of width W and $2W$ is defined by

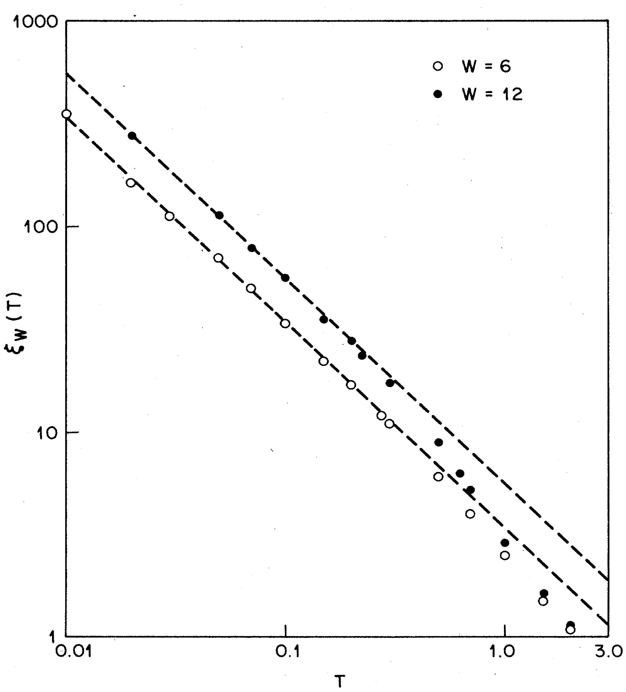

FIG. 1. The finite-size correlation length $\xi_W(T)$ for $W=6$ and $W=12$. The dashed lines on this log-log plot indicate the low-temperature one-dimensional behavior $\xi_W(T)\sim T^{-1}$. The crossover to two-dimensional behavior occurs in the vicinity of $\xi_W(T)=W$. The sampling errors are estimated to be less than or equal to the size of the data points.

$$\frac{\xi_{2W}(T)}{2W}=\frac{\xi_W(T'_W(T))}{W} . \quad (3)$$

Let us first consider what sort of recursion relations we should expect near a lower critical dimension. Since we are dealing with a spin glass, we do not really have much guidance on what to expect, so we make the simplest ansatz, namely that the RG equation for continuous rescaling can be expanded about $T=0$ as

$$-\frac{\partial T}{\partial \ln\xi}=\frac{1}{\nu}T(1+BT^{\theta}+\cdots) , \quad (4)$$

with $\theta>0$. The solution to this equation is

$$\xi=AT^{-\nu}\left[1+\frac{\nu}{\theta}BT^{\theta}+\cdots\right] , \quad (5)$$

where A is a constant of integration. If we then rescale by a factor of $b=2$, as in our phenomenological RG, a convenient way of presenting the result is

$$\frac{T'(T)}{T}=2^{1/\nu}+\frac{B}{\theta}(2^{2/\nu}-2^{1/\nu})T^{\theta}+\cdots . \quad (6)$$

Of course, keeping terms at higher order in T in (4) will yield the higher-order terms in (6).

The phenomenological recursion relations obtained from (3) with $W=2$ and $W=4$ are shown in Fig. 2, where the data are presented as $T'_W(T)/T$ versus T as in (6) and the results for both Gaussian and exponential distributions are shown. We have also obtained recursion relations for $W=6$, but because of the limited length strips sampled for $W=12$ the statistical uncertainties in the data are too large to make them useful when compared to the data in Fig. 2. The results in Fig. 2 differ from the bulk recursion relations (6) due to finite-size effects, which are quite noticeable when comparing the data for $W=2$ and $W=4$. Finite-size estimates, ν_W, for the exponent ν may be obtained from the intercepts in Fig. 2 via

$$\lim_{T\to0}[T'_W(T)/T]=2^{1/\nu_W} . \quad (7)$$

Extrapolation of our data yields $\nu_2^{(G)}=3.3\pm0.2$ and $\nu_4^{(G)}=3.7\pm0.2$ for the Gaussian distribution and $\nu_2^{(E)}=4.5\pm0.3$ and $\nu_4^{(E)}=4.4\pm0.3$ for the exponential distribution. By universality, we expect $\nu_W\to\nu$ for $W\to\infty$ for any continuous distribution, $P(J)$. Taking our finite-size estimates together and noting that the variation of ν_W is stronger for the Gaussian case we estimate the bulk exponent as $\nu=4.2\pm0.5$.

One noteworthy feature of Fig. 2 is that the difference between the results for $W=2$ and $W=4$ are significantly larger with the Gaussian distribution than they are with the exponential distribution. This suggests that the latter may be a closer approximation to the zero-temperature fixed-point Hamiltonian distribution. The fixed-point distribution for the one-dimensional chain random exchange Ising model is precisely the exponential distribution,[11] so it might be expected to still be a good approximation for two dimensions. Also, Bray and Moore[11] have obtained a renormalized distribution $P_5(J_{\rm eff})$ from the effective couplings $J_{\rm eff}$ across blocks of linear dimension $W=5$. They start with a Gaussian distribution of nearest-neighbor

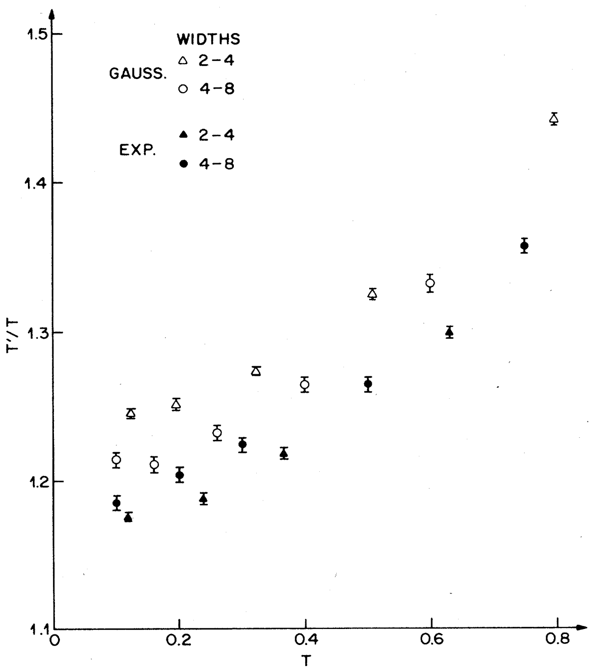

FIG. 2. Phenomenological RG recursion relations $T'_W(T)$ obtained by comparing data from strips of widths 2, 4, and 8 for Gaussian and exponentially distributed nearest-neighbor couplings. The intercept in this plot for $T \to 0$ is $T'/T = 2^{1/\nu}$ for $W \to \infty$, and so finite-size estimates of ν are obtained from extrapolating these data to $T = 0$.

couplings and end up with a renormalized distribution looking more like the exponential, with a sharp change in $dP(J)/dJ$ near $J = 0$, as is shown in Fig. 2 of Ref. 11. Thus we feel that the exponential distribution probably gives better estimates of the critical exponent ν than the Gaussian distribution.

We would like to point out a possible reason why our estimate of ν may still be incorrect. It is obtained from data at low temperatures where the excitations that determine the finite-size correlations lengths and therefore the recursion relations $T'_W(T)$ are short domain walls passing across the strips, as discussed above. In the bulk system these excitations are not available and it is presumably other, much more extended "domain-wall" excitations that determine the true bulk correlation lengths at these low temperatures. Therefore it is not impossible that the bulk recursion at low T is actually rather different from the finite-size estimates shown in Fig. 2. However, since the estimate of ν has systematically increased as more careful studies have been done, we suspect that if our estimate is wrong, it is because the correct bulk ν is actually still larger.

It is worth noting how Bray and Moore[11] and McMillan[12] obtained estimates of ν with such small error bars. Our recursion relations indicate corrections to simple power-law scaling that are already substantial by $T = 0.5$ where the bulk correlation length is more than 10 lattice spacings. Allowing for such corrections the finite-size "domain-wall" energies calculated in Refs. 11 and 12 should scale as

$$\tilde{J}(W) \approx F(W^{-1/\nu}) , \tag{8}$$

where the scaling function $F(x)$ is smooth and behaves as $F(x) \approx F_0 x$ for $x \to 0$, but shows significant corrections to this linear form for $x = W^{-1/\nu}$ of order unity. In Refs. 11 and 12 they forced the data into the form $F(x) = F_0 x$ without allowing for such corrections, which was possible because of the very limited range of $W^{-1/\nu}$ accessible numerically.

In conclusion, we have performed a careful finite-size scaling study of the two-dimensional Ising spin glass. Phenomenological RG recursion relations show that there are strong corrections to the $\xi \sim T^{-\nu}$ scaling expected for a system below its lower critical dimension (LCD). Allowing for these corrections we estimate $\nu = 4.2 \pm 0.5$. We also find evidence that the exponential distribution of couplings is a better approximation to the zero-temperature fixed-point distribution than the Gaussian.

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