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Time-dependent Ginzburg-Landau equations for a dirty gapless superconductor

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A complete set of charge-conserving, gauge-invariant, time-dependent Ginzburg-Landau equations is derived for a gapless, dirty, type-II superconductor using the method of Gor'kov and Eliashberg. Pair breaking by magnetic impurities and by a static magnetic field is permitted in any ratio, in contrast to the equations of Hu and Thompson, where the magnetic impurities dominate. Our extension of the range of validity requires a sizable increase in complexity. The space and time derivatives of the order parameter and the dynamic electromagnetic fields must be included to one higher order than in previous work. The equations are solved for a plane geometry in perpendicular magnetic field to first order in a parallel time-independent electric field, including all screening or backflow effects. The nonlinear terms due to normal-state Joule heating found by Larkin and Ovchinnikov are also obtained, and a precise connection is made between the vertex functions appearing in our equations and the electron distribution function appearing in the Boltzmann equation in the normal state.

I. INTRODUCTION

The first time-dependent Ginzburg-Landau (TDGL) equations were derived by Schmid[1] for the case of weak pair breaking. This seminal work had two significant shortcomings: (1) as later emphasized by Gor'kov and Eliashberg,[2] TDGL equations require slow variations in space and time and can only be valid in the gapless regime, which is vanishingly small for weak pair breaking, and (2) space and time derivatives were not carried out to high enough order to include the anomalous terms. The first anomalous contribution to the conductivity was discovered by Maki[3] and shown by Thompson[4] to be of the same order of magnitude as the regular terms derived by Schmid. The anomalous terms are caused by the slow decay of phase coherence of time-reversed electronic trajectories, which also contribute importantly to electron localization in the normal state.[5]

Schmid's work was generalized to arbitrary pair breaking by a static magnetic field by Caroli and Maki.[6] Although the first shortcoming of Schmid's work was removed near the upper critical magnetic field, the second remained. The anomalous terms were still not included. Furthermore, as shown by Takayama and Ebisawa,[7] a new error was introduced because they were not careful about the correct ordering of the gauge-invariant space and time derivatives. The derivatives do not commute because of the space-time dependence of the electromagnetic potentials.

The first completely correct set of TDGL equations was obtained by Gor'kov and Eliashberg[2] in the limit of strong pair breaking due to magnetic impurities. This treatment was extended by Eliashberg[8] to the case of weak pair breaking by magnetic impurities and by Hu and Thompson[9] to arbitrary pair breaking by magnetic impurities. The situation is relatively simple for these cases where the pair breaking is dominated by magnetic impurities. The anomalous terms are not important, and the space and time derivatives are only needed in the lowest order where the question of their relative ordering does not arise.

Later Gor'kov and Eliashberg[10] included the pair-breaking effect of a static magnetic field within the structure of their TDGL. However, they also did not carry out the calculation to high enough order to include the anomalous terms necessary to calculate correctly flux-flow conductivity. The necessary calculation is carried out in the present work. The complete TDGL equations we obtain are presented in Sec. II.

As an example of an application of our equations we solve in Sec. III for the complete structure, including all screening or backflow effects, for a plane superconductor in a strong perpendicular magnetic field and a weak parallel electric field. Some results for these screening effects were given earlier by Thompson and Hu for two-dimensional features[11] and some of the three-dimensional features[12] for the simpler case of strong pair breaking by magnetic impurities, although not all of the three-dimensional features were written explicitly even for this case.

Finally, we consider nonlinear features of our equations in Sec. IV. As has been emphasized by Larkin and Ovchinnikov,[13] the main effect near the upper critical field is normal-state Joule heating because of the extremely long time for the energy absorbed from the electric field to escape from the electrons at low temperatures.

II. TDGL EQUATIONS

Our purpose is to present TDGL equations of sufficient accuracy to calculate the rate of change with respect to the magnetic field B of the conductivity of a superconductor from its normal-state value σ near the upper critical field H_{c2}. The lowest-order TDGL equations include contributions to the current $\mathbf{j}$ up to order $|\Delta|^2\mathbf{A}$, where Δ is the order parameter and $\mathbf{A}$ is the electromagnetic vector potential. An implicit contribution to the change in the conductivity is induced by the change in Δ caused by the presence of an electric field $\mathbf{E}$. However, a change in the conductivity of order $|\Delta|^2$ could also result from an explicit term in the current of order $|\Delta|^2\mathbf{E}$, where $\mathbf{A}$ contributes to $\mathbf{E}$ through its time derivative $\partial\mathbf{A}/\partial t$. It happens that the explicit contributions are negligible compared with the implicit ones when pair breaking is dominated by magnetic impurities, so the extra terms were not needed in Refs. 2, 8, and 9. However, when the pair breaking is dominated by the magnetic field both contributions are important. To develop a consistent set of equations which will satisfy charge conservation $\nabla\cdot\mathbf{j}+\partial\rho/\partial t=0$, where ρ is the charge density, we must also calculate the other TDGL equations for the order parameter, the charge density, and the vertex functions along with the current to one higher dynamic order.

Our derivation uses exactly the methods explained by Gor'kov and Eliashberg.[2,8,10] Employing standard techniques of quantum-field theory, particular attention is paid to certain vertex functions, called Γ_1 and Γ_2, which are anomalously large, being inversely proportional to the diffusion propagator. These vertices arise from diagrams in which an electronic energy changes from one side of the Fermi level to the other because of the addition of the frequency of the external perturbation causing the system to leave equilibrium. We omit the details of the derivation and directly present our results:

$$\left[\ln\left[\frac{T}{T_{c0}}\right]+\psi(\tfrac{1}{2}+\hat{\rho})-\psi(\tfrac{1}{2})\right]\Delta-\frac{1}{4\pi T}\frac{\partial}{\partial t}[\psi'(\tfrac{1}{2}+\hat{\rho})\Delta]+\hat{U}\Delta-\frac{1}{32\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0)\left[\frac{\partial^2\Delta}{\partial t^2}-i2e\left[\frac{\partial\Phi}{\partial t}\right]\Delta\right]$$
$$+\frac{1}{2}\frac{\partial}{\partial t}(U'\Delta)+\frac{\Delta}{16\pi^2T^2}\left[\psi''(\tfrac{1}{2}+\rho_0)+\tfrac{1}{3}\psi'''(\tfrac{1}{2}+\rho_0)\left[\rho_0+\frac{D\nabla^2}{4\pi T}\right]\right]|\Delta|^2=0\,, \quad (1)$$

$$\mathbf{j}=-\sigma\left[\frac{\partial\mathbf{A}}{\partial t}+\nabla\left[\Psi+\frac{|\Delta|^2U_2'}{2e}\right]\right]-\sigma|\Delta|^2\left[\frac{g_1(\rho_0)}{4\pi^2T^2}\frac{\partial\mathbf{A}}{\partial t}-\frac{\nabla}{2e}\left[\frac{U_2}{\alpha_0}+U_2'\right]\right]$$
$$+\frac{\sigma}{4e\pi T}\sum_{n=0}^{\infty}\left[\psi^{(n+1)}(\tfrac{1}{2}+\rho_0)+\frac{1}{4\pi T}\psi^{(n+2)}(\tfrac{1}{2}+\rho_0)\frac{\partial}{\partial t}\right]\left[\sum_{m=0}^{n}[(\hat{\rho}-\rho_0)^m\Delta^*\mathbf{q}(\hat{\rho}-\rho_0)^{n-m}\Delta\right.$$
$$\left.+(\hat{\rho}-\rho_0)^{n-m}\Delta\mathbf{q}(\hat{\rho}-\rho_0)^m\Delta^*]\right]\,, \quad (2)$$

$$\rho=\frac{\sigma}{D}\left\{\left[\Psi+\frac{|\Delta|^2U_2'}{2e}-\Phi\right]+\left[\frac{i}{32e\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0)\left[\Delta\frac{\partial\Delta^*}{\partial t}-\Delta^*\frac{\partial\Delta}{\partial t}\right]-\frac{U_2'|\Delta|^2}{2e}\right]\right\}\,, \quad (3)$$

$$e\Psi=-\frac{i\tau}{4}\int d\epsilon(\Gamma_1-\Gamma_2)\,,$$
$$\hat{U}=\hat{U}_1+i\hat{U}_2\,,$$
$$\hat{U}_1=\frac{i\tau}{4}\int d\epsilon(\Gamma_1+\Gamma_2)\frac{\epsilon}{\epsilon^2+\hat{\alpha}^2}\,, \quad (4)$$
$$\hat{U}_2=\frac{i\tau}{4}\int d\epsilon(\Gamma_1-\Gamma_2)\frac{\hat{\alpha}}{\epsilon^2+\hat{\alpha}^2}$$

$$g_1(\rho_0)=\frac{1}{2\rho_0}\psi'(\tfrac{1}{2}+\rho_0)+\tfrac{1}{2}\psi''(\tfrac{1}{2}+\rho_0)\,. \quad (5)$$

In these equations T is temperature, T_{c0} the critical temperature in the absence of pair-breaking effects, ψ the digamma function, ψ' and ψ'' its first and second derivatives, e the electronic charge, Φ the electromagnetic scalar potential, τ the normal-state single-electron lifetime, and D the normal-state diffusion constant. ($\hbar=c=k_B=1$.) The dirty-limit condition $2\pi T_{c0}\tau\ll 1$ is required. $\hat{\alpha}$ is an operator acting on Δ or Δ^*, $\hat{\alpha}=\tau_s^{-1}+\frac{1}{2}Dq^2$, where τ_s is the normal-state single-electron lifetime for spin-flip scattering, and $\mathbf{q}$ is a gauge-invariant derivative defined by $\mathbf{q}\Delta=(-i\nabla-2e\mathbf{A})\Delta$ and $\mathbf{q}\Delta^*=(i\nabla-2e\mathbf{A})\Delta^*$. α_0 is the lowest eigenvalue of $\hat{\alpha}$, which is $\alpha_0=\tau_s^{-1}+\frac{1}{2}\epsilon_0$. $\epsilon_0=2eDB_0$ near the upper critical field for the vortex state. B_0 is the average magnetic field. $\hat{\rho}=\hat{\alpha}/2\pi T$ and $\rho_0=\alpha_0/2\pi T$. $\hat{\alpha}$ must be kept as an operator in certain terms because the dynamic corrections to Δ may have different eigenvalues than α_0. Some other terms already have enough explicit time derivatives so that these corrections are not important to the order we are working. The operator $\hat{U}$ becomes a number U when $\hat{\alpha}$ is replaced by α_0. U is defined as in Ref. 8 so that both U_1 and U_2 are real, in distinction from Ref. 10 where U_2 is imaginary.

The terms linear in Δ in Eq. (1) resemble the expansion of the operator

$$\{\ln(T/T_{c0})+\psi[\tfrac{1}{2}+\hat{\rho}-(\partial/\partial t+ie\Phi)/4\pi T]-\psi(\tfrac{1}{2})\}\Delta$$

proposed in Ref. 6, except for the ordering of the operators and the appearance of more complicated functions replacing Φ. Equation (2) for the current is again complicated by the appearance of some new functions replacing the scalar potential in the first set of large square brackets, which is the normal-state conductivity, and in the second set of large square brackets, which is the anomalous term of Ref. 4. The last sum of terms in Eq. (2) reduces to the usual supercurrent in the static case. The explicit time derivative of the supercurrent does not contribute to the average conductivity but does affect the local current distribution. Γ_1 and Γ_2 are the vertex functions defined by Gor'kov and Eliashberg. They obey the following equations:

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]\left[\left[1-\frac{(\epsilon^2-\alpha_0^2)}{(\epsilon^2+\alpha_0^2)^2}\frac{|\Delta|^2}{2}\right](\Gamma_1-\Gamma_2)\right]$$

$$=-\frac{4ief_0'(\epsilon)}{\tau}\frac{\partial}{\partial t}(D\nabla\cdot\mathbf{A}+\Phi)+2\left[\left[\frac{f_0'(\epsilon)}{\tau}\frac{\partial}{\partial t}-\tfrac{1}{2}(\Gamma_1-\Gamma_2)\right]\Delta\left[\frac{\hat{\alpha}}{\epsilon^2+\hat{\alpha}^2}\right]\Delta^*\right.$$

$$\left.-\left[\frac{f_0'(\epsilon)}{\tau}\frac{\partial}{\partial t}+\tfrac{1}{2}(\Gamma_1-\Gamma_2)\right]\Delta^*\left[\frac{\hat{\alpha}}{\epsilon^2+\hat{\alpha}^2}\right]\Delta\right]$$

$$+\frac{(\epsilon^2-\alpha_0^2)}{(\epsilon^2+\alpha_0^2)^2}\frac{\partial}{\partial t}\left[\frac{f_0(\epsilon)}{\tau}\left[\Delta^*\frac{\partial\Delta}{\partial t}-\Delta\frac{\partial\Delta^*}{\partial t}\right]-(\Gamma_1-\Gamma_2)|\Delta|^2\right]$$

$$+D\frac{\epsilon^2}{(\epsilon^2+\alpha_0^2)^2}\nabla\cdot\left[|\Delta|^2\left[-\frac{4ie}{\tau}f_0'(\epsilon)\frac{\partial\mathbf{A}}{\partial t}+\nabla(\Gamma_1-\Gamma_2)\right]\right]$$

$$-\frac{D}{2}\frac{(\epsilon^2-\alpha_0^2)}{(\epsilon^2+\alpha_0^2)^2}|\Delta|^2\nabla\cdot\left[-\frac{4ie}{\tau}f_0'(\epsilon)\frac{\partial\mathbf{A}}{\partial t}+\nabla(\Gamma_1-\Gamma_2)\right], \quad (6)$$

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right](\Gamma_1+\Gamma_2)=-\frac{2i}{\tau}\left[\frac{\epsilon}{\epsilon^2+\alpha_0^2}\right]f_0'(\epsilon)\frac{\partial|\Delta|^2}{\partial t}+\frac{i4e^2D}{\tau}f_0''(\epsilon)\left[\frac{\partial\mathbf{A}}{\partial t}\right]^2$$

$$-e\left[\frac{\partial}{\partial t}(D\nabla\cdot\mathbf{A}+\Phi)+2D\left[\frac{\partial\mathbf{A}}{\partial t}\right]\cdot\nabla\right]\frac{\partial(\Gamma_1-\Gamma_2)}{\partial\epsilon}. \quad (7)$$

Here $f_0(\epsilon)$ is the normal-state electronic distribution function, and f_0' and f_0'' are its first and second derivatives with respect to ϵ,

$$f_0(\epsilon)=\frac{1}{2}\left[1-\tanh\left[\frac{\epsilon}{2T}\right]\right],$$

$$f_0'(\epsilon)=-\frac{1}{4T}\cosh^{-2}\left[\frac{\epsilon}{2T}\right], \quad (8)$$

$$f_0''(\epsilon)=\frac{1}{4T^2}\tanh\left[\frac{\epsilon}{2T}\right]\cosh^{-2}\left[\frac{\epsilon}{2T}\right].$$

We obtain differential equations for Ψ and U_1 by multiplying Eq. (6) by $-(i\tau/2)\int d\epsilon$ and Eq. (7) by

$$(i\tau/4)\int\epsilon\, d\epsilon/(\epsilon^2+\alpha_0^2)$$

and then integrating

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right](2e\Psi+|\Delta|^2U_2')=2e\frac{\partial}{\partial t}(D\nabla\cdot\mathbf{A}+\Phi)$$
$$+2i\left[\left[\frac{\partial\Delta}{\partial t}\right]\frac{\psi'(\frac{1}{2}+\hat{\rho})}{4\pi T}\Delta^*-i\Delta\hat{U}_2\Delta^*-\left[\frac{\partial\Delta^*}{\partial t}\right]\frac{\psi'(\frac{1}{2}+\hat{\rho})}{4\pi T}\Delta-i\Delta^*\hat{U}_2\Delta\right]$$
$$+2i\frac{\partial}{\partial t}\left[\frac{\psi''(\frac{1}{2}+\rho_0)}{16\pi^2T^2}\left[\Delta^*\frac{\partial\Delta}{\partial t}-\Delta\frac{\partial\Delta^*}{\partial t}\right]-iU_2'|\Delta|^2\right]$$
$$+D\nabla\cdot\left\{|\Delta|^2\left[\frac{eg_1(\rho_0)}{2\pi^2T^2}\frac{\partial\mathbf{A}}{\partial t}-\nabla\left[\frac{U_2}{\alpha_0}+U_2'\right]\right]\right\}$$
$$-D|\Delta|^2\nabla\cdot\left[\frac{e\psi''(\frac{1}{2}+\rho_0)}{4\pi^2T^2}\frac{\partial\mathbf{A}}{\partial t}-\nabla U_2'\right], \quad (9)$$

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]U_1=-\frac{g_1(\rho_0)}{8\pi^2T^2}\frac{\partial|\Delta|^2}{\partial t}+\frac{e^2D\psi''(\frac{1}{2}+\rho_0)}{4\pi^2T^2}\left[\frac{\partial\mathbf{A}}{\partial t}\right]^2-e\left[\frac{\partial}{\partial t}(D\nabla\cdot\mathbf{A}+\Phi)+2D\frac{\partial\mathbf{A}}{\partial t}\cdot\nabla\right]U_2' . \quad (10)$$

With these equations we can now consider charge conservation. Using Eqs. (1)—(3), (9), and (10), we obtain directly

$$\nabla\cdot\mathbf{j}+\partial\rho/\partial t=-\left[\frac{\sigma}{2eD}\right]|\Delta|^2\left[\left[\frac{\partial}{\partial t}-D\nabla^2\right]U_2'+\frac{e}{4\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0)\frac{\partial}{\partial t}(D\nabla\cdot\mathbf{A}+\Phi)\right] . \quad (11)$$

Because of the explicit appearance of $|\Delta|^2$ in Eq. (11), we only need a differential equation for U_2' to zeroth order in Δ, i.e., in the normal state. Setting $\Delta=0$ in Eq. (6) and then multiplying by

$$(i\tau/4)\int\alpha_0 d\epsilon/(\epsilon^2+\alpha_0^2)$$

and integrating,

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]U_2=-\frac{e}{2\pi T}\psi'(\tfrac{1}{2}+\rho_0)\frac{\partial}{\partial t}(D\nabla\cdot\mathbf{A}+\Phi) . \quad (12)$$

Differentiating Eq. (12) with respect to α_0 gives

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]U_2'=-\frac{e}{4\pi T}\psi''(\tfrac{1}{2}+\rho_0)\frac{\partial}{\partial t}(D\nabla\cdot\mathbf{A}+\Phi) . \quad (13)$$

This result shows that the right-hand side of Eq. (11) is zero to the order we are working, $|\Delta|^2$.

In general, we would need a more accurate equation than (11) to determine U_2 in the superconducting state. However, applying the above integral to the terms of order $|\Delta|^2$ in Eq. (6) would generate some functions which we have not yet defined. To avoid an endless set of coupled equations one really would need to solve Eq. (6) for $\Gamma_1-\Gamma_2$ first and then integrate to get U_2. In the special case we consider in the next section, where the screening of the electric field is very weak, we need only the normal-state value of U_2. Comparing Eqs. (12) and (13) with Eq. (9) with $\Delta=0$ we find, in the normal state,

$$\hat{U}_2=-\frac{e\Psi}{2\pi T}\psi'(\tfrac{1}{2}+\hat{\rho}) ,$$
$$U_2'=-\frac{e\Psi}{4\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0) . \quad (14)$$

Finally, we want to remark that our equations are gauge invariant. The basic transformations are $\mathbf{A}\to\mathbf{A}-\nabla\chi$, $\Phi\to\Phi+\partial\chi/\partial t$, and $\Delta\to\Delta\exp(-i2e\chi)$. Since U_1, U_2, and U_2' are always multiplied by Δ, we only need their transformations in the normal state. Using Eqs. (10), (12), and (13),

$$\hat{U}_2\to\hat{U}_2-\frac{e}{2\pi T}\left[\frac{\partial\chi}{\partial t}\right]\psi'(\tfrac{1}{2}+\hat{\rho}) ,$$
$$U_2'\to U_2'-\frac{e}{4\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0)\frac{\partial\chi}{\partial t} , \quad (15)$$
$$U_1\to U_1-eU_2'\left[\frac{\partial\chi}{\partial t}\right]+\frac{e^2}{8\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0)\left[\frac{\partial\chi}{\partial t}\right]^2 .$$

Finally, we need Ψ not only in the normal state but to order $|\Delta|^2$ for $\mathbf{j}$ and ρ to be accurate to order $|\Delta|^2$. Using Eq. (9),

$$\Psi\to\Psi+\left[1+\frac{|\Delta|^2}{8\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0)\right]\frac{\partial\chi}{\partial t} . \quad (16)$$

Although Ψ has the same gauge transformation as Φ in the normal state, the transformations of Ψ and Φ are different to order $|\Delta|^2$. For further calculations it is more convenient to replace Ψ in Eqs. (2) and (3) by a potential Υ, which has simpler gauge properties,

$$\Upsilon=\Psi+|\Delta|^2U_2'/2e . \quad (17)$$

Then Υ has the same gauge transformation as Φ, and $e\Upsilon$ may be identified as the change in the electrochemical potential from the equilibrium Fermi energy.

With the above transformation relations, Eqs. (1)—(3) are verified to be invariant. Each of the terms in large square brackets in Eqs. (2) and (3) is separately gauge invariant. The invariance properties of Eq. (1) are most easily seen in the case of weak screening of the electric field. Then we can use Eq. (14) and group the terms into gauge-invariant combinations. Equation (1) linearized in Δ then reads

$$-[\ln(T/T_{c0})+\psi(\tfrac{1}{2}+\hat{\rho})-\psi(\tfrac{1}{2})]\Delta-\frac{1}{4\pi T}\left[\frac{\partial}{\partial t}+i2e\Upsilon\right][\psi'(\tfrac{1}{2}+\hat{\rho})\Delta]$$
$$-\frac{1}{32\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0)\left[\left[\frac{\partial}{\partial t}+i2e\Upsilon\right]^2+i2e\left[\frac{\partial\Upsilon}{\partial t}-\frac{\partial\Phi}{\partial t}\right]\right]\Delta+\left[U_1-\frac{e^2}{8\pi^2T^2}\psi''(\tfrac{1}{2}+\rho_0)\Upsilon^2\right]\Delta=0\ . \quad (18)$$

The contents of the first three sets of square brackets are easily seen to be invariant, whereas the invariance of the term in the last set of square brackets, which involves U_1, is now easy to verify using Eq. (15). Since U_1 is of second order in the potentials, it is important for nonlinear effects, which are discussed in Sec. IV. The terms of order $|\Delta|^2$ appearing on the right-hand sides of Eqs. (6), (7), (9), and (10) are all grouped into gauge-invariant combinations.

III. DYNAMIC RESPONSE, LINEAR IN ELECTRIC FIELD

To apply the TDGL equations derived in Sec. II to a specific example, we consider a plane type-II superconductor of thickness $2d$ located in the region $-d<z<d$. A uniform magnetic field $\mathbf{B}_0$ is applied perpendicular to the film in the $\hat{\mathbf{e}}_z$ direction. The magnetic field is close to the upper critical field H_{c2} determined by

$$\ln(T/T_{c0})+\psi(\tfrac{1}{2}+\rho_0)-\psi(\tfrac{1}{2})=0\ . \quad (19)$$

A lattice of current vortices is formed in the film. An electric field with average value $\mathbf{E}_0$ is applied in the plane of the film and causes the vortices to move in a direction perpendicular to both $\mathbf{E}_0$ and $\mathbf{B}_0$. The solution to the dynamic equations to first order in E is obtained following the procedure of Thompson and Hu.[11,12]

First we solve for the order parameter Δ. For type-II superconductors the electric and magnetic fields are weakly screened near H_{c2}. We can use Eq. (18) with the potentials for the normal-state fields and simplify the equation to keep only terms linear in E,

$$\left[\psi(\tfrac{1}{2}+\hat{\rho})-\psi(\tfrac{1}{2}+\rho_0)+\frac{1}{4\pi T}\psi'(\tfrac{1}{2}+\rho_0)\left[\frac{\partial}{\partial t}+i2e\Upsilon\right]\right]\Delta=0\ . \quad (20)$$

The solution is obtained in two stages. First we ignore the magnetic field generated by the transport current $\mathbf{j}_t=\sigma\mathbf{E}_0$ and obtain a two-dimensional solution Δ_2, which is independent of z in the metal. Then we find the z-dependent correction $\delta\Delta_3$, so $\Delta=\Delta_2+\delta\Delta_3$.

The operator $\hat{\rho}$ acting on Δ_2 gives $\rho_0\Delta_2$ plus a correction of order E, which involves the first-excited eigenfunction. To first order in E,

$$(\hat{\rho}-\rho_0)^n\Delta_2=(\rho_1-\rho_0)^{n-1}(\hat{\rho}-\rho_0)\Delta_2\ , \quad (21)$$

where $\rho_1=3\rho_0$. Expanding the first term in Eq. (20) in powers of $\hat{\rho}-\rho_0$, we get

$$\left[P^{-1}(Dq^2-\epsilon_0)+\left[\frac{\partial}{\partial t}+i2e\Upsilon\right]\right]\Delta_2=0\ ,$$
$$P^{-1}=\frac{\psi(\tfrac{1}{2}+\rho_1)-\psi(\tfrac{1}{2}+\rho_0)}{(\rho_1-\rho_0)\psi'(\tfrac{1}{2}+\rho_0)}\ . \quad (22)$$

The function P was introduced by Takayama and Ebisawa.[7] The solution of Δ_2 is just the static solution Δ_0 translating uniformly with velocity $\mathbf{v}=\mathbf{E}_0\times\mathbf{B}_0/B_0^2$ with no distortions:

$$\Delta_2(\mathbf{x},t)=\Delta_0(\mathbf{x}-\mathbf{v}t)=\exp[-eB_0(x-x_0-v_xt)^2+i2eB_0x_0(y-v_yt)]\ . \quad (23)$$

We have picked our gauge so the normal-state potentials are

$$\mathbf{A}_0=B_0(x-v_xt)\hat{\mathbf{e}}_y-P\mathbf{E}_0/2\epsilon_0\ ,$$
$$\Upsilon_0=v_yB_0(x-v_xt)\ . \quad (24)$$

All time dependence here and throughout our solution appears in the combination $\mathbf{x}-\mathbf{v}t$. Other choices of gauge may obscure the simplicity of the solution given in Eq. (23). Although P was called a polarization in Ref. 7, the order parameter Δ_2 is not polarized.

To find $\delta\Delta_3$ we must replace $\mathbf{A}_0$ by $\mathbf{A}_0+\mathbf{A}_t$ in Eq. (20). In the metal

$$\mathbf{A}_t=-2\pi\mathbf{j}_t(z^2+f_0)\ ,$$
$$\nabla\times\mathbf{A}_t=\mathbf{B}_t=-4\pi\mathbf{z}\times\mathbf{j}_t\ , \quad (25)$$

where f_0 is a constant. (Outside the metal

$\mathbf{B}_t=-4\pi d\hat{\mathbf{e}}_z\times\mathbf{j}_t\mathrm{sgn}(z)$ is a constant.) To linear order in E,

$$\delta\Delta_3=-f(z)8\pi e\,\mathbf{j}_t\cdot\mathbf{q}\Delta_2\ ,$$
$$\left[-\frac{\partial^2}{\partial z^2}+4eB_0\right]f(z)=z^2+f_0\ . \tag{26}$$

The boundary condition at the surfaces $z=\pm d$ is that $q_z\Delta=0$, which means that $df/dz=0$. Introducing the coherence length ξ by $2eB_0=\xi^{-2}$, the solution to Eq. (26) is

$$f(z)=\frac{\xi^2}{2}\left[f_0+\xi^2+z^2-\sqrt{2}d\xi\frac{\cosh(\sqrt{2}z/\xi)}{\sinh(\sqrt{2}d/\xi)}\right]. \tag{27}$$

We choose f_0 so that $f(\pm d)=0$. This choice simplifies the expression for the magnetic field in Eq. (44). The change in the magnitude $|\Delta|^2$ to linear order in E is

$$\delta|\Delta|^2=(\Delta_2^*)(\delta\Delta_3)+(\Delta_2)(\delta\Delta_3^*)$$
$$=-8\pi ef(z)(\mathbf{j}_t\times\hat{\mathbf{e}}_z)\cdot\nabla|\Delta_2|^2 \tag{28}$$

which is a translation or bending of the vortex lines to follow the lines of the total field $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_t$, except within a coherence length of the surface, which the vortices intersect perpendicularly. We require $B_t\ll B_0$ at the surface for the linearized solution to be valid.

Next we consider the current using Eq. (2). It is convenient to introduce a vector field $\mathscr{E}$ in the metal by

$$\mathscr{E}=-\nabla\Upsilon-\frac{\partial\mathbf{A}}{\partial t}\ . \tag{29}$$

$\mathscr{E}$ differs locally from the Maxwell field $\mathbf{E}$, but the average of $\mathscr{E}$ is the applied field $\mathbf{E}_0$. Using the operator properties of $\hat{\rho}$ as above, we get

$$\mathbf{j}=\sigma\mathscr{E}+\sigma\left[\frac{g_1(\rho_0)}{4\pi^2T^2}+\frac{\psi'(\frac{1}{2}+\rho_0)}{2\pi T\epsilon_0}\right]|\Delta_2|^2\mathbf{E}_0$$
$$+\mathbf{j}_{s0}+\left[\frac{\psi''(\frac{1}{2}+\rho_0)}{\psi'(\frac{1}{2}+\rho_0)}-\frac{(1-P)2\pi T}{\epsilon_0}\right]\frac{1}{4\pi T}\frac{\partial\mathbf{j}_{s0}}{\partial t}$$
$$+\frac{2\sigma}{T}\psi'(\tfrac{1}{2}+\rho_0)\nabla\times[\nabla\times\mathbf{j}_tf(z)|\Delta_2|^2]\ , \tag{30}$$

where $\mathbf{j}_{s0}$ is the current distribution of the equilibrium vortex lattice translating with velocity $\mathbf{v}$,

$$\mathbf{j}_{s0}=-\frac{\sigma}{4e\pi T}\psi'(\tfrac{1}{2}+\rho_0)\nabla\times(\hat{\mathbf{e}}_z|\Delta_2|^2)\ . \tag{31}$$

The spatial variation of $\mathscr{E}$ is determined below. However, if one is only interested in the average conductivity for flux flow, Eq. (30) can now be averaged over space, $\langle\ \rangle$,

$$\langle\mathbf{j}\rangle=\sigma'\mathbf{E}_0\ ,$$
$$\sigma'=\sigma\left[1+\left[\frac{g_1(\rho_0)}{4\pi^2T^2}+\frac{\psi'(\frac{1}{2}+\rho_0)}{2\pi T\epsilon_0}\right]\langle|\Delta|^2\rangle\right]. \tag{32}$$

Equation (32) agrees with the result of Ovchinnikov.[14] The value of $\langle|\Delta|^2\rangle$ is found from the static solution

$$\langle|\Delta|^2\rangle=\left[\frac{eT}{\sigma\psi'(\frac{1}{2}+\rho_0)}\right]\left[\frac{H_{c2}-B_0}{2\beta_A\kappa_2^2-\eta}\right], \tag{33}$$

where

$$\beta_A=\langle|\Delta|^4\rangle/\langle|\Delta|^2\rangle^2=1.16$$

and κ_2 is a generalized Ginzburg-Landau parameter:[15]

$$\kappa_2^2=\frac{[-\psi''(\frac{1}{2}+\rho_0)-\psi'''(\frac{1}{2}+\rho_0)/6\pi T\tau_s]}{16\pi\sigma D[\psi'(\frac{1}{2}+\rho_0)]^2}\ . \tag{34}$$

η is a function calculated by Lasher.[16] For thick films with $d\gg\xi$, $\eta=\beta_A-1=0.16$. For thin films η goes monotonically to zero.

For later economy of notation, we can generalize the dynamic screening length ζ defined in Ref. 11:

$$\left[\frac{\lambda_2}{\zeta}\right]^2=\frac{1}{4\pi\sigma D}\left[1+\frac{\epsilon_0g_1(\rho_0)}{2\pi T\psi'(\frac{1}{2}+\rho_0)}\right], \tag{35}$$

where $\lambda_2=\kappa_2\xi$. The ratio ξ/ζ has simple limits at $T=0$ and $T=T_{c0}$,

$$\left[\frac{\xi}{\zeta}\right]^2=\begin{cases}12\left[1-\dfrac{2\epsilon_0}{3\epsilon_0+2/\tau_s}\right] & \text{at } T=0,\\[2ex] 5.8\left[1+\dfrac{\epsilon_0}{\epsilon_0+2/\tau_s}\right] & \text{at } T=T_{c0}\ .\end{cases} \tag{36}$$

Overall ζ has a rather limited range of values, $0.29\xi\le\zeta\le0.50\xi$. In terms of ζ the normalized slope of the conductivity may be written

$$\frac{H_{c2}}{\sigma}\left[\frac{d\sigma'}{dB_0}\right]_{H_{c2}}=-\left[\frac{\xi}{\zeta}\right]^2\frac{\kappa_2^2}{2\beta_A\kappa_2^2-\eta}\ . \tag{37}$$

To find the spatial variation of $\mathscr{E}$ and $\mathbf{j}$ we develop a differential equation for Υ. We start with $\nabla\cdot\mathbf{j}=0$ because $\partial\rho/\partial t$ is of order E^2 and use Eq. (30) for $\mathbf{j}$. The time dependence of $\mathbf{A}$ can be rewritten $\partial\mathbf{A}/\partial t=-(\mathbf{v}\cdot\nabla)\mathbf{A}$. Using a vector identity

$$\nabla^2(\mathbf{v}\cdot\mathbf{A})=(\mathbf{v}\cdot\nabla)(\nabla\cdot\mathbf{A})-\mathbf{v}\cdot(\nabla\times\mathbf{B})$$

and substituting $\nabla\times\mathbf{B}=4\pi\mathbf{j}$, we get, to first order in E,

$$\nabla^2(\Upsilon-\mathbf{v}\cdot\mathbf{A})=[1-(\lambda_2/\zeta)^2]4\pi\mathbf{v}\cdot\mathbf{j}_{s0}\ . \tag{38}$$

The solution to Eq. (38) is most easily obtained by using Fourier transforms in the variables x and y. Any function F which has the periodicity of the vortex lattice can be written

$$F(x,y,z)=\sum_k(F)_{k,z}\exp(i\mathbf{k}\cdot\mathbf{x})\ , \tag{39}$$

where $\mathbf{k}$ is the set of two-dimensional reciprocal-lattice vectors for the vortex lattice in the x-y plane.[17] A two-dimensional solution to Eq. (38) is then obtained immediately since the two-dimensional part of the operator ∇^2 is transformed into a number $-k^2$. To the two-dimensional solution $(\Upsilon-\mathbf{v}\cdot\mathbf{A})_2$ can be added any three-dimensional function $(\delta\Upsilon)_3$ which satisfies $\nabla^2(\delta\Upsilon)_3=0$,

$$(\delta\Upsilon)_3=\sum_k [C_k\exp(kz)+D_k\exp(-kz)] \times\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)] , \tag{40}$$

where the factor k multiplying the z coordinate is the magnitude of the vector $\mathbf{k}$. The boundary condition which determines the coefficients C_k and D_k is that $j_z=0$ at the metal surfaces $z=\pm d$. To find $\mathbf{j}$ from Eq. (30), we substitute $\mathscr{E}$ from

$$\begin{aligned}\mathscr{E}&=-\nabla\Upsilon+(\mathbf{v}\cdot\nabla)\mathbf{A}\\&=-\nabla(\Upsilon-\mathbf{v}\cdot\mathbf{A})-\mathbf{v}\times\mathbf{B} .\end{aligned} \tag{41}$$

In the last term of Eq. (41), we only need the magnetic field to zeroth order in E, $\mathbf{B}_{s0}$. $\mathbf{B}_{s0}$ is the function obtained previously for the static lattice by Lasher.[16] To $\mathbf{B}_{s2}$, the two-dimensional solution of Eq. (31) and $\nabla\times\mathbf{B}_{s0}=4\pi\mathbf{j}_{s0}$, is added the gradient of a scalar of the form of Eq. (40) to satisfy the boundary condition that $\mathbf{B}$ is continuous at $z=\pm d$:

$$\mathbf{B}_{s0}=\mathbf{B}_0+(\delta B_{s2})\hat{\mathbf{e}}_z\Theta(d^2-z^2)-\sum_{k\,(\neq 0)}\frac{(\delta B_{s2})_k}{2k}\nabla\{[\exp(-k\,|z-d\,|)-\exp(-k\,|z+d\,|)]\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)]\} , \tag{42}$$

where $\delta B_{s2}=-(\sigma/eT)\psi'(\frac{1}{2}+\rho_0)\delta\,|\,\Delta_2\,|^2$, $\delta\,|\,\Delta_2\,|^2=|\,\Delta_2\,|^2-\langle\,|\,\Delta\,|^2\rangle$, and Θ is the step function. Finally, we obtain the result

$$\mathscr{E}=-\mathbf{v}\times\mathbf{B}_{s0}-\nabla\sum_{k\,(\neq 0)}\frac{4\pi\mathbf{v}\cdot(\mathbf{j}_{s0})_k}{k^2}\left[\left[\frac{\lambda_2}{\xi}\right]^2-1+\exp(-kd)\cosh(kz)\right]\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)] . \tag{43}$$

To obtain an explicit expression, one substitutes $4\pi\mathbf{v}\cdot\mathbf{j}_{s0}=\mathbf{E}_0\cdot\nabla(\delta B_{s2})/B_0$ with δB_{s2} given above.

Substituting Eq. (43) into Eq. (30), the resulting expression for $\mathbf{j}$ is divergenceless and can be rewritten $4\pi\mathbf{j}=\nabla\times\mathbf{B}$.

The solution for $\mathbf{B}$ is obtained as above by erasing the curl operator and adding the gradient of a scalar of the form of Eq. (40):

$$\begin{aligned}\mathbf{B}=&\mathbf{B}_{s0}-2\pi\hat{\mathbf{e}}_z\times(\sigma'\mathbf{E}_0)(\,|z+d\,|-|z-d\,|\,)\\&-\sum_{k\,(\neq 0)}(\delta B_{s2})_k(i\mathbf{k}\cdot\mathbf{v})\left\{\frac{4\pi\sigma}{k^2}\left[\left[\frac{\lambda_2}{\xi}\right]^2-\exp(-kd)\sinh(kd)\right]+\left[\frac{\psi''(\frac{1}{2}+\rho_0)}{4\pi T\psi'(\frac{1}{2}+\rho_0)}-\frac{1-P}{2\epsilon_0}\right]\right\}\\&\times\left[\hat{\mathbf{e}}_z\Theta(d^2-z^2)\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)]-\frac{1}{2k}\nabla\{[\exp(-k\,|z-d\,|)-\exp(-k\,|z+d\,|)]\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)]\}\right]\\&-\sum_{k\,(\neq 0)}(\delta B_{s2})_k(i\mathbf{k}\cdot\mathbf{v})\left[\frac{4\pi\sigma}{k^2}\right]\exp(-kd)[\cosh(kz)-\cosh(kd)]\hat{e}_z\Theta(d^2-z^2)-8\pi e\nabla\times[\mathbf{j}_t f(z)\delta B_{s2}]\Theta(d^2-z^2) .\end{aligned} \tag{44}$$

The first term on the right-hand side of Eq. (44) is given in Eq. (42). The second term is the field generated by the average transport current, which is $\mathbf{j}_t=\sigma'\mathbf{E}_0$ in the superconducting state. The two-dimensional part of the third term was called the backflow field in Ref. 11 and illustrated in Ref. 17. Most of the weight of the Fourier sum is included in the set of the six shortest $\mathbf{k}\neq\mathbf{0}$. The fourth term is a new contribution resulting from the three-dimensional contributions in Eq. (43). The last term, which includes the bending of the vortex lines, was discussed and illustrated in Ref. 12.

We can now derive a differential equation for the charge density ρ by applying the operator ∇^2 to Eq. (3) and using $\nabla\cdot\mathbf{E}=4\pi\rho$ and Eq. (43):

$$\left[1-\frac{D\nabla^2}{4\pi\sigma}\right]\rho=\left[\left[\frac{\lambda_2}{\xi}\right]^2-\frac{\psi''(\frac{1}{2}+\rho_0)D\nabla^2}{8\pi T\psi'(\frac{1}{2}+\rho_0)}\right]\mathbf{v}\cdot\mathbf{j}_{s0} . \tag{45}$$

The quantity $D/4\pi\sigma=\lambda_{\text{TF}}^2$, where λ_{TF} is the Thomas-Fermi screening length. ρ_2, the two-dimensional solution to Eq. (45), is

$$(\rho_2)_k = \left[\left[\frac{\lambda_2}{\xi} \right]^2 + \frac{\psi''(\frac{1}{2}+\rho_0)Dk^2}{8\pi T\psi'(\frac{1}{2}+\rho_0)} \right] \frac{\mathbf{v}\cdot(\mathbf{j}_{s0})_k}{1+\lambda_{\rm TF}^2 k^2} . \quad (46)$$

To ρ_2 may be added any three-dimensional function $(\delta\rho)_3$ which satisfies $(1-\lambda_{\rm TF}^2\nabla^2)(\delta\rho)_3=0$:

$$(\delta\rho)_3 = \sum_k [F_k \exp(z/\Lambda) + G_k \exp(-z/\Lambda)] \times \exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)] , \quad (47)$$

where $\Lambda=(\lambda_{\rm TF}^{-2}+k^2)^{-1/2}$. Physically, $\lambda_{\rm TF} \ll \xi$ and $\Lambda \approx \lambda_{\rm TF}$, so the contributions in Eq. (47) are surface charges at $z=\pm d$. The coefficients F_k and G_k are determined by the boundary condition on the electric field $\mathbf{E}$.

Inside the metal, $\mathbf{E}$ is found from our previous results using Eq. (3),

$$\mathbf{E} = \mathscr{E} + \nabla(\Upsilon - \Phi) = \mathscr{E} + \frac{D\nabla\rho}{\sigma} + \frac{\psi''(\frac{1}{2}+\rho_0)D\nabla(\mathbf{E}_0\cdot\nabla\,|\Delta_2|^2)}{16\pi^2 T^2 \epsilon_0} . \quad (48)$$

We note that $\mathbf{E}=-\nabla(\Phi-\mathbf{v}\cdot\mathbf{A})-\mathbf{v}\times\mathbf{B}$ and $\nabla^2(\Phi-\mathbf{v}\cdot\mathbf{A})=4\pi(\mathbf{v}\cdot\mathbf{j}-\rho)$. Outside the metal the last equation equals zero, so $\Phi-\mathbf{v}\cdot\mathbf{A}$ has the form of Eq. (40). Making $\mathbf{E}$ continuous at $z=\pm d$ determines the coefficients.

Our final results are

$$\rho = \sum_{k\,(\neq 0)} (\rho_2)_k \left[1 + \frac{1}{\lambda_{\rm TF}^2 k^2} \frac{\cosh(z/\Lambda)}{\cosh(d/\Lambda)+(k\Lambda)^{-1}\sinh(d/\Lambda)} \right] \exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)] . \quad (49)$$

Inside the metal,

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B}_{s0} + \nabla \sum_{k\,(\neq 0)} \frac{4\pi}{k^2} \left[(\mathbf{v}\cdot\mathbf{j}_{s0}-\rho_2)_k + (\rho_2)_k \frac{\cosh(z/\Lambda)}{\cosh(d/\Lambda)+(k\Lambda)^{-1}\sinh(d/\Lambda)} - \mathbf{v}\cdot(\mathbf{j}_{s0})_k \exp(-kd)\cosh(kz) \right] \exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)] . \quad (50)$$

Outside the metal,

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B}_{s0} + \nabla \sum_{k\,(\neq 0)} \frac{4\pi}{k^2} \left[\mathbf{v}\cdot(\mathbf{j}_{s0})_k \sinh(kd) - \frac{(\rho_2)_k \exp(kd)}{1+k\Lambda\coth(d/\Lambda)} \right] \exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t) - k\,|z|\,] . \quad (51)$$

IV. DYNAMIC RESPONSE, NONLINEAR IN ELECTRIC FIELD

In the preceding section we present a complete dynamic solution to first order in the electric field for a superconducting film in a perpendicular magnetic field. Now we want to consider stronger electric fields which give rise to nonlinear effects, including destruction of the superconductivity. These effects have been considered previously by Thompson and Hu[18] and by Larkin and Ovchinnikov.[13] However, in the work of Thompson and Hu only pair-breaking effects were considered. Larkin and Ovchinnikov showed that energy-relaxation effects are usually more important. We can now show how both types of nonlinear effects appear in the Gor'kov-Eliashberg formalism. We also can relate the vertex functions to the electron distribution function in the normal state by using the Boltzmann equation.

We do not obtain a complete solution for the system as in Sec. III, but limit ourselves to consideration of the average conductivity $\sigma'(E)=\mathbf{j}_t/\mathbf{E}_0$ as in Refs. 13 and 18. The main effect of nonlinearity is a reduction in the magnitude of the order parameter Δ. The part of Eq. (1) which is linear in Δ, Eq. (6), and the normal-state part of Eq. (7), are, in fact, accurate to order E^2. Equation (2) for $\mathbf{j}$ is not accurate to order E^3. However, the direct E^3 corrections to $\mathbf{j}$ are not important, since they are to be compared with unity relative to the corrections to Δ which are compared with small quantities of order $|\Delta|^2$ or $H_{c2}-B_0$.

We consider sufficiently weak screening so that we can use the normal-state values of the fields to find the average conductivity. For the magnetic field this requires $\kappa_2 \gg 0.26\eta^{1/2}$. The terms linear in Δ in Eq. (1) are simplified as in Eq. (18) and compared with the term of order Δ^3 in Eq. (1) to determine $|\Delta|^2$. The contributions of the first three square-bracketed terms in Eq. (18) are identified as pair-breaking effects due to the motion of the vortex lattice. The contents of the last set of square brackets are unusual and are related to the escape from the electronic system of the energy absorbed from the field by dissipation.

To understand the quantities U_1 and Υ appearing in Eq. (18), which are integrals of the vertex functions Γ_1 and Γ_2, it is useful to compare Eqs. (6) and (7) with the Boltzmann kinetic equation for the electron distribution function $\widetilde{f}(\mathbf{x},t,\epsilon,\hat{\mathbf{n}})$. ϵ is the electron energy, and $\hat{\mathbf{n}}$ is a unit vector in the direction of its motion. In the normal state,

$$\frac{\partial\tilde{f}}{\partial t}+v_F\hat{\mathbf{n}}\cdot\nabla\tilde{f}+ev_F\hat{\mathbf{n}}\cdot\mathbf{E}\frac{\partial\tilde{f}}{\partial\epsilon}=\frac{1}{\tau}\left[-\tilde{f}+\frac{1}{4\pi}\int d\Omega\tilde{f}\right],$$

$$\mathbf{j}=2N(0)ev_F\int d\epsilon\, d\Omega\,\hat{\mathbf{n}}\tilde{f}, \tag{52}$$

$$\rho=2N(0)e\int d\epsilon\, d\Omega(\tilde{f}-f_0),$$

where $N(0)$ is the density of states per spin at the Fermi energy, and v_F is the Fermi velocity. In the dirty limit for slow variations, where $v_F\tau\nabla$ and $\tau\partial/\partial t$ are small, a solution can be constructed as

$$\tilde{f}=f-\tau v_F\hat{\mathbf{n}}\cdot(e\mathbf{E}\partial f/\partial\epsilon+\nabla f),$$

where f is isotropic.[19] To order E^2,

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]f=eD\frac{\partial f}{\partial\epsilon}\nabla\cdot\mathbf{E}+2eD\mathbf{E}\cdot\nabla\frac{\partial f}{\partial\epsilon}+e^2D\mathbf{E}^2\frac{\partial^2 f}{\partial\epsilon^2}. \tag{53}$$

Expanding $f=f_0+f_1+f_2$, where f_i is of order E^i,

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]f_1=eDf_0'\nabla\cdot\mathbf{E},$$

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]f_2=eD[(\nabla\cdot\mathbf{E})+2\mathbf{E}\cdot\nabla]\left[\frac{\partial f_1}{\partial\epsilon}\right]+e^2D\mathbf{E}^2f_0'', \tag{54}$$

where f_0 and its derivatives are the same as in Eq. (8). Noting that f_1 is even and f_2 is odd in ϵ and comparing Eq. (54) with Eqs. (6) and (7) in the normal state with $\Delta=0$, we find that the equations are the same if the following identifications are made:

$$\Gamma_1-\Gamma_2=\frac{i4}{\tau}(f_1-ef_0'\Phi),$$

$$\Gamma_1+\Gamma_2=\frac{i4}{\tau}\left[f_2-e\Phi\frac{\partial f_1}{\partial\epsilon}+\frac{(e\Phi)^2}{2}f_0''\right]. \tag{55}$$

These equations give precise physical meaning to Γ_1 and Γ_2 in the normal state. Also, the gauge transformations presented in Eq. (15) can be reverified, noting that the f_i are gauge invariant.

Integration of the first of Eqs. (55) with respect to ϵ to form $\Psi=\Upsilon$ gives Eq. (3) in the normal state for the charge density $\rho=2N(0)e\int f_1 d\epsilon$. $e\Psi=e\Upsilon$ is therefore the change in the electrochemical potential from its equilibrium value.

To interpret U_1, it is helpful to introduce a quantity U_0 which only depends on normal-state parameters,

$$U_0=-\frac{i\tau}{4}\int(\Gamma_1+\Gamma_2)\epsilon\, d\epsilon. \tag{56}$$

Integrating Eq. (7) in the normal state with respect to $\epsilon\, d\epsilon$, the resulting equation for U_0 is the same as Eq. (10) for U_1 in the normal state with

$$U_1=\frac{\psi''(\frac{1}{2}+\rho_0)}{4\pi^2T^2}U_0. \tag{57}$$

The combination appearing in the last pair of square brackets of Eq. (18) is then proportional to $U_0-(e\Upsilon)^2/2$, which, using Eq. (55), is

$$U_0-\frac{(e\Upsilon)^2}{2}=\int f_2\epsilon\, d\epsilon-\frac{e^2}{2}(\Upsilon-\Phi)^2. \tag{58}$$

The differential equation for this combination is

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]\left[U_0-\frac{(e\Upsilon)^2}{2}\right]=D(e\mathscr{E})^2. \tag{59}$$

The right-hand side of Eq. (59) is proportional to the rate of heat dissipation $\sigma\mathscr{E}^2$, and the second factor on the left-hand side in large parentheses is apparently growing linearly in time proportional to the heat input. This situation arises because we did not provide any mechanism in our equations for heat to leave the electronic system. Phenomenologically, we may replace the diffusion operator on the left-hand side of Eq. (59) by $(\partial/\partial t-D\nabla^2+\tau_\epsilon^{-1})$, where τ_ϵ is a time for energy escape from the electronic system. Since τ_ϵ is very long at low temperatures, $\tau_\epsilon\approx\Theta_D^2/T^3$ for electron-phonon processes, U_1 is unusually large with a new scale factor which does not appear in the other terms. Θ_D is the Debye temperature.

If the electrons were in thermal equilibrium at a slightly elevated temperature $T+\delta T$, one could expect heating effects to appear by the addition to the first term in square brackets in Eq. (18) of

$$\delta T\frac{\partial}{\partial T}[-\ln T+\psi(\tfrac{1}{2}+\rho_0)]=-\frac{\delta T}{T}[1-\rho_0\psi'(\tfrac{1}{2}+\rho_0)], \tag{60}$$

with $\delta T=\tau_\epsilon\sigma\mathscr{E}^2/C$, where the electronic specific heat $C=(2\pi^2/3)N(0)T$. If we set the last factor in square brackets in Eq. (18) equal to Eq. (60), but with δT replaced by an effective temperature shift δT^*, the ratio is

$$\frac{\delta T^*}{\delta T}=-\frac{\psi''(\frac{1}{2}+\rho_0)}{12[1-\rho_0\psi'(\frac{1}{2}+\rho_0)]}. \tag{61}$$

This ratio goes to 1 at $T=0$ and to 1.4 at $T=T_{c0}$. The two temperature shifts are different because we have not included electron-electron interactions which would allow the electrons to thermalize with respect to each other. Our f_2 is proportional to f_0'', whereas a change in temperature gives a correction to f proportional to $\epsilon f_0'$. If the characteristic time for electron-electron interactions became shorter than τ_ϵ, we would expect δT^* to equal δT also near T_{c0}.

To calculate the nonlinear pair-breaking terms in Eq. (18), one may solve the differential equations to order E^2 as before. Fortunately though, the pair-breaking terms are only significant relative to the U_1 term for small magnetic fields B_0. For small magnetic fields we only need to expand the ψ function to first order in Dq^2, so the operator ordering and higher eigenvalue problems do not arise. The nonlinear value of the simple operator Dq^2 was found by Caroli and Maki,[6]

$$Dq^2=D[2eB_0+(E_0/2DB_0)^2].$$

The combination of the two terms gives us an equation for the conductivity of the same form as obtained by Thompson and Hu:

$$\sigma'=\sigma\left\{1+\frac{1}{2\beta_A}\left[\frac{\xi}{\zeta}\right]^2\left[\frac{H_{c2}}{B_0}-1-\left[\frac{E_0}{E_c}\right]^2\right]\right\}. \quad (62)$$

However, the characteristic electric field E_c is now given by

$$E_c^{-2}=\frac{1}{2eB_0}\left[\frac{1}{2DB_0}\right]^2+\frac{\psi''(\frac{1}{2}+\rho_0)}{\pi T\psi'(\frac{1}{2}+\rho_0)}\frac{e\tau_\epsilon}{B_0}. \quad (63)$$

Only the first term on the right-hand side of Eq. (63) was included in Ref. (18). From the ratio of the two contributions to E_c, we see that the pair-breaking term dominates only for weak magnetic fields,

$$2eDB_0<(T_{c0}/\tau_\epsilon)^{1/2}\approx T^{3/2}T_{c0}^{1/2}/\Theta_D .$$

This condition becomes especially hard to satisfy for very low temperatures $T \ll T_{c0}$. Nevertheless, if the corrected value of E_c from Eq. (63) is substituted in the parameter $\epsilon=E/E_c$, the results illustrated in Fig. 1 of Ref. 18 remain qualitatively the same. The j_t-versus-E_0 curve can be continuous and single-valued only if

$$B_0/H_{c2}\geqq(1+\zeta^2\beta_A/\xi^2)^{-1} .$$

For smaller values of B_0/H_{c2} the slope $\partial j_t/\partial E_0$ is negative at $\Delta=0$, and a discontinuous jump in the electric field is predicted at the transition to the normal state from the superconducting state carrying the maximum current, which occurs for $\Delta\neq0$. Owing to the range of variation of ζ/ξ, this critical field lies in the range $0.78\leqq B_0/H_{c2}\leqq0.91$. The corresponding critical values of E_0 lie in the range $0.27\leqq E_0/E_c\leqq0.47$.

Larkin and Ovchinnikov also evaluated an additional nonlinear term, which corresponds to a contribution to U_1 of order $|\Delta|^2E_0^2$. If this nonlinear effect were simply due to heating, one would only need to adjust the heating rate from $\sigma\mathscr{E}^2$ to $\sigma'\mathscr{E}^2$. However, the contribution they found, $\approx(\sigma'-\sigma)\mathscr{E}^2(T_{c0}/\alpha_0)^2$, is much larger for weak pair breaking. They obtained a formula which can be written

$$\sigma'=\sigma\left\{1+\frac{1}{2\beta_A}\left[\frac{\xi}{\zeta}\right]^2\left[\frac{H_{c2}}{B_0}-1-\left[\frac{E_0}{E_c}\right]^2\right]\Big/\left[1+\left[\frac{E_0}{E^*}\right]^2\right]\right\}. \quad (64)$$

The new characteristic field E^* is related to E_c by the parameter $\gamma=(E_c/E^*)^2$. For weak pair breaking near T_{c0}, they found $\gamma=1.46(T_{c0}/\alpha_0)^2$. The heating model, which may be applied for strong pair breaking near $T=0$, gives $\gamma=\xi^2/2\zeta^2\beta_A$.

For strong pair breaking when $\gamma\leqq3\xi^2/4\zeta^2\beta_A$ our previous results are only slightly modified. Single-valued j_t-versus-E_0 curves are obtained only if

$$B_0/H_{c2}\geqq\{1+[(\xi^2/\zeta^2\beta_A)-\gamma]^{-1}\}^{-1} .$$

For smaller B_0 the curves have a maximum for $\Delta\neq0$ and a minimum at $\Delta=0$. When $\gamma=3\xi^2/4\zeta^2\beta_A$, both $\partial j_t/\partial E_0$ and $\partial^2 j_t/\partial E_0^2$ vanish at $\Delta=0$ for $B/H_{c2}=(1+4\zeta^2\beta_A/\xi^2)^{-1}$. Then for weaker pair breaking when $\gamma>3\xi^2/4\zeta^2\beta_A$, a new type of curve is obtained with both a maximum and a minimum in j_t versus E_0 occurring for $\Delta\neq0$ when $B<B^*$. Unfortunately, one cannot determine the new characteristic field B^* using only information about the conductivity near $\Delta=0$.

V. CONCLUSION

We have developed a set of TDGL equations for gapless superconductors in the dirty limit, which for the first time include all dynamic effects to the first order in an electric field for pair breaking by a magnetic field and by magnetic impurities in an arbitrary ratio. These equations are especially useful for deriving a complete description of all the vortex deformations and field-screening effects which occur when the vortex lattice in a type-II superconductor moves.

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