Quasiparticle motion in superfluid ^{3}He and Kapitza resistance of ^{3}He A-B phase boundary

S. Yip
Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801
(Received 19 February 1985)

In this paper we investigate the motion of quasiparticles in superfluid ^{3}He. The Bogoliubov—de Gennes equation is generalized to an arbitrary type of pairing. For the case of a unitary p-wave state, the ballistic description of Greaves and Leggett is shown to follow when the spatial variation of the order parameter is slow on the scale of a coherence length. We also investigate the equation of motion of the excitation spin in this case. We apply our equations to investigate the transport of energy across the ^{3}He A-B phase boundary, where in particular we show that a Kapitza resistance exists. The feasibility of measuring this resistance is discussed.

I. INTRODUCTION

The motion of quasiparticle excitations in a superfluid has already been discussed in s-wave singlet pairing by Andreev.[1] The case of p-wave triplet pairing appropriate to superfluid ^{3}He has been discussed by Greaves and Leggett[2] when the textural variation is slow enough that ballistics applies. More recently, Kieselmann and Rainer study this more generally, using the quasiclassical approach.[3] This approach, though powerful, is much more difficult and less transparent than the work by Andreev. In this paper, we shall generalize the "Bogoliubov—de Gennes" equations used by Andreev to an arbitrary type of pairing, though we shall quickly specialize to superfluid ^{3}He. It is then shown that the ballistic description of Ref. 2 follows if the spatial variation of the order parameter is slow. We shall also investigate the question of branch conversion, in particular the equation of motion for the excitation spin, in which we obtain results contradicting Ref. 3.

It is known that the two superfluid phases A and B of ^{3}He coexist at a particular temperature T_{AB}, which is a function of the pressure and applied magnetic field.[4] We shall apply our genrealized equation of motion to the phase boundary. In particular we investigate the transport of energy when there is a temperature difference between the two phases. We show that owing to the "Andreev reflection" of excitations by the variation of the order parameter, a Kapitza type of resistance arises. This resistance is calculated at low temperatures assuming the order parameter of Kaul and Kleinert.[5] We finally discuss the experimental feasibility of measuring this resistance, and show in particular that the arrangement of Osheroff and Cross[6] is a potential candidate.

II. GENERALIZED BOGOLIUBOV—DE GENNES EQUATION

In the first part of this section we shall generalize the Bogoliubov—de Gennes equation to arbitrary pairing of a superfluid. Our derivation parallels the evaluation of the quasiparticle energy by Balian and Werthamer in momentum space:[7] We shall transform their calculation back to relative space $\rho\equiv r_1-r_2$ while taking appropriately the dependence on the center-of-mass coordinate $r=(r_1+r_2)/2$ into account (see also Ref. 1).

First, we consider a uniform system. We take a unit volume and simplify our notation for spatial integration by putting the dummy variable as a subscript on the integral sign. Recall first that, in a uniform system, the gap matrix $\Delta^{\mathbf{k}}_{\alpha\beta}$ is related to the order parameter $F^{\mathbf{k}}_{\alpha\beta}\equiv\langle a_{\mathbf{k}\alpha}a_{-\mathbf{k}\beta}\rangle$ by

$$\Delta^{\mathbf{k}}_{\alpha\beta}=-\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}F^{\mathbf{k}'}_{\alpha\beta}\ ,$$

where $V_{\mathbf{k}\mathbf{k}'}$ is the Fourier-transformed pair-interaction potential $V(\boldsymbol{\rho})$, $a_{\mathbf{k}\alpha}$ annihilates a normal-fluid excitation of momentum $\mathbf{k}$, spin α, and the angular brackets denote an average over the ground state (or, more generally, the thermal average). On transforming back to relative coordinate space $\boldsymbol{\rho}\equiv\mathbf{r}_1-\mathbf{r}_2$ (recall that we are in a uniform system), we get

$$F_{\alpha\beta}(\boldsymbol{\rho})\equiv\sum_{\mathbf{k}'}F^{\mathbf{k}'}_{\alpha\beta}e^{i\mathbf{k}'\cdot\boldsymbol{\rho}}=\langle\psi_\alpha(\mathbf{r}_1)\psi_\beta(\mathbf{r}_2)\rangle\ ,$$

$$\Delta_{\alpha\beta}(\boldsymbol{\rho})\equiv\sum_{\mathbf{k}}\Delta^{\mathbf{k}}_{\alpha\beta}e^{i\mathbf{k}\cdot\boldsymbol{\rho}}=-V(\boldsymbol{\rho})F_{\alpha\beta}(\boldsymbol{\rho})\ ,$$

where $\psi_\alpha(\mathbf{r}_1)$ annihilates a normal-fluid excitation of spin α at $\mathbf{r}_1$.

In the case of a non-uniform system, then, we have to give the dependence on the center-of-mass coordinate as well as on the relative coordinate $\boldsymbol{\rho}$. Thus the order parameter at $\mathbf{r}=(\mathbf{r}_1+\mathbf{r}_2)/2$ for relative distance $\boldsymbol{\rho}$ is

$$F_{\alpha\beta}(\mathbf{r},\boldsymbol{\rho})\equiv\langle\psi_\alpha(\mathbf{r}_1)\psi_\beta(\mathbf{r}_2)\rangle\ , \tag{1}$$

and the gap is therefore

$$\Delta_{\alpha\beta}(\mathbf{r},\boldsymbol{\rho})=-V(\boldsymbol{\rho})F_{\alpha\beta}(\mathbf{r},\boldsymbol{\rho})\ . \tag{2}$$

We note that generally the "gap" is a nonlocal object.

We next consider the motion of an excitation. Let $|\Phi_0\rangle$, $|\Phi_1\rangle$ denote the superfluid ground state and the state with an excitation in which we are interested, respectively. The excitation is uniquely specified by the values

$$f_\alpha(\mathbf{x},t)\equiv\langle\Phi_0|\psi_\alpha(\mathbf{x},t)|\Phi_1\rangle\ ,$$
$$g_\alpha(\mathbf{x},t)\equiv\langle\Phi_0|\psi_\alpha^\dagger(\mathbf{x},t)|\Phi_1\rangle\ .$$

The equation of motion for $f_\alpha(\mathbf{x})$ can be obtained by using

$$i\hbar\frac{\partial}{\partial t}\psi_\alpha(\mathbf{x})=[\psi_\alpha(\mathbf{x}),H]\ , \tag{3}$$

with the reduced Hamiltonian H given by

$$H=\int_{\mathbf{x}}\psi_\alpha^\dagger(\mathbf{x})\left[-\frac{\hbar^2}{2m}\nabla^2-\mu\right]\psi_\alpha(\mathbf{x})+\tfrac{1}{2}\int_{\mathbf{x},\mathbf{x}'}\psi_\alpha^\dagger(\mathbf{x})\psi_\beta^\dagger(\mathbf{x}')V(\mathbf{x}-\mathbf{x}')\psi_\beta(\mathbf{x}')\psi_\alpha(\mathbf{x})\ , \tag{4}$$

where m is the effective mass, and by also using the usual mean-field-theory results

$$i\hbar\frac{\partial}{\partial t}f_\alpha(\mathbf{x})=\left[-\frac{\hbar^2}{2m}\nabla_{\mathbf{x}}^2-\mu\right]f_\alpha(\mathbf{x})+\int_{\rho}\Delta_{\alpha\beta}\left[\mathbf{x}-\frac{\rho}{2},\rho\right]g_\beta(\mathbf{x}-\rho)\ , \tag{5}$$

$$i\hbar\frac{\partial}{\partial t}g_\alpha(\mathbf{x})=\left[\frac{\hbar^2}{2m}\nabla_{\mathbf{x}}^2+\mu\right]g_\alpha(\mathbf{x})-\int_{\rho}\Delta_{\alpha\beta}^*\left[\mathbf{x}-\frac{\rho}{2},\rho\right]f_\beta(\mathbf{x}-\rho)\ . \tag{6}$$

Usually the gap varies slowly on the scale of its range (this assumption will be justified below); we can then expand the gap in a Taylor series in the c.m. variable, keeping only the first two terms. We do the same thing for $f_\beta(\mathbf{x}-\rho)$ and $g_\beta(\mathbf{x}-\rho)$, but now do not restrict the number of expansion terms (see below) and obtain the last term of (5) as

$$\int_\rho[\Delta_{\alpha\beta}(\mathbf{x},\rho)-\frac{\rho_\nu}{2}\partial_\nu(\Delta_{\alpha\beta}(\mathbf{x},\rho))]\times\left[g_\beta(\mathbf{x})-\rho_\mu\partial_\mu g_\beta(\mathbf{x})+\frac{\rho_\mu\rho_\lambda}{2!}\partial_\mu\partial_\lambda g_\beta(\mathbf{x})+\cdots\right]\ , \tag{7}$$

where $\partial_\mu\equiv\partial/\partial x^\mu$. We define, with n the number of $\mu,\nu,\ldots,\lambda$,

$$(D_n)_{\mu\nu\cdots\lambda,\alpha\beta}(\mathbf{x})\equiv(-1)^n\int_\rho\Delta_{\alpha\beta}(\mathbf{x},\rho)\rho_\mu\rho_\nu\cdots\rho_\lambda\ , \tag{8}$$

and thus rewrite (5) as

$$i\hbar\frac{\partial}{\partial t}f=\left[-\frac{\hbar^2}{2m}\nabla^2-\mu\right]f+D_0 g+\{(D_1)_\mu\partial_\mu g+\tfrac{1}{2}[\partial_\nu(D_1)_\nu]g\}+\{\tfrac{1}{2}(D_2)_{\mu\lambda}\partial_\mu\partial_\lambda g+\tfrac{1}{2}[\partial_\nu(D_2)_{\mu\nu}]\partial_\mu g\}+\cdots\ , \tag{9}$$

where we have dropped the spin indices α and β, and position $\mathbf{x}$ for easier notation. An analogous equation for g is obtained by interchanging f and g, changing the sign of the right-hand side and complex conjugating all D_n's.

There are a number of simplifications to the seemingly complicated Eq. (9). For singlet pairing, F and Δ are proportional to $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, and are even under $\rho\to-\rho$. Thus all D_n for n odd vanish. If we further keep only the D_0 term we obtain

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix}f\uparrow\\f\downarrow\\g\uparrow\\g\downarrow\end{bmatrix}=\begin{bmatrix}-\frac{\hbar^2}{2m}\nabla^2-\mu & -D_0\begin{bmatrix}0&1\\-1&0\end{bmatrix}\\ -D_0^*\begin{bmatrix}0&1\\-1&0\end{bmatrix} & \frac{\hbar^2}{2m}\nabla^2+\mu\end{bmatrix}\begin{bmatrix}f\uparrow\\f\downarrow\\g\uparrow\\g\downarrow\end{bmatrix}\ ,$$

which decomposes into the two sets of equations of Andreev.[1] For triplet pairing, F and Δ are symmetric matrices and are odd under $\rho\to-\rho$, so only D_n for odd n remains.

As soon as the gap varies only slowly on the scale of its range, we can define the gap $\Delta^{\mathbf{k}}(\mathbf{r})$ for wave vector $\mathbf{k}$ at position $\mathbf{r}$ by

$$\Delta_{\alpha\beta}(\mathbf{r},\rho)\equiv\sum_{\mathbf{k}}\Delta_{\alpha\beta}^{\mathbf{k}}(\mathbf{r})e^{i\mathbf{k}\cdot\rho}\ , \tag{10}$$

analogous to the uniform case. D_n then becomes

$$(D_n)_{\mu\nu\cdots\lambda}=\frac{1}{i^n}\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\nu}\cdots\frac{\partial}{\partial k_\lambda}\Delta^k\Big|_{k=0}\ , \tag{11}$$

and hence the right-hand side of (8) is essentially an expansion of gap as a polynomial in k. The order of derivative of g to be kept in (8) depends thus on the form of gap chosen. For the form chosen by Greaves and Leggett:[2]

$$\Delta^{\mathbf{k}}=i(\sigma_i\sigma_y)d_{i\mu}k_\mu/k_F \tag{12}$$

then only D_1 need be kept. For the "cutoff" version

$$\Delta^{\mathbf{k}}=\begin{cases}i(\sigma_i\sigma_y)d_{i\mu}k_\mu & \text{if } \mathbf{k}\text{ "on shell"}\\ 0 & \text{otherwise}\end{cases} \tag{13}$$

our formula still applies for those (physical) momentum values well within the cutoff, to lowest order in $|k-k_F|/k_F$.

If $\Delta^{\mathbf{k}}$ spreads over $\mathbf{k}$ values of thickness Δk, then the range of the gap, from (10) is roughly $\hbar/\Delta k$. Even for the form (13) it is $\hbar v_F/\tilde{\omega}$ where $\tilde{\omega}$ is the cutoff energy, and is much less than $\hbar v_F/\Delta_0\approx\xi_0$, where Δ_0 is the maximum of gap and ξ_0 the coherence length. Thus, except when the gap varies appreciably in a distance of the order of a few angstroms, we can say that the gap varies slowly on the scale of its range. This justifies the argument given before in Eqs. (7) and (10). If we confine ourselves to the case (12), as will be done in Sec. III, the equation of motion has the simple form

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix}f\\g\end{bmatrix}=\begin{bmatrix}-\frac{\hbar^2}{2m}\nabla^2-\mu & -\frac{1}{2}(\nabla\cdot\mathbf{D}_1^*)-\mathbf{D}_1^*\cdot\nabla\\ \frac{1}{2}(\nabla\cdot\mathbf{D}_1)+\mathbf{D}_1\cdot\nabla & \frac{\hbar^2}{2m}\nabla^2+\mu\end{bmatrix}\begin{bmatrix}f\\g\end{bmatrix}\equiv\mathscr{H}\begin{bmatrix}f\\g\end{bmatrix}\ . \tag{14}$$

Note that the operator $\mathscr{H}$ is Hermitian.

Before we proceed, we shall discuss the spin of the quasiparticle. The simplest way to find the necessary spin operator would be to use the operational definition $\boldsymbol{\mu}=\gamma/2\mathbf{s}$, with $\boldsymbol{\mu}$ the "magnetic moment" $\mathbf{s}$ the "spin," and where γ is the gyromagnetic ratio of ^{3}He. This is well defined (by considering a fictitious magnetic field) and clearly reduces to the ordinary definition when the fluid is normal.

Under a magnetic field $\mathbf{H}$, the Hamiltonian H in (4) then acquires an additional term-$(\gamma\hbar/2)\psi_\alpha^\dagger\sigma_{\alpha\beta}\psi_\beta\cdot\mathbf{H}$. The equations of motion for ψ and $\psi^\dagger$ then have extra terms-$(\gamma\hbar/2)\mathbf{H}\cdot\boldsymbol{\sigma}_{\alpha\beta}\psi_\beta$, $(\gamma\hbar/2)\mathbf{H}\cdot\boldsymbol{\sigma}^*_{\alpha\beta}\psi^\dagger_\beta$, respectively. Hence (14) becomes

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} f \\ g \end{bmatrix} = \left[\mathscr{H}-\frac{\gamma\hbar}{2}\mathbf{H}\cdot\begin{bmatrix}\boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma}^*\end{bmatrix}\right]\begin{bmatrix} f \\ g \end{bmatrix}, \tag{15}$$

from which we identify the spin-density vector

$$\boldsymbol{\Sigma}=\begin{bmatrix}\boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma}^*\end{bmatrix}.$$

This result can also be obtained by the more conventional but less easy approach of calculating the expectation value $\langle\Phi_1|\psi_\alpha^\dagger\sigma_{\alpha\beta}\psi_\beta|\Phi_1\rangle-\langle\Phi_0|\psi_\alpha^\dagger\sigma_{\alpha\beta}\psi_\beta|\Phi_0\rangle$, as we show in Appendix A.

It may be of interest to note some of the properties of the spin-density operator. We consider the possibility of an energy eigenstate being also an eigenvector of a component of Σ. Writing

$$\mathscr{H}=\begin{bmatrix}H_{11} & H_{12} \\ H_{21} & H_{22}\end{bmatrix}$$

where

$$-H_{22}=H_{11}=-(\hbar^2/2m)\nabla^2-\mu\ ,$$

$$-H_{21}^*=H_{12}=\tfrac{1}{2}(\nabla\cdot D_1)+(\mathbf{D}_1\cdot\nabla)\ ,$$

we easily compute the commutator

$$[\boldsymbol{\Sigma},\mathscr{H}]=\begin{bmatrix} 0 & -(\boldsymbol{\sigma}^*H_{21}+H_{21}\boldsymbol{\sigma}) \\ (\boldsymbol{\sigma}H_{12}+H_{12}\boldsymbol{\sigma}^*) & 0\end{bmatrix}.$$

Hence $[\boldsymbol{\Sigma},\mathscr{H}]=0$ if $\mathbf{M}\equiv\frac{1}{2}(\boldsymbol{\sigma}\mathbf{D}_1+\mathbf{D}_1\boldsymbol{\sigma}^*)=0$. Consider in particular the A phase, where $d_{i\mu}=\Delta_A d_i(\phi_\mu^{(1)}+i\phi_\mu^{(2)})$ with $\mathbf{d}$, $\boldsymbol{\phi}^{(1)}$, $\boldsymbol{\phi}^{(2)}$ specifying the spin and orbital parameters, respectively, hence $\mathbf{D}_1\propto i(\sigma\sigma_y)\cdot\mathbf{d}$. We can easily show that $\boldsymbol{\Sigma}\cdot\mathbf{d}$ commutes with $\mathscr{H}$ but not with $\boldsymbol{\Sigma}-(\boldsymbol{\Sigma}\cdot\mathbf{d})\mathbf{d}$. We can understand this physically as follows: if $\mathbf{d}$ is in x-y plane (not necessarily constant) then we have pairing between $k\uparrow$ and $-k\uparrow$ (similarly for down spins). An excitation in the occupation number representation of these two states, say $|10\rangle$, can be regarded as creating a $k\uparrow$, therefore of spin up character (amplitude $f_\uparrow$) or annihilating a $-k\uparrow$ from $|11\rangle$ (amplitude $g_\uparrow$), therefore of spin down character. Thus, Σ_z is not a good quantum number. However, if $\mathbf{d}=\hat{\mathbf{z}}$, $k\uparrow$ pairs with $-k\downarrow$; then both ways of creating the $|10\rangle$ state would have spin up, and Σ_z is a good quantum number. This is related to an interesting result for the excitation spin in a A-phase texture, as we shall see in Sec. III E.

III. QUASIPARTICLE BALLISTICS

It has already been argued by Greaves and Leggett,[2] in the context of ^{3}He-A, that when the relevant spatial variation of the order parameter is on a scale long compared with the zero-temperature coherence length ξ_0, we can speak of an excitation of wave vector $\mathbf{k}$ at position $\mathbf{r}$, whose equation of motion, from the Hamiltonian formalism, is

$$\dot{\mathbf{r}}=\frac{1}{\hbar}\frac{\partial E^{\mathbf{k}}}{\partial\mathbf{k}}\ ,\quad \hbar\dot{\mathbf{k}}=-\frac{\partial E^{\mathbf{k}}}{\partial\mathbf{r}}\ , \tag{16}$$

where $E^{\mathbf{k}}(\mathbf{r})=[\epsilon_{\mathbf{k}}^2+\Delta^{\mathbf{k}}(\mathbf{r})\Delta^{\mathbf{k}\dagger}(\mathbf{r})]^{1/2}$ is the quasiparticle energy and $\epsilon_{\mathbf{k}}\equiv(\hbar^2k^2/2m)-\mu$. It can be realized immediately that (16) is valid for any other unitary states and for nonunitary states as well if we replace $\Delta^{\mathbf{k}}\Delta^{\mathbf{k}\dagger}$ by the relevant one of its eigenvalues, provided the direction in spin space diagonalizing $\Delta^{\mathbf{k}}\Delta^{\mathbf{k}\dagger}$ is uniform and the spin is along that direction.

As expected, the ballistics must follow from the generalized Bogoliubov—de Gennes equation derived under the slow-variation condition. This will be shown below. For simplicity we confine ourselves to the unitary case, though the argument can be readily generalized to the aforementioned nonunitary case, since then we can decompose (14) into two sets of equations (cf. the singlet case mentioned) with $\mathbf{D}_1$ becoming diagonal by performing a rotation. Note for the general nonunitary case, (16) is incomplete, even for slow variations.[8,9] First we obtain some preliminary results which will be useful to our demonstration.

A. Uniform solutions

Assuming a uniform gap linear in $\mathbf{k}$ of the form (12),[10] we obtain from (11)

$$(D_1)_\mu=(\sigma_i\sigma_y)d_{i\mu}/k_F\ . \tag{17}$$

We shall be interested in the A phase where $d_{i\mu}=\Delta_A d_i\Phi_\mu$, $\Phi_\mu\equiv\phi_\mu^{(1)}+i\phi_\mu^{(2)}$ as already described, and the B phase, where $d_{i\mu}=\Delta_B R_{i\mu}e^{i\phi}$. $R_{i\mu}$ is the rotation matrix of angle $\Theta=\cos^{-1}(-\frac{1}{4})$ along $\hat{\omega}$ and ϕ is the phase angle.

In this case an eigenfunction of (14) can be easily obtained in the form

$$\begin{bmatrix} f(\mathbf{r},t) \\ g(\mathbf{r},t)\end{bmatrix}=\begin{bmatrix} f_{\mathbf{k}} \\ g_{\mathbf{k}}\end{bmatrix}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{18}$$

with $\omega^{\mathbf{k}}\equiv E^{\mathbf{k}}/\hbar$, where, by observing $\Delta^{\mathbf{k}}=i\mathbf{k}\cdot\mathbf{D}_1$

$$\mathscr{E}^{\mathbf{k}}\begin{bmatrix} f_{\mathbf{k}} \\ g_{\mathbf{k}}\end{bmatrix}\equiv\begin{bmatrix}\epsilon_{\mathbf{k}} & \Delta^{\mathbf{k}} \\ \Delta^{\mathbf{k}\dagger} & -\epsilon_{\mathbf{k}}\end{bmatrix}\begin{bmatrix} f_{\mathbf{k}} \\ g_{\mathbf{k}}\end{bmatrix}=E^{\mathbf{k}}\begin{bmatrix} f_{\mathbf{k}} \\ g_{\mathbf{k}}\end{bmatrix}. \tag{19}$$

We see that

$$\begin{bmatrix} f_{\mathbf{k}} \\ g_{\mathbf{k}}\end{bmatrix}$$

is proportional to a column of the unitary matrix $U^{\mathbf{k}}$ of Balian and Werthamer,[7] which diagonalizes the Hermitian matrix $\mathscr{E}^{\mathbf{k}}$ and transforms from superfluid excitations to normal-fluid excitations. $E^{\mathbf{k}}>0$ solutions for (19) are physical excitations ($|\Phi_1\rangle$ has higher energy than $|\Phi_0\rangle$) and $E^{\mathbf{k}}<0$ solutions correspond to lowering of the energy when an excitation is destroyed [obtained by considering $f'\equiv\langle\Phi_1|\psi|\Phi_0\rangle$ and $g'\equiv\langle\Phi_1|\psi^\dagger|\Phi_0\rangle$, note the resulting equation is identical to (14)].

We can solve for $g_{\mathbf{k}}$ from (19)

$$g_{\mathbf{k}}=\frac{1}{E^{\mathbf{k}}+\epsilon_{\mathbf{k}}}\Delta^{\mathbf{k}\dagger}f_{\mathbf{k}} \tag{20a}$$

whence, in the unitary case

$$g_{\mathbf{k}}^\dagger g_{\mathbf{k}}=\frac{E^{\mathbf{k}}-\epsilon^{\mathbf{k}}}{E^{\mathbf{k}}+\epsilon^{\mathbf{k}}}f_{\mathbf{k}}^\dagger f_{\mathbf{k}}\ . \tag{20b}$$

If we normalize our solution, for unit volume, by

$$\int_{\mathbf{r}}(f^\dagger f+g^\dagger g)=1$$

then

$$f_{\mathbf{k}}^\dagger f_{\mathbf{k}}=\frac{1}{2}\left[1+\frac{\epsilon_{\mathbf{k}}}{E^{\mathbf{k}}}\right], \tag{21a}$$

$$g_{\mathbf{k}}^\dagger g_{\mathbf{k}}=\frac{1}{2}\left[1-\frac{\epsilon_{\mathbf{k}}}{E^{\mathbf{k}}}\right]. \tag{21b}$$

We shall show in the next section that $\int_{\mathbf{r}}f^\dagger f+g^\dagger g$ is a constant in time, thus the above normalization is possible.

B. Density, current, and momentum density

Consider the quantity ρ

$$\rho(r,t)\equiv f^\dagger f+g^\dagger g\ . \tag{22}$$

Application of (14) yields

$$i\hbar\frac{\partial}{\partial t}\rho(r,t)=-\frac{\hbar^2}{2m}[(f^\dagger\nabla^2 f-\tilde{f}\nabla^2 f^*)-(g^\dagger\nabla^2 g-\tilde{g}\nabla^2 g^*)]$$
$$+[f^\dagger(\mathbf{D}_1\cdot\nabla)g+\tilde{g}(\mathbf{D}_1\cdot\nabla)f^*+\tfrac{1}{2}f^\dagger(\nabla\cdot\mathbf{D}_1)g+\tfrac{1}{2}\tilde{g}(\nabla\cdot\mathbf{D}_1)f^*-\text{c.c.}]\ . \tag{23}$$

Hence

$$\frac{\partial}{\partial t}\rho(r,t)=-\nabla\cdot(\mathbf{j}_1+\mathbf{j}_2)\ , \tag{24}$$

where

$$\mathbf{j}_1=\frac{\hbar}{2mi}[(f^\dagger\nabla f-\tilde{f}\nabla f^*)-(g^\dagger\nabla g-\tilde{g}\nabla g^*)]\ , \tag{25a}$$

$$\mathbf{j}_2=-\frac{2}{\hbar}\text{Im}(f^\dagger\mathbf{D}_1 g)\ . \tag{25b}$$

Equation (24) is in the form of a continuity equation and allows us to interpret ρ as an excitation density (a conserved quantity), and $\mathbf{j}_1+\mathbf{j}_2$ as the excitation current. $\mathbf{j}_1$ is as in the singlet case of Andreev's whereas an additional term $\mathbf{j}_2$ arises for higher-l pairing. The presence of the $\mathbf{j}_2$ term need not surprise us (if one recalls Dirac's treatment of relativistic spin-$\frac{1}{2}$ particles), and, as we shall show in the subsection D, it is a term which arises from the dependence of $E^{\mathbf{k}}$ on $\mathbf{k}$ through $\Delta^{\mathbf{k}}$.

The currents (24) should be distinguished from the real ^{3}He-particle current associated with the excitation, which is given by $(m/m_{\text{He}})\mathbf{j}_1'$, where m_{He} is the mass of (bare) ^{3}He, and

$$m\mathbf{j}_1'(r,t)\equiv\frac{\hbar}{2i}[(f^\dagger\nabla f-\tilde{f}\nabla f^*)+(g^\dagger\nabla g-\tilde{g}\nabla g^*)]\ . \tag{26}$$

Such an identification is justified by the fact that for the normalized plane-wave solutions (18), $m\mathbf{j}_1'=\hbar\mathbf{k}$ is the momentum of the particle [cf. (34) below]. Note $\mathbf{j}_1'\neq\mathbf{j}_1$ (because of the sign).

C. Spin density and spin current

We have already seen in Sec. II that the spin density of the excitation is

$$\mathbf{S}(r,t)=f^\dagger\boldsymbol{\sigma}f-g^\dagger\boldsymbol{\sigma}^*g\ . \tag{27}$$

Using (14), we can easily show that

$$\frac{\partial}{\partial t}S_i(r,t)=-\nabla\cdot\mathbf{j}_i+\dots\ , \tag{28}$$

where the ellipsis represents source terms, and where

$$\mathbf{j}_i\equiv\frac{\hbar}{2mi}(f^\dagger\sigma_i\nabla f-(\nabla f^\dagger)\sigma_i f-\nabla g^\dagger\sigma_i^* g+g^\dagger\sigma_i^*\nabla g)$$
$$=\frac{\hbar}{2mi}(f^\dagger\sigma_i\nabla f-\nabla f^\dagger\sigma_i f+(\nabla\tilde{g})\sigma_i g^*-\tilde{g}\sigma_i\nabla g^*)\ , \tag{29}$$

is the i-direction spin-current density (Appendix A). The second equality follows by taking the transpose of the last two terms. The source terms are given by

$$\frac{1}{i\hbar}[\tfrac{1}{2}f^\dagger(\sigma_i(\nabla\cdot\mathbf{D}_1)-(\nabla\cdot\mathbf{D}_1)\sigma_i^*)g$$
$$+(f^\dagger\sigma_i\mathbf{D}_1\cdot\nabla g-\nabla f^\dagger\cdot\mathbf{D}_1\sigma_i^*g)-c.c.]\ . \tag{30}$$

This is the reaction due to the Cooper pairs. For the total spin of the excitation, we integrate (28). After integration by parts, we get

$$\frac{d}{dt}\langle\mathbf{S}\rangle=\frac{1}{i\hbar}\int_{\mathbf{r}}[\tfrac{1}{2}f^\dagger\nabla\cdot\mathbf{M}g+f^\dagger\mathbf{M}\cdot\nabla g$$
$$+g^\dagger(\tfrac{1}{2}\nabla\cdot\mathbf{M}^*)f+g^\dagger(\mathbf{M}^\dagger\cdot\nabla)f]\ , \tag{31}$$

where the tensor $\mathbf{M}\equiv\frac{1}{2}(\boldsymbol{\sigma}\mathbf{D}_1+\mathbf{D}_1\boldsymbol{\sigma}^*)$ is as defined previously and the dot products with ∇ are always understood to be between the spatial parts. [This, in fact, could have been obtained by using $i\hbar(d/dt)\langle\mathbf{S}\rangle=\langle[\boldsymbol{\Sigma},H]\rangle$.] (31) is the same form as (23) when the latter is integrated, if $\mathbf{M}$ is replaced by $\mathbf{D}_1$ and $\mathbf{M}^*$ by $-\mathbf{D}_1^*$. This sign difference prevents us from putting the expression in square brackets equal to a divergence and integrating it to zero. It represents the action of the superfluid on the excitation spin (Sec. III E for further discussions).

D. Derivation of ballistics

We now demonstrate that ballistics in unitary states follows from our general formulation. We consider wave packets of the form

$$f(\mathbf{r},t)=\sum_{\Delta k} f_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega^{\mathbf{k}}t)} ,$$
$$g(\mathbf{r},t)=\sum_{\Delta k} g_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega^{\mathbf{k}}t)} , \quad (32)$$

where sum is over a momentum range Δk for k_x,k_y,k_z near the average wavevector $\mathbf{k}$. When the typical textural-variation scale R_0 is large compared with ξ_0, then it is possible to choose the size of the wave packet $\Delta r \sim (\hbar/\Delta k)$ such that $R_0 \gg \Delta r \gg \xi_0 \sim (\hbar v_F/\Delta_0)$, where Δ_0 is the maximum of the gap. This allows us to speak of the position of the excitation. The uncertainty in energy, $\Delta E^{\mathbf{k}}$, is then $\Delta E^{\mathbf{k}} \approx \hbar v_F(\Delta k) \ll \Delta_0 \ll$ typical energy scale in which $\Delta^{\mathbf{k}}$ varies. This allows us to speak of the gap $\Delta^{\mathbf{k}}$ and energy $E^{\mathbf{k}}$ appropriate to the excitation. Hence the wave packet (well) defines a quasiparticle in ballistics.

With this, the rest is just simple algebra. The mean position of the quasiparticle is (the angular brackets shall henceforth denote expectation values for the excitation)

$$\langle \mathbf{r} \rangle = \int_{\mathbf{r}} \mathbf{r}(f^\dagger f + g^\dagger g) ,$$

whose rate of change is

$$\frac{d}{dt}\langle \mathbf{r}\rangle = \int_{\mathbf{r}} \mathbf{r}\frac{\partial}{\partial t}\rho = \int_{\mathbf{r}} (\mathbf{j}_1+\mathbf{j}_2)$$
$$\equiv \langle \mathbf{j}_1+\mathbf{j}_2 \rangle \quad (33)$$

where we have substituted (23), integrated by parts, and dropped the surface term due the fact that the wave packet is localized in space. This is in fact just the Ehrenfest theorem in quantum mechanics.

The average momentum for the wave packet is, following (26),

$$\langle \hbar\mathbf{k}\rangle = \frac{\hbar}{2i}\int_{\mathbf{r}} [(f^\dagger \nabla f - \tilde{f}\nabla f^*) + (g^\dagger \nabla g - \tilde{g}\nabla g^*)] . \quad (34)$$

Taking the time derivative and noting that we have a wave packet so that we can integrate by parts and drop surface terms, we get

$$\frac{d}{dt}\langle \hbar\mathbf{k}\rangle = -\int_{\mathbf{r}} \left\{ \left[f^\dagger \nabla \left[i\hbar\frac{\partial f}{\partial t}\right] - \tilde{f}\nabla\left[i\hbar \frac{\partial f^*}{\partial t}\right]\right] + \left[g^\dagger \nabla \left[i\hbar\frac{\partial g}{\partial t}\right] - \tilde{g}\nabla\left[i\hbar \frac{\partial g^*}{\partial t}\right]\right]\right\} .$$

(14) is now substituted, and further integrated by parts. The terms involving only f and g cancel exactly. We are left with

$$\frac{d}{dt}\langle \hbar k_\mu \rangle = -\int_{\mathbf{r}} \{ [f^\dagger \nabla_\mu(\tfrac{1}{2}(\nabla\cdot\mathbf{D}_1)g + (\mathbf{D}_1\cdot\nabla)g) - \tilde{g}\nabla_\mu(\tfrac{1}{2}(\nabla\cdot\mathbf{D}_1)f^* + (\mathbf{D}_1\cdot\nabla)f^*)] + \text{c.c.}\} .$$

Thus all terms involve two derivatives. We recall our assumption that the order parameter, thus $\mathbf{D}_1$, varies only in a distance $R_0 \gg \xi_0 \gg 1/k_F$. Thus we can ignore second derivatives of D_1 when compared with other terms. The third term combines with the first to yield $\int_{\mathbf{r}} f^\dagger(\nabla\cdot\mathbf{D}_1)\nabla_\mu g$, whereas the second one, integrated by parts twice, combines with the fourth to yield

$$\int_{\mathbf{r}} [\nabla_\mu f^\dagger(\nabla_\nu D_{1\nu})g - \tilde{g}(\nabla_\mu D_{1\nu})\nabla_\nu f^*]$$
$$= -\int_{\mathbf{r}} [f^\dagger(\nabla\cdot\mathbf{D}_1)\nabla_\mu g + \tilde{g}(\nabla_\mu D_{1\nu})\nabla_\nu f^*] .$$

Hence

$$\frac{d}{dt}\langle \hbar k_\mu\rangle = \int_{\mathbf{r}} [\tilde{g}(\nabla_\mu D_{1\nu})\nabla_\nu f^* + \text{c.c.}] .$$

Here we can substitute our wave-packet form (32) since $\nabla_\mu D_{1\nu}$ is already a first-order gradient. Since the order parameter is slowly varying, $R_0 \gg \Delta r$, we can take $\nabla_\mu D_{1\nu}$ to be the value at the position of the wave packet, and get, with $A_{\mathbf{k}}$ the amplitude of the $\mathbf{k}$ wave,

$$\frac{d}{dt}\langle \hbar k_\mu\rangle = 2\text{Re}\sum_{\mathbf{k}} |A_{\mathbf{k}}|^2 g_{\mathbf{k}}^\dagger(\nabla_\mu D_{1\nu}^*)_{\langle r\rangle} i k_\nu f_{\mathbf{k}} .$$

The relation (20) can be substituted, and using $i\mathbf{k}\cdot\mathbf{D}(r)=\Delta^{\mathbf{k}}(r)$ we get

$$\frac{d}{dt}\langle \hbar k_\mu\rangle = -2\sum_{\mathbf{k}}\text{Re}\left[\frac{1}{E^{\mathbf{k}}+\epsilon_{\mathbf{k}}} f_{\mathbf{k}}^\dagger \Delta^{\mathbf{k}}(\nabla_\mu \Delta^{\mathbf{k}\dagger})f_{\mathbf{k}}\right]|A_{\mathbf{k}}|^2$$
$$= -\sum_{\mathbf{k}}\frac{1}{E^{\mathbf{k}}+\epsilon_{\mathbf{k}}} f_{\mathbf{k}}^\dagger \nabla_\mu(\Delta^{\mathbf{k}}\Delta^{\mathbf{k}\dagger})f_{\mathbf{k}}|A_{\mathbf{k}}|^2$$

and when the state is unitary and the wave packet normalized, with (21) we get

$$\frac{d}{dt}\langle \hbar k_\mu\rangle = -\nabla_\mu E^{\langle k\rangle} ,$$

which is just the second of (16).

To complete our demonstration of (16) we have to evaluate $\mathbf{j}_1+\mathbf{j}_2$. $\mathbf{j}_1$ can be evaluated easily by our wave packet (31) for unitary states, using (20) and (21) and replacing averages of products by products of averages

$$\mathbf{j}_1 = \frac{\hbar\mathbf{k}}{m}\frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}} ,$$

where we have just written $\mathbf{k}$ for $\langle \mathbf{k}\rangle$ for simpler notation. Note this is just

$$\frac{\partial E^{\mathbf{k}}}{\partial \epsilon_{\mathbf{k}}}\frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}} .$$

For $\mathbf{j}_2$, we have

$$\mathbf{j}_2=-\frac{2}{\hbar}\int_{\mathbf{r}}\mathrm{Im}f^{\dagger}(\mathbf{D}_1)g \ .$$

We can evaluate $\mathbf{D}_1$ at the position of the wave packet and substitute our plane-wave expansion (32), and moreover use (20) and replace the products of functions of k by products at $\langle \mathbf{k} \rangle$, we obtain

$$\mathbf{j}_2=-\frac{1}{i\hbar}f_{\mathbf{k}}^{\dagger}\left[(\mathbf{D}_1)\frac{\Delta^{\mathbf{k}\dagger}}{E^{\mathbf{k}}+\epsilon_{\mathbf{k}}}-\frac{\Delta_{\mathbf{k}}(\mathbf{D}_1^{\dagger})}{E^{\mathbf{k}}+\epsilon_{\mathbf{k}}}\right]f_{\mathbf{k}} \ ,$$

where we have again used $\mathbf{k}$ for $\langle \mathbf{k} \rangle$, $\mathbf{D}_1=\mathbf{D}_1(\langle \mathbf{r} \rangle)$, $\Delta^{\mathbf{k}}=\Delta^{\mathbf{k}}(\langle \mathbf{r} \rangle)$ for the sake of a simplified notation. Using the identity

$$(\sigma_i\sigma_y)(\sigma_y\sigma_j)=\delta_{ij}+i\epsilon_{ijm}\sigma_m \ ,$$

we find

$$D_{1\mu}\Delta^{\mathbf{k}\dagger}-\Delta^{\mathbf{k}}D_{1\mu}^{\dagger}=-\frac{ik_\nu}{k_\rho^2}(d_{i\mu}d_{i\nu}^{*}+d_{i\nu}d_{i\mu}^{*}) + \frac{k_\nu}{k_F^2}(d_{i\mu}d_{j\nu}^{*}-d_{i\mu}^{*}d_{j\nu})\epsilon_{ijm}\sigma_m \ . \quad (35)$$

If the state is unitary, the condition that $\Delta^{\mathbf{k}}\Delta^{\mathbf{k}\dagger}$ is proportional to an identity matrix can be written as

$$k_\mu d_{i\mu}k_\nu d_{j\nu}^{*}\epsilon_{ijm}\sigma_m=0 \ . \quad (36)$$

We can differentiate this with respect to k_μ and find

$$(d_{i\mu}k_\nu d_{j\nu}^{*})+k_\nu d_{i\nu}d_{j\mu}^{*})\epsilon_{ijm}\sigma_m=0 \ ,$$

which implies that the last term of (35) is identically zero. Thus

$$D_{1\mu}\Delta^{\mathbf{k}\dagger}-\Delta^{\mathbf{k}}D_{1\mu}^{\dagger}=\tfrac{1}{2}\mathrm{tr}(D_{1\mu}\Delta^{\mathbf{k}\dagger}-\Delta^{\mathbf{k}}D_{1\mu}^{\dagger}) \ , \quad (37)$$

and so, with the help of (21)

$$\mathbf{j}_2=\frac{i}{2\hbar E^{\mathbf{k}}}\mathrm{tr}(D_{1\mu}\Delta^{\mathbf{k}\dagger}-\Delta^{\mathbf{k}}D_{1\mu}^{\dagger}) \ .$$

Now, since $E^{\mathbf{k}}=(\epsilon_{\mathbf{k}}^2+\Delta^{\mathbf{k}}\Delta^{\mathbf{k}\dagger})^{1/2}$,

$$\frac{1}{\hbar}\frac{\partial E^{\mathbf{k}}}{\partial \mathbf{k}}=\frac{\partial E^{\mathbf{k}}}{\partial \epsilon_{\mathbf{k}}}\frac{\partial \epsilon_{\mathbf{k}}}{\hbar\partial \mathbf{k}}+\frac{1}{\hbar}\frac{\partial E^{\mathbf{k}}}{\partial \Delta_{\alpha\beta}^{\mathbf{k}}}\frac{\partial \Delta_{\alpha\beta}^{\mathbf{k}}}{\partial \mathbf{k}}+\frac{1}{\hbar}\frac{\partial E^{\mathbf{k}}}{\partial \Delta_{\alpha\beta}^{\mathbf{k}\dagger}}\frac{\partial \Delta_{\alpha\beta}^{\mathbf{k}\dagger}}{\partial \mathbf{k}} \ .$$

The first term has already seen to be $\mathbf{j}_1$. The last two terms are

$$\frac{1}{\hbar}\frac{i}{2E^{\mathbf{k}}}[\Delta_{\alpha\beta}^{\mathbf{k}\dagger}(\mathbf{D}_1)_{\beta\alpha}-\Delta_{\beta\alpha}^{\mathbf{k}}(\mathbf{D}_1)_{\alpha\beta}^{\dagger}] =\frac{i}{2E^{\mathbf{k}}\hbar}\mathrm{tr}(\mathbf{D}_1\Delta^{\mathbf{k}\dagger}-\Delta^{\mathbf{k}}\mathbf{D}_1^{\dagger})=\mathbf{j}_2 \ .$$

This completes our proof.

We also see that if the state is not unitary, then (36) fails and $D_{1\mu}\Delta^{\mathbf{k}\dagger}-\Delta^{\mathbf{k}}D_{1\mu}^{\dagger}$ is no longer proportional to an identity matrix. $\mathbf{j}_1,\mathbf{j}_2$ then depend directly on the amplitudes $f_\uparrow$ and $f_\downarrow$, i.e., on the spin of the excitation.

Before concluding this section we should mention that we have implicitly used the plane-wave solutions for the stationary condensate in our expansion (31). More precisely, if $\mathbf{v}_s\neq 0$, the plane-wave solutions should be of the form

$$\begin{bmatrix} f_{\mathbf{k}}e^{i(\mathbf{k}+m\mathbf{v}_s/\hbar)\cdot\mathbf{r}} \\ g_{\mathbf{k}}e^{i(\mathbf{k}-m\mathbf{v}_s/\hbar)\cdot\mathbf{r}} \end{bmatrix} e^{-i[E^{\mathbf{k}}(\mathbf{v}_s=0)-\hbar\mathbf{k}\cdot\mathbf{v}_s]t} \quad (38)$$

as one may verify explicitly on substitution into (14) when

$$\mathbf{D}_1(r)=\mathbf{D}_1(0)e^{(2im\mathbf{v}_s\cdot\mathbf{r}/\hbar)} \ .$$

But our procedure is valid provided $\xi_0/R_0 \ll 1$. This has already been pointed out be Greaves and Leggett in the ballistic case. We have thus shown that ballistics is true in unitary phases (or nonunitary phases with restrictions mentioned before) in lowest order in ξ_0/R_0, in which case the Hamiltonian equations (16) hold.

E. Comments on branch conversion

In the Sec. III D we only considered the position, momentum, and energy of the excitation and left out completely comments on the internal degrees of freedom. This we shall now turn out attention to.

The internal degrees of freedom correspond to the product space of (superfluid) particle-hole system, characterized by whether the group velocity is parallel or antiparallel to the momentum, and the spin degree of freedom.[3] Since $\Delta^{\mathbf{k}}$ usually has a much weaker dependence on $\mathbf{k}$ than $\epsilon_{\mathbf{k}}$ except very close to the Fermi surface,[2] we shall simply distinguish particles and holes by $p \gtrless p_F$.

Consider first the particle-hole degree of freedom of an excitation of (mean) energy E and momentum $\mathbf{p}$. If the typical textural length is R_0, we saw at the end of the Sec. III D that the texture will mix the plane-wave solutions of different momentum, the range of momentum being $\sim mv_s \sim \hbar/R_0$. Thus unless E is very close to $|\Delta|$ such that the difference between the particle and the hole momentum is $\lesssim \hbar/R_0$, there is no appreciable mixing of particle and hole. In other words, for $R_0 \gg \xi_0$ in the ballistic case, particle-hole conversion occurs only when $E\approx|\Delta|$, i.e., where Andreev reflection occurs; in other situations the particle-hole degree of freedom follows adiabatically that of the texture,[3] as implicitly assumed in Ref. 2. If $R_0 \sim \xi_0$, then the range of momentum is $\sim(\hbar/\xi_0)$ and corresponds to an energy $\sim \Delta_0$. In this case detailed solution of (14) is necessary and particle-hole conversion occurs appreciably.

However, the spin internal degree of freedom behaves entirely differently, since for given $E,\mathbf{p}$, there is nothing to prevent the conversion between the two spin directions, at least for the unitary case. Looking at the formula for $d\langle \mathbf{S} \rangle/dt$ in Sec. III C, in general we expect that probably we have to specify the amplitudes and phases of f and g, or equivalently the expectation values of $\langle \mathbf{S} \rangle$ in order to know its evolution in time (cf. Ref. 3).[9]

We can do this explicitly, first in a uniform system, for our wave packet (31), for which we must now demand that it has an average spin $\langle \mathbf{S} \rangle$. Ignoring now the $\nabla\cdot\mathbf{M}$ terms as they are of higher order, we obtain from (30), since $\Delta=i\mathbf{k}\cdot\mathbf{D}_1$, $\Delta^{\dagger}=-i\mathbf{k}\cdot\mathbf{D}_1^{*}$,

$$\frac{d}{dt}\langle \mathbf{S}\rangle = \frac{1}{i\hbar}\frac{1}{E+\epsilon}f^{\dagger}[\sigma,\Delta\Delta^{\dagger}]f \ ,$$

where we have dropped the implicit $\mathbf{k}$ dependences. For a general nonunitary state with $\Delta^{\mathbf{k}}=\Delta_0 i\mathbf{d}(\mathbf{k})\cdot\sigma\sigma_y$, with $\mathbf{d}$ now complex and $\mathbf{d}\cdot\mathbf{d}^*=1$ and Δ_0 the root mean square of the gap, we easily find

$$\Delta\Delta^{\dagger}=\Delta_0^2[1+i\sigma\cdot(\mathbf{d}\times\mathbf{d}^*)] \ ,$$

$$[\sigma,\Delta\Delta^{\dagger}]=-2\Delta_0^2(\mathbf{d}\times\mathbf{d}^*)\times\sigma \ ,$$

and so

$$\frac{d}{dt}\langle \mathbf{S}\rangle = -\frac{2}{i\hbar}\frac{\Delta_0^2}{E+\epsilon}(\mathbf{d}\times\mathbf{d}^*)\times f^{\dagger}\sigma f \ . \quad (39a)$$

On the other hand, if we use $f=\Delta/(E-\epsilon)g$ and express f in terms of g, we get

$$\frac{d}{dt}\langle \mathbf{S}\rangle = -\frac{1}{i\hbar}\frac{1}{E+\epsilon}g^{\dagger}[\sigma,\Delta\Delta^{\dagger}]^*g$$

$$=-\frac{2}{i\hbar}\frac{\Delta_0^2}{E-\epsilon}g^{\dagger}(\mathbf{d}\times\mathbf{d}^*)\times\sigma^*g \ . \quad (39b)$$

(39) can be combined to yield

$$\frac{d}{dt}\langle \mathbf{S}\rangle = -\frac{1}{\epsilon(i\hbar)}\Delta_0^2(\mathbf{d}\times\mathbf{d}^*)\times\langle \mathbf{S}\rangle \ . \quad (40)$$

Thus $\langle \mathbf{S}\rangle$ precesses around the direction given by $(1/i)(\mathbf{d}\times\mathbf{d}^*)$. For example, in the $A1$ phase with $\mathbf{d}(\mathbf{k})=\mathbf{d}=(1/\sqrt{2})(\hat{\mathbf{x}}+i\hat{\mathbf{y}})$, then $\langle \mathbf{S}\rangle$ precesses around the z axis. This can be understood as the result of the tunneling between the $\pm\hat{\mathbf{z}}$ amplitudes (cf. precession of a spin-$\frac{1}{2}$ electron under a magnetic field).

In a uniform unitary state then, $\langle \mathbf{S}\rangle$ is a constant. We now investigate the effect of a long-wavelength texture ($R_0 \gg \xi_0$). Since the time of flight across the texture is proportional to R_0, we shall need $(d/dt)\langle \mathbf{S}\rangle$ to first order in the gradients of the order parameter. By this token, we can investigate $(d/dt)\langle \mathbf{S}\rangle$ in cases where only part of the order parameter is changing uniformly, and add up our final result.

We shall consider first the A phase in which $\hat{\mathbf{d}}$ is a constant and the orbital parts $\phi^{(1)},\phi^{(2)}$ are changing. First we let $\hat{\mathbf{d}}=\hat{\mathbf{z}}$ so that only opposite-spin pairs form.[11] The consideration of Sec. III B shows that $\int_{\mathbf{r}}|f_{\uparrow}|^2+|g_{\downarrow}|^2$ and $\int_{\mathbf{r}}|f_{\downarrow}|^2+|g_{\uparrow}|^2$ are constants. It follows then trivially that $\langle S^z\rangle=$constant.

Next let $\hat{\mathbf{d}}$ be a constant in the x-y plane so that we have equal spin pairing (ESP).[11] Now $\int_{\mathbf{r}}(|f_{\uparrow}|^2+|g_{\uparrow}|^2)=C_1$, $\int_{\mathbf{r}}(|f_{\downarrow}|^2+|g_{\downarrow}|^2)=C_2$, where C_1 and C_2 are constants. If we are in the uniform $\phi^{(1)},\phi^{(2)}$ case then (20b) implies[12] $\langle S^z\rangle=(C_1-C_2)\epsilon_k/E_k$. This formula is valid then also in nonuniform $\phi^{(1)},\phi^{(2)}$ case to lowest order in ξ_0/R_0. With now $E_k=E_k(r)$, $d\langle S^z\rangle/dt$, to lowest order in gradients, is hence given by

$$\frac{(C_1-C_2)}{E_k}\frac{d\epsilon_k}{d\mathbf{k}}\frac{d\mathbf{k}}{dt} \ ,$$

which by (16), is

$$\frac{d\langle S^z\rangle}{dt}=-\frac{\Delta_A^2}{E_k\epsilon_k}\frac{1}{\hbar}\frac{d\epsilon_k}{dk}\frac{\partial}{\partial r}(\hat{\mathbf{k}}\times\hat{\mathbf{l}})^2S^z \ .$$

Combining the two cases above, our result for $\hat{\mathbf{d}}=$const, while $\phi^{(1)},\phi^{(2)}$ is nonuniform, is then

$$\frac{d\langle \mathbf{S}\rangle}{dt}=-\frac{\Delta_A^2}{E_k\epsilon_k}\frac{1}{\hbar}\frac{d\epsilon_k}{dk}\frac{\partial}{\partial r}(\hat{\mathbf{k}}\times\hat{\mathbf{l}})^2[\mathbf{S}-\hat{\mathbf{d}}(\hat{\mathbf{d}}\cdot\hat{\mathbf{S}})] \ . \quad (41)$$

Note that only the component of $\mathbf{S}$ perpendicular to $\mathbf{d}$ changes (cf. the argument in Sec. II).

Next consider $\hat{\boldsymbol{\phi}}^{(1)},\hat{\boldsymbol{\phi}}^{(2)}=$const and $\hat{\mathbf{d}}$ nonuniform. We define the tensor Ω_μ^i by

$$\delta d^i=\epsilon^{ijm}\delta R_\mu\Omega_\mu^j d^m \ . \quad (42)$$

It is reasonable that $\hat{\mathbf{l}}$ enters in $(d/dt)\langle \mathbf{S}\rangle$ through the magnitude of the gap only. Then the only other nonscalar quantities on which $(d/dt)\langle \mathbf{S}\rangle$ can depend are the wave vector $\hat{\mathbf{k}}$, the spin $\mathbf{S}$ of the wave packet, and the parameters $\hat{\mathbf{d}},\Omega_\mu^i$ of the condensate. Since $\langle \mathbf{S}\rangle$ is in spin space, Ω_μ^i and $\hat{\mathbf{k}}$ always occur in the form $\Omega_{\hat{k}}^i\equiv k_\mu\Omega_\mu^i$. $\mathbf{\Omega}$, $\mathbf{S}$, and $\mathbf{d}$ are all odd under time reversal and even under inversion. Taking into account also that the order parameter of the A phase has the ambiguity $\hat{\boldsymbol{\phi}}^{(1)},\hat{\boldsymbol{\phi}}^{(2)}\to-\hat{\boldsymbol{\phi}}^{(1)},-\hat{\boldsymbol{\phi}}^{(2)}$, $\hat{\mathbf{l}}\to\hat{\mathbf{l}},\hat{\mathbf{d}}\to-\hat{\mathbf{d}}$, we conclude that in each term $\hat{\mathbf{d}}$ can only occur an even number of times. Further, since we are interested only in the first-order gradients, then

$$\frac{d}{dt}\langle \mathbf{S}\rangle = a_1\hat{\mathbf{d}}[\mathbf{\Omega}_k\cdot(\hat{\mathbf{d}}\times\mathbf{S})]+a_2(\hat{\mathbf{d}}\times\mathbf{\Omega}_k)(\hat{\mathbf{d}}\cdot\mathbf{S})$$

$$+b_1(\mathbf{d}\times\mathbf{S})(\hat{\mathbf{d}}\cdot\mathbf{\Omega}_k)+b_2(\mathbf{\Omega}_k\times\mathbf{S}) \ , \quad (43)$$

where a_1, a_2, b_1, and b_2 are scalars. The third term can be dropped as $\hat{\mathbf{d}}\cdot\mathbf{\Omega}_k$ can always be chosen to be zero.

We must now use (14) to obtain the coefficients. For this we consider a wave packet at $t=0$ passing through the origin, where, by suitable choice of axes, $\hat{\mathbf{d}}$ is rotating in the x-y plane

$$\hat{\mathbf{d}}=\hat{\mathbf{d}}(\mathbf{r})=\hat{\mathbf{x}}\cos(\boldsymbol{\alpha}\cdot\mathbf{r}+\beta)+\hat{\mathbf{y}}\sin(\boldsymbol{\alpha}\cdot\mathbf{r}+\beta) \ . \quad (44)$$

In this case we have ESP along $\hat{\mathbf{z}}$ and a superfluid spin counterflow along $\hat{\boldsymbol{\alpha}}$. The gap matrix is, with $\hat{\mathbf{k}}\cdot(\boldsymbol{\phi}^{(1)}+i\boldsymbol{\phi}^{(1)})=|\hat{\mathbf{k}}\times\hat{\mathbf{l}}|e^{i\delta}$,

$$\Delta^{\mathbf{k}}=\Delta_A|\hat{\mathbf{k}}\times\hat{\mathbf{l}}|e^{i\delta}\begin{bmatrix}-e^{-i(\boldsymbol{\alpha}\cdot\mathbf{r}+\beta)} & 0\\ 0 & e^{i(\boldsymbol{\alpha}\cdot\mathbf{r}+\beta)}\end{bmatrix} \ , \quad (45)$$

It is necessary to keep the effect of the spin velocity in our eigenstates to first order (cf. end of Sec. III D). Thus we write them as

$$A_k\begin{bmatrix}f_{k\uparrow}e^{i\mathbf{k}_+^{\uparrow}\cdot\mathbf{r}}\\ 0\\ g_{k\uparrow}e^{i\mathbf{k}_-^{\uparrow}\cdot\mathbf{r}}\\ 0\end{bmatrix}e^{-iE_{k\uparrow}t/\hbar} \ , \quad B_k\begin{bmatrix}0\\ f_{k\downarrow}e^{i\mathbf{k}_+^{\downarrow}\cdot r}\\ 0\\ g_{k\downarrow}e^{i\mathbf{k}_+^{\downarrow}\cdot r}\end{bmatrix}e^{-iE_{k\downarrow}t/\hbar} \ . \quad (46)$$

From the physical interpretation and comparing with (38), we have

$$k_{\pm}^{\uparrow}=k\mp\tfrac{1}{2}\boldsymbol{\alpha}\ ,\quad E_{k^{\uparrow}}=E_k^{\alpha=0}-\frac{v_F}{2}\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}\ ,$$

$$k_{\pm}^{\downarrow}=k\pm\tfrac{1}{2}\boldsymbol{\alpha}\ ,\quad E_{k^{\downarrow}}=E_k^{\alpha=0}+\frac{v_F}{2}\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}\ . \tag{47}$$

One can also verify this by direct substitution in (14). Note that the texture leads to $E_{k^{\uparrow}}\neq E_{k^{\downarrow}}$ and so a nonconstant spin, as we shall see.

Our wave packet is constructed by superpositions of the solutions in (46). The width of the k values must satisfy $\Delta\mathbf{k}\cdot\hat{\boldsymbol{\alpha}}\gg\alpha$ so that the size of the wave packet is much smaller than the texture. That the wave packet is passing the origin at $t=0$ allows us to choose $f_{k^{\uparrow}}$, $f_{k^{\downarrow}}$, $g_{k^{\uparrow}}$, $g_{k^{\downarrow}}$, A_k, and B_k to be nearly constant at the average value p of k (on a scale of α).

We shall evaluate directly $(d/dt)\langle\mathbf{S}\rangle|_{t=0}$; by (43) we shall need $\langle\mathbf{S}\rangle|_{t=0}$ to zeroth order in α. For example,

$$\langle\mathbf{S}\rangle|_{t=0}=\sum_{k',k''}A_{k'}^{*}B_{k''}\left(f_{k'^{\uparrow}}^{*}f_{k''^{\downarrow}}e^{i(k_{+}''^{\downarrow}-k_{+}'^{\uparrow})\cdot r}-g_{k'^{\uparrow}}^{*}g_{k''^{\downarrow}}e^{i(k_{-}''^{\downarrow}-k_{+}'^{\uparrow})\cdot r}\right)+\text{c.c.}$$

In these terms we need $k'-k''\mp\boldsymbol{\alpha}=0$, respectively. To zeroth order in α, we can use [see (45), (20a)]

$$g_{k^{\uparrow}}^{*}g_{k^{\downarrow}}=-\frac{|\Delta^k|^2}{(E_k+\epsilon_k)^2}e^{-2i\beta}f_{k^{\uparrow}}^{*}f_{k^{\downarrow}}\ ,$$

and factor out the integral $C\equiv\sum_k A_k^{*}B_k f_{k^{\uparrow}}^{*}f_{k^{\downarrow}}$ (using the approximate constancy of the factors) to get

$$\langle S^x\rangle_{t=0}=C\left[1+\frac{|\Delta_p|^2}{(E_p+\epsilon_p)^2}e^{-2i\beta}\right]+\text{c.c.} \tag{48a}$$

Similarly

$$\langle S^y\rangle_{t=0}=-iC\left[1-\frac{|\Delta_p|^2}{(E_p+\epsilon_p)^2}e^{-2i\beta}\right]+\text{c.c.} \tag{48b}$$

We can verify that $\langle S^z\rangle$, whose explicit expression we shall not need, is constant. Using the same technique and (47), we get

$$\frac{d}{dt}\langle S^x\rangle|_{t=0}=-\frac{i}{\hbar}v_F\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}C\left[1-\frac{\epsilon_p}{E_p}\right](1+e^{-2i\beta})+\text{c.c.}\ , \tag{49a}$$

$$\frac{d}{dt}\langle S^y\rangle|_{t=0}=\frac{i}{\hbar}v_F\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}C\left[1-\frac{\epsilon_p}{E_p}\right](-1+e^{-2i\beta})+\text{c.c.} \tag{49b}$$

(48) can be substituted to get

$$\frac{d}{dt}\langle S^x\rangle=\frac{v_F|\Delta_p|^2}{\hbar E_p\epsilon_p}d^x(\Omega_\mu^z\hat{\mathbf{p}}_\mu)(\hat{\mathbf{d}}\times\mathbf{S})^z\ ,$$

$$\frac{d}{dt}\langle S^y\rangle=\frac{v_F|\Delta_p|^2}{\hbar E_p\epsilon_p}d^y(\Omega_\mu^z\hat{\mathbf{p}}_\mu)(\hat{\mathbf{d}}\times\mathbf{S})^z\ ,$$

and we conclude, by comparison with (43),

$$\frac{d}{dt}\langle\mathbf{S}^i\rangle=\frac{v_F|\Delta_p|^2}{\hbar E_p\epsilon_p}d^i(\hat{\mathbf{p}}_\mu\Omega_\mu^j)(\hat{\mathbf{d}}\times\mathbf{S})^j\ . \tag{50}$$

We must have $a_2=b_2=0$ as $(\hat{\mathbf{d}}\times\boldsymbol{\Omega}_k)(\hat{\mathbf{d}}\cdot\mathbf{S})$, $\boldsymbol{\Omega}_k\times\mathbf{S}$ are nonzero. We may rewrite this as

$$\frac{d}{dt}\langle\mathbf{S}^i\rangle=-a_1\hat{\mathbf{d}}[\hat{\mathbf{d}}\cdot(\boldsymbol{\Omega}_p\times\mathbf{S})]\ ,$$

and interpret the result as follows: the spin is "precessing" around the same axis $\boldsymbol{\Omega}_p$ as that of $\hat{\mathbf{d}}$ as we move along the quasiparticle trajectory (along $\hat{\mathbf{p}}$), at a rate specified by (50), while only its component parallel to $\hat{\mathbf{d}}$ can change.

(41) and (50) is our final A-phase result.[9] Note that $\langle S\rangle^2\neq$const. In the B phase the result is of the same form as (50) with $\hat{\mathbf{d}}$ replaced by $\hat{\mathbf{d}}(\hat{\mathbf{p}})$ [also in (42)]: only the "texture" of $\hat{\mathbf{d}}(\hat{\mathbf{p}})$ for the wave vector $\hat{\mathbf{p}}$ will be relevant, as a consideration of Eq. (38) will show.[9]

IV. TRANSPORT OF ENERGY ACROSS PHASE BOUNDARY

A. Theory

It has already been pointed out in the Introduction that for excitations incident on the boundary, the variation of the order parameter leads to Andreev reflection, an effect discovered by Andreev in studying the normal-metal—superconductor boundary. This additional reflection brings about a Kapitza resistance of the phase boundary. We shall do our calculation in the absence of a magnetic field. We believe that the magnetic field necessary for the experiment (see below) will not qualitatively affect our results provided it is not strong enough to cause appreciable distortion of the phases. Kaul and Kleinert[5] (abbreviated hereafter as KK) assumed an order parameter for the ^{3}He A-B phase boundary of the form ($\hat{\mathbf{z}}$ is perpendicular to boundary and points from A to B)

$$d_{i\mu}=\lambda(z)d_{i\mu}^A+\kappa(z)d_{i\mu}^B\ , \tag{51}$$

where

$$\lambda(z)+\kappa(z)=1\ ,$$

and $d_{i\mu}^A,d_{i\mu}^B$ are the order parameters of the uniform A and B phases, respectively, as described after (17). By minimizing the energy in the Ginzburg-Landau region, they find that

$$\lambda(z)=\frac{1}{2}\left[1-\tanh\frac{z}{R(T)}\right]\ , \tag{52}$$

where $R(T)=1.267\xi(T)$, $\xi(T)=\xi_0/(1-T/T_c)^{1/2}$ is the temperature-dependent coherence length, and if we define

$$\tilde{\boldsymbol{\Phi}}\equiv\boldsymbol{\Phi}e^{-i\phi}=\tilde{\boldsymbol{\phi}}^{(1)}+i\tilde{\boldsymbol{\phi}}^{(2)} \tag{53}$$

then

$$d_i R_{i\mu} = \tilde{\phi}^{(1)}_\mu \tag{54a}$$

and

$$\tilde{\phi}^{(1)}_\mu = \pm\hat{\mathbf{z}} \ . \tag{54b}$$

The orientation for these vectors are shown in Fig. 1 for the case $\tilde{\phi}^{(1)} = \hat{\mathbf{z}}$. We shall assume below that the phase factors are chosen so that $d^B_{i\mu}$ is real.

We shall be mainly interested in the transport of heat across the phase boundary. For the region $1 - T/T_c \ll 1$, then the mean free path ($\sim 10^{-4}$ cm) would be the same order as $R(T)$ [if $(1-T/T_c) \approx 10^{-2}$]. This means that a quasiparticle incident on the boundary has a high probability of interacting with the condensate or other quasiparticles. Hence the resistance of the boundary itself is nothing different from that of a slab of a normal ^{3}He fluid of thickness $\sim R_0(T)$.

For more interesting properties, we therefore turn to the low-temperature limit. This also allows us to regard the condensate inert, except that it sets up the "potential" in which our quasiparticles move. Though the argument of KK would not be true in this case as $R(T) \approx R(0) \equiv 1.267\xi_0$, we shall simply assume that (51)–(54) gives the correct order parameter.

To determine the transmission coefficient of excitations incident on the boundary, in principle we have to integrate the Eq. (14), with $(D_1)_\mu$ determined by (17) and (51). However, we observe a number of simplifications of the problem. Since we are interested in excitations of wave vector close to k_F and $E_F \gg \Delta_B$, we can write the excitation wave as, with $\hat{\mathbf{n}}$ the unit vector along the incident direction,

$$\begin{bmatrix} f \\ g \end{bmatrix} = e^{i[k_F \hat{\mathbf{n}}\cdot\mathbf{r} - (E/\hbar)t]} \begin{bmatrix} \xi(\mathbf{r}) \\ \eta(\mathbf{r}) \end{bmatrix} , \tag{55}$$

where ξ, and η will be expected to be slowly varying, compared with $e^{ik_F\hat{\mathbf{n}}\cdot\mathbf{r}}$, since the kinetic energies $\epsilon_{\mathbf{k}}$ of interest are at most $\sim\Delta_B$ and the deviation of the wave-propagation direction is small. (14) can then be approximated by leaving out terms of order Δ/E_F or smaller,

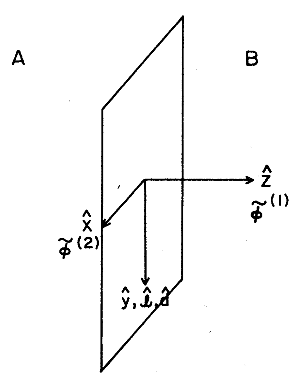

FIG. 1. Orientations of the order-parameter vectors of the ^{3}He A-B phase boundary.

$$\begin{bmatrix} -i\hbar v_F \hat{\mathbf{n}}\cdot\dfrac{\partial}{\partial\mathbf{r}} - E & \Delta^{\hat{\mathbf{n}}\dagger} \\ \Delta^{\hat{\mathbf{n}}\dagger} & i\hbar v_F \hat{\mathbf{n}}\cdot\dfrac{\partial}{\partial\mathbf{r}} - E \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0 , \tag{56}$$

where, since we shall be interested only in excitations near the Fermi surface, we have approximated Δ^k by $\Delta^{k_F\hat{\mathbf{n}}}$.

This is still four coupled first-order complex differential equations. We can further simplify this by using the fact that the incident and transmitted waves are both along $\pm\hat{\mathbf{n}}$ in our approximation. We note that, for the A phase, if we choose our spin axis perpendicular to $\mathbf{d}$, then $\mathbf{D}_1$ is diagonal and (14) would decompose into two sets: in this case we have equal spin pairing (ESP),[11] and $f_\uparrow, g_\uparrow$ do not couple with $f_\downarrow, g_\downarrow$, and vise versa. For the B phase and a given $\hat{\mathbf{n}}$, then the vector[11] $R_{i\mu}\hat{\mathbf{n}}_\mu \equiv \hat{\mathbf{n}}^R$ exists in which case we have ESP for spin axes perpendicular to $\hat{\mathbf{n}}^R$.[13] Thus if we choose a new set of axes x_s, y_s, and z_s for the spin directions, where the subscripts s remind us that these are axes for the spin, with $\hat{\mathbf{z}}_s(n)$ given by

$$\begin{aligned} \hat{\mathbf{z}}_s(n) &= \mathbf{d}\times\hat{\mathbf{n}}^R \\ &= (\mathbf{d}^R \times \hat{\mathbf{n}})^R , \end{aligned} \tag{57}$$

then (56) decomposes into two independent sets, each of the same form as itself, with new $\Delta^{\hat{\mathbf{n}}}$ replaced by scalars $\Delta^{\hat{\mathbf{n}}}_\pm$ for the spin $S_{z_s(\hat{\mathbf{n}})} \gtrless 0$, and ξ, η the appropriate scalar components. We shall not use separate symbols to distinguish these from the previous matrices. Note that $z_s(\hat{\mathbf{n}})$ depends on $\hat{\mathbf{n}}$, and our decomposition only holds for order parameters of the form (51) so that there exists a constant, common ESP $z_s(\hat{\mathbf{n}})$ axis, though $\lambda + \kappa = 1$ is not necessary. Note also it is vital that $\hat{\mathbf{n}} =$ const (approximately) for both incident and transmitted waves.

With this choice, one can show that (Appendix B)

$$\begin{aligned} \Delta^{\hat{\mathbf{n}}}_\pm(z) = {} & \lambda(z)\Delta_A(-n_x + in_z) \\ & + \kappa(z)\Delta_B(\pm(1-n_z^2)^{1/2} + in_z) \ . \end{aligned} \tag{58}$$

Note $\Delta^{\hat{\mathbf{n}}}_+ \neq \Delta^{\hat{\mathbf{n}}}_-$, i.e., the intermediate states are nonunitary. This leads to the possibility of a spin flux resulting from a temperature difference across the boundary, an effect which, however, will not be examined in the present paper.

To solve (56), we shall make use of the translational invariance parallel to the boundary to reduce it to a one-dimensional differential equation. We let

$$\begin{bmatrix} \xi(\mathbf{r}) \\ \eta(\mathbf{r}) \end{bmatrix} = e^{iq_I \hat{\mathbf{n}}_{\parallel}\cdot\mathbf{r}} e^{-iq_I \hat{\mathbf{n}}^2_{\parallel} z/n_z} \begin{bmatrix} Z_1(z) \\ Z_2(z) \end{bmatrix} , \tag{59}$$

where $\hat{\mathbf{n}}_{\parallel}$ denotes $\hat{\mathbf{n}} = \hat{\mathbf{n}} - \hat{\mathbf{z}}(\hat{\mathbf{z}}\cdot\hat{\mathbf{n}})$. (56) then becomes (we shall drop $\mathbf{k}$ in $E^{\mathbf{k}}$ and $\epsilon_{\mathbf{k}}$ whenever no confusion will arise)

$$\begin{bmatrix} v_F n_z \dfrac{\hbar}{i}\dfrac{\partial}{\partial z} - E & \Delta^{\hat{\mathbf{n}}} \\ \Delta^{\hat{\mathbf{n}}\dagger} & -v_F n_z \dfrac{\hbar}{i}\dfrac{\partial}{\partial z} - E \end{bmatrix} \begin{bmatrix} Z_1(z) \\ Z_2(z) \end{bmatrix} = 0 \ . \tag{60}$$

The plane-wave solution of (60) when $\Delta^{\hat{\mathbf{n}}}$ is uniform is of the form

$$\begin{bmatrix} Z_1(z) \\ Z_2(z) \end{bmatrix} = G_1 \begin{bmatrix} 1 \\ \frac{(\Delta^{\hat{n}})^*}{E+\epsilon} \end{bmatrix} e^{i\nu_1 z} + G_2 \begin{bmatrix} \frac{\Delta^{\hat{n}}}{E+\epsilon} \\ 1 \end{bmatrix} e^{i\nu_2 z} , \tag{61}$$

where $\nu_1 \nu_2$ can be determined by substituting this back to (50) and identifying the vector $\mathbf{q}$. We find

$$\frac{\epsilon}{\hbar v_F} \equiv \mathbf{q}\cdot\hat{\mathbf{n}} = \nu n_z ,$$

where $\nu=\nu_1$ or ν_2. Thus if $n_z>0$, $\nu_1=|\nu| =(E^2-|\Delta^{\hat{n}}|^2)^{1/2}$ $(\epsilon>0)$ refers to a particle propagating towards $+\hat{\mathbf{z}}$ $(\mathbf{q}\cdot\hat{\mathbf{n}}>0)$, whereas $\nu_2=-|\nu|$ $(\epsilon<0)$ is a hole propagating towards $-\hat{\mathbf{z}}$ $(\mathbf{q}\cdot\hat{\mathbf{n}}<0)$; if $n_z<0$, then $\nu_2=+|\nu|$ $(\epsilon<0)$ is a hole propagating towards $+\hat{\mathbf{z}}$, and $\nu_1=-|\nu|$ $(\epsilon>0)$ is a particle propagating towards $-\hat{\mathbf{z}}$.

Before we discuss (60) with $\Delta^{\hat{n}}$ given by KK, we make a detour and solve it analytically for the following "piecewise-constant gap" model:

$$d_{i\mu} = \begin{cases} d_{i\mu}^A & \text{if } z<-\xi_0/2 \\ \frac{1}{2}(d_{i\mu}^A+d_{i\mu}^B) & \text{if } |z|<\xi_0/2 \\ d_{i\mu}^B & \text{if } z>\xi_0/2 . \end{cases} \tag{62}$$

We shall denote these three regions by A, C, and B, respectively. In each region the gap $\Delta^{\hat{n}}_{A\,(B)\,(C)}$ are constant and the solutions are as shown in (61). We shall call the corresponding coefficients A,B,C instead of G. Since we have a first-order differential equation (60),[14] the wave functions Z_1,Z_2 are continuous across the boundaries. Define

$$M_{B,z} \equiv \begin{bmatrix} e^{i\nu_B z} & \frac{\Delta_B^{n\dagger}}{E+\epsilon_B} e^{i\nu_B z} \\ \frac{\Delta_B^{n}}{E+\epsilon_B} e^{-i\nu_B z} & e^{-i\nu_B z} \end{bmatrix} ,$$

and similar matrices for A and C. The continuity equations give

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = M^{-1}_{A,-\xi_0/2} M_{C,-\xi_0/2} M^{-1}_{C,\xi_0/2} M_{B,\xi_0/2} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} .$$

We can extract the transmission coefficients easily from this equation. For a particle incident from the A phase, $n_z>0$ and $B_2=0$. The transmission coefficient, being defined as $(j_1)_T/(j_1)_I$ (where we have neglected the j_2 which is of order Δ/ϵ_F smaller), is

$$\tilde{T} = \left. \frac{|B_1|^2 S_B^2}{|A_1|^2 S_A^2} \right|_{B_2=0} , \tag{63}$$

and hence

$$\tilde{T} = \frac{S_A^2 S_B^2 S_C^4}{\left| (1-\rho_A\rho_C^*)(1-\rho_C\rho_B^*) e^{-i\nu_c\xi_0} + (\rho_C-\rho_A)(\rho_B^*-\rho_C^*) e^{i\nu_c\xi_0} \right|^2} , \tag{64}$$

where $\rho_A \equiv \Delta_A^{\hat{n}}/(E+\epsilon_A)$, $S_A^2 \equiv 1-|\rho_A|^2$, etc. For a hole incident from the A phase $(n_z<0)$ then a similar argument leads to

$$\tilde{T} = \left. \frac{|B_2|^2 S_B^2}{|A_2|^2 S_A^2} \right|_{B_1=0} . \tag{65}$$

$\tilde{T}$ is given explicitly as in (64) except for the new ρ's and S's. Thus $\tilde{T}_\pm(\hat{\mathbf{n}})$ for particles are the same as $\tilde{T}_\mp(-\hat{\mathbf{n}})$ for holes (as $\Delta_\pm^{-\hat{n}} = -\Delta_\mp^{\hat{n}}$). Similarly, one can show that, for given E, $\hat{\mathbf{n}}$, and spin, the $\tilde{T}$ values are the same for excitations incident from the B phase.[15] The $\pm$ signs here denote whether $\Delta_\pm^{\hat{n}}$ is used. It has the physical meaning that the z_s component of spin (defined in Sec. II) for the particles involved $(|\xi|^2>|\eta|^2)$ has expectation values $\gtrless 0$, while the holes involved $(|\xi|^2<|\eta|^2)$ would have expectation values $\lesseqgtr 0$.

For the KK order parameter (58), we integrate (60) numerically. $\tilde{T}$ is found from (63) and (65). We find that the properties of the transmission coefficients just mentioned also hold.[15]

After obtaining the relevant transmission coefficients, it is easy to obtain the transport properties. If we denote by $N(\epsilon)$ the density of states for the normal fluid at quasiparticle or quasihole energy ϵ, then, since the group velocity of the excitations if $v_F(\epsilon/E)\hat{\mathbf{n}}$ to lowest order in Δ/ϵ_F, the transmitted flux of excitations incident from A to B of a particular spin is (since $n_z>0$ for particles and $n_z<$ for holes)

$$dj_\pm = d\epsilon \frac{d\Omega}{4\pi} N(0) v_F n_z \frac{\epsilon}{E} f(E) \tilde{T}_\pm , \tag{66}$$

where $\tilde{T}$ is the relevant transmission coefficient, $f(E)$ is the Fermi distribution function, given by

$$f(E) = \frac{1}{e^{\beta E}+1}$$

$(\beta^{-1}=k_B T$, T is temperature) and we have assumed particle-hole symmetry and replaced $N(\epsilon)$ by $N(0)$. The net flux of quasiparticle or hole energy from A to B cancels exactly with its backward counterpart: thus no net flow of energy exists when the two phases have the same temperature, an expected result.

However, when a temperature difference of ΔT exists, the energy flux is given by

$$W = \Delta T \int_{-\infty}^{\infty} d\epsilon \int_{>} \frac{d\Omega}{4\pi} \frac{E^2/k_B T^2 e^{\beta E}}{(e^{\beta E}+1)^2} \times N(0) v_F |n_z| \frac{|\epsilon|}{E} (\tilde{T}_+ + \tilde{T}_-) \tag{67}$$

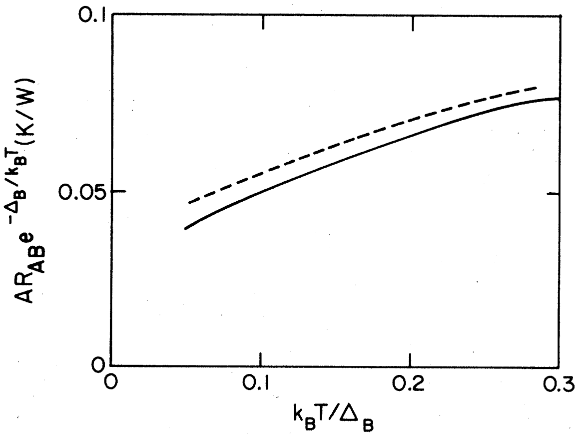

FIG. 2. Resistance of the ^{3}He A-B phase boundary for unit area multiplied by $e^{-\Delta_B/k_BT}$ plotted against k_BT/Δ_B. Solid line, KK order parameter; dashed line, piecewise-constant model.

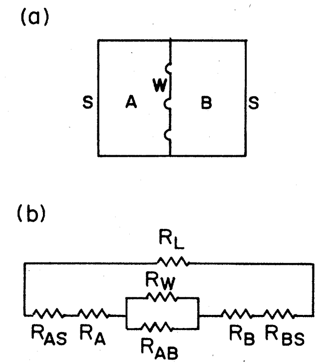

FIG. 3. Schematic diagram for a feasible set up for measuring the A-B boundary resistance (a) and its "circuit diagram" for heat flow (b).

where the $>$ sign indicates that the integral is confined to directions of $\hat{\mathbf{n}}$ pointing from A to B (using the properties of $\tilde{T}$ mentioned). The evaluation of the integral can be made more efficient by noting that the gap $\Delta^{\hat{n}}$ is independent of the sign of n_y. Since we have also assumed particle-hole symmetry and replaced $N(\epsilon)$ by $N(0)$, particles and holes contribute an equal amount. Also by the symmetry[16] $\tilde{T}_{\pm}(n_x,n_y,n_z)=\tilde{T}_{\mp}(-n_x,-n_y,n_z)$, the $\tilde{T}_{\pm}$ contributes equally. The Kapitza resistance of the phase boundary is given by

$$R_{AB}=\frac{\Delta T}{WA} \tag{68}$$

where A is the area.

The results for both the piecewise-constant model (62) and the KK order parameter [(51), (52)] are given in Fig. 2, where we have plotted $AR_{AB}e^{-\beta\Delta_B}$ versus k_BT/Δ_B. Note that R_{AB} varies roughly exponentially with temperature.[17]

B. Experimental feasibility

Here we consider a practical way of measuring the resistance calculated in the Sec. IV A. We shall in particular show that an experimental setup similar to that used by Osheroff and Cross[6] for surface-tension measurements is favorable. This arrangement is schematically as shown in Fig. 3(a), with the "circuit diagram" for heat flow in Fig. 3(b). We shall almost exclusively consider the case that the pressure is the melting pressure without further warning.

Let us consider the particular example in which the middle Cu plate W, 12-μm thick, is drilled with about 150 40-μm-diam holes. A small temperature gradient causes the A and B phases to exist in separate compartments, with the phase boundary just at the holes. The ends of the set up are sinters S which allow cooling and temperature measurement. Since we are just examining the experimental feasibility we ignore complications like the bulging of the phase boundary and possible effects of textures on the boundary due to the proximity to the walls. The resistance of the wall separating the two phases R_W is dominated by the Kapitza resistance between the Cu plate and the superfluid. To estimate this, we use the Kapitza resistance R_K of sintered Cu with ^{3}He-B in the millikelvin region,[18] $AR_KT\approx10^3$ m^2 K^2 W^{-1}, where A is the actual area of the sinter. We shall assume this to be true also for the Cu plate, which has an area of order $A\sim1$ cm^2. Thus the resistance R_W has order 10^{10} K/W. We shall see that this is much larger than the R_{AB} involved in the experimental situations in mind, and hence we can assume that the wall is practically an "open circuit."

The resistance of the bulk B phase is already treated theoretically by Pethick *et al.*[19] As $T\rightarrow0$, the thermal conductivity K roughly obeys $KT\approx6.9$ erg/sec cm, and assuming the B-phase dimension is 1×1 cm^2, we give a resistance of order 1×10^3 K/W in the millikelvin region. For the A phase we shall only give a very rough order-of-magnitude estimate. Let us assume $K\sim K(T_c)\times(T/T_c)^{\alpha}$, for both $K=K_{\parallel},K_{\perp}$, $\alpha=\alpha_{\parallel},\alpha_{\perp}$, where $\parallel$ and $\perp$ indicates conduction parallel and perpendicular to the $\hat{\mathbf{l}}$ vector, and $K(T_c)$ is the value at T_c, which is given by (the normal-state value) $K(T_c)T_c=10.7$ erg/sec cm^3.[4] Thus except for the $(T/T_c)^{\alpha}$ factor, the resistance would be the same order as that of B phase. Now $\alpha_{\parallel}$ and $\alpha_{\perp}$ are likely to be around -2 to -1 and 0 to 1, respectively (Refs. 20 and 21). Thus the resistance of A phase is about (at most a few times higher than) that of B phase. The bulk phases are in series with the sinter, and assuming a 10-g sinter of 70-nm Ag powder, the resistance R_{BS} is of order 1.7×10^5 K/W in the millikelvin region.[18] Hence we need the resistance R_{AB} to be $R_{AB}\gtrsim1.7\times10^5$ K/W so that an experiment is at all possible. We can easily estimate that at $k_BT=0.1\Delta_B, 0.12\Delta_B, 0.15\Delta_B$, R_{AB} for our arrangement (area of A-B surface $\approx2\times10^{-3}$ cm^2) is about 5.5×10^5, 1.1×10^5, 2.2×10^4 K/W, respectively, for the KK order parameter. Thus for $T\lesssim0.12\Delta_B\approx0.21T_c$ R_{AB} dominates and hence is measurable. Note that we still have $R_{AB}\ll R_w$, as claimed.

We have implicitly assumed that the mean free path of the quasiparticle, λ, is much shorter than the sample size, so that we can use the concept of bulk heat conductivity K. For the B phase, Pethick *et al.*[19] estimate $\lambda \approx (0.2\ \mu\text{m}) e^{\Delta_B/k_B T}$, and is about 0.44, 0.8, and 0.06 cm at $k_B T = 0.1\Delta_B, 0.12\Delta_B, 0.15\Delta_B$. Thus at $0.12\Delta_B$, the previous argument applies. At $T \lesssim 0.1 \times \Delta_B \approx 0.17 T_c$, however, the bulk heat conductivity is meaningless. We need to reconsider then the resistance of the bulk phases. Let us for simplicity consider the opposite limit $\lambda \gg 1$ cm. In this case the ballistics of Greaves and Leggett[2] applies in the bulk phase. Clearly for the B phase the resistance is practically zero, as the excitations are never texturally reflected. For the A phase, we actually need to know the texture to make an estimate. However, we can easily obtain an upper bound for the resistance by referring to the work of Greaves and Leggett: In their texture II A, where all the excitations must have at least Δ_A to pass through, the resistance is estimated to be $(0.15)(T/T_c) e^{\Delta_A/k_B T}$ cm^2 KW^{-1}, or about 2.6×10^3 KW^{-1} for our sample at $k_B T = 0.1\Delta_B$ ($\Delta_A = \sqrt{1.32}\Delta_B$). In our case the "resistance of A phase" would be much lower, as the texture will generally allow some "easier" paths for the excitations. The important point here is that it still would not dominate the R_{AB} contribution.

We note that we have also implicitly assumed that the resistance R_{AB} is dominated by quasiparticle and hole transport. One may worry about whether the contribution by the collective modes would appreciably affect our calculation of R_{AB}. To give an upper bound for the contribution to the energy transfer, we assume that all collective modes propagate without hindrance across the phase boundary. We use the radiation formula for the power transfer per unit area $(\pi^2/60)(k_B^4/c^2\hbar^3)T^4$; this gives a transfer of at most $(\pi^2/15)(k_B^4/c^2\hbar^3)T^3\Delta T$ when a temperature difference ΔT exists, where c is the velocity of the mode concerned. Even for spin waves (lowest c), at melting pressure and $k_B T \sim 0.1\Delta_B$ we estimate a resistance for unit area of $\sim 10^6$ K cm^2/W, which is much larger than that of quasiparticles ($\sim 10^3$ K cm^2/W). Thus the heat conduction is mainly by the quasiparticle or quasiholes, and the contribution from collective modes are negligible.

Lastly, we worry about the heat leak R_L. We do not have a very reliable estimate of it. However, in view of the heat-flow experiment by Johnson *et al*,[22] if we assume that all the (magnetic) temperature differences (~ 0.5 mK) are due to the heat leak (estimated to be 0.73×10^{-3} erg/sec) we obtain $R_L \sim 10^7$ K/W $\gg R_{AB}$ at $k_B T = 0.12\Delta_B$ which is as required for a feasible experiment.

We should remark that though we have given our estimates all at the melting pressure, there is practically no difficulty in doing the experiment at a slightly lower pressure, provided we can achieve the necessary low temperature for R_{AB} to dominate and a sufficiently large magnetic field for stabilization of the A phase.

Thus we can conclude that we have a possible experimental setup for measuring our predicted Kapitza resistance of the phase boundary, the experimental conditions are $T \lesssim 0.21 T_c$ and magnetic field of several kilogcauss, which is possible with present cryogenics.

ACKNOWLEDGMENTS

The author would like to thank Professor A. J. Leggett for his suggestion of the problem and his patient guidance throughout the course of this work. This research is supported by the National Science Foundation under Contract No. DMR83-15550.

APPENDIX A

In this appendix we "prove" that the spin of a quasiparticle is $\mathbf{S} = \int_r (f^\dagger \boldsymbol{\sigma} f - g^\dagger \boldsymbol{\sigma}^* g)$ using the more conventional way of taking expectation values. The spin of the quasiparticle is defined by subtracting the ground-state expectation value from the $|\Phi_1\rangle$ value

$$\langle \mathbf{S} \rangle \equiv \int_r (\langle \Phi_1 | \psi_\alpha^\dagger \sigma_{\alpha\beta} \psi_\beta | \Phi_1 \rangle - \langle \Phi_0 | \psi_\alpha^\dagger \sigma_{\alpha\beta} \psi_\beta | \Phi_0 \rangle) \ .$$

To evaluate this, we insert $1 - \sum_m |\Phi_m\rangle\langle\Phi_m|$, where $|\Phi_m\rangle$ is a complete set of states. $|\Phi_0\rangle\langle\Phi_0| + |\Phi_1\rangle\langle\Phi_1|$ gives the terms (note $\langle\Phi_1|\psi|\Phi_1\rangle = \langle\Phi_0|\psi|\Phi_0\rangle = 0$)

$$\int_r (f_\alpha^* \sigma_{\alpha\beta} f_\beta - \tilde{g}_\alpha \sigma_{\alpha\beta} g_\beta^*) = \int_r (f^\dagger \boldsymbol{\sigma} f - g^\dagger \boldsymbol{\sigma}^* g)$$

(the equality follows by observing that $g^\dagger \boldsymbol{\sigma}^* g$ is real). For a state $|\Phi_m\rangle \neq |\Phi_0\rangle$, such that $\langle \Phi_m | \psi_\beta | \Phi_1 \rangle \neq 0$, then we must be annihilating a particle from the condensate other than the one in $|\Phi_1\rangle$. Then there must exist a corresponding state $|\Phi_{m'}\rangle$, corresponding to annihilating the "same particle" from the condensate in $|\Phi_0\rangle$, so that $\langle \Phi_m | \psi_\beta | \Phi_1 \rangle = \langle \Phi_m | \psi_\beta | \Phi_0 \rangle + O(1/N)$, where N is the (approximate) number of particles in the system. Thus these states do not contribute significantly to $\langle \mathbf{S} \rangle$. This completes the proof.

The same procedure can be used to justify our expressions for the number density, number current, and spin current of the excitation by using the field operators

$$\psi_\alpha^\dagger \psi_\alpha \ ,$$

$$\left[\frac{\hbar}{2mi}\right][\psi_\alpha^\dagger \nabla \psi_\alpha - (\nabla \psi_\alpha^\dagger)\psi_\alpha] \ ,$$

and

$$\left[\frac{\hbar}{2mi}\right][\psi_\alpha^\dagger \sigma_{\alpha\beta} \nabla \psi_\beta - (\nabla \psi_\alpha^\dagger)\sigma_{\alpha\beta}\psi_\beta] \ ,$$

respectively.

APPENDIX B

Here we provide some useful geometric information about the order parameter of the A-B phase boundary and calculate the $\Delta_\pm^n$ defined in text. The boundary conditions are already given in (54). In the A phase, the dipole energy is minimized by $\hat{\mathbf{d}} = \hat{\mathbf{l}}$ ($\hat{\mathbf{d}} = -\hat{\mathbf{l}}$ corresponds just to choosing $\phi^{(1)}, \phi^{(2)}$ in the opposite directions), while in the B phase, we have $R_{i\mu} = R_{i\mu}(\hat{\omega}, \Theta)$ where $\Theta = \cos^{-1}(-\frac{1}{4})$.

We shall first consider the case $\hat{\phi}_\mu^{(1)} = \hat{\mathbf{z}}$ (54b) as shown in Fig. 1. $R_{i\mu}$ then rotates $\hat{\mathbf{y}}$ to $\hat{\mathbf{z}}$. Thus we see that $\hat{\omega}$ must be of the form

$$(\cos\beta)\hat{\mathbf{x}}+(1/\sqrt{2})(\sin\beta)\hat{\mathbf{y}}+(1/\sqrt{2})(\sin\beta)\hat{\mathbf{z}} .$$

Now $\sin\Theta=\sqrt{15}/4$ (the other sign corresponds to choosing an opposite $\hat{\omega}$). Hence

$$R_{i\mu}=\tfrac{5}{4}\omega_i\omega_\mu-\tfrac{1}{4}\delta_{i\mu}+\epsilon_{i\mu\nu}\omega_\nu\frac{\sqrt{15}}{4} .$$

The conditions $\hat{\mathbf{y}}^R=d^R=\hat{\mathbf{z}}$ are then $R_{yx}=R_{yy}=0$, $R_{yz}=1$ and hence $\cos\beta=\sqrt{3/5}$, $\sin\beta=\pm\sqrt{2/5}$. The R matrices are

$$R=\begin{bmatrix} \frac{1}{2} & \pm\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \\ \pm\sqrt{3}/2 & -\frac{1}{2} & 0 \end{bmatrix}, \quad \text{(B1)}$$

respectively.

Now we choose the new spin axes as described in the text, which we shall always distinguish by subscripts s. Since $(\hat{\mathbf{n}}^R)_i=R_{i\mu}n_\mu$, $\hat{\mathbf{z}}_s(n)=\hat{\mathbf{d}}\times\hat{\mathbf{n}}^R=\hat{\mathbf{y}}\times\hat{\mathbf{n}}^R$ implies

$$\hat{\mathbf{z}}_s(n)=(R_{z\mu}n_\mu,0,-R_{x\mu}n_\mu)/\sqrt{L} ,$$

where $L\equiv(R_{x\mu}n_\mu)^2+(R_{z\mu}n_\mu)^2$. We choose $\hat{\mathbf{y}}_s=\hat{\mathbf{y}}$, which is always possible as shown in the text. Hence

$$\hat{\mathbf{x}}_s(n)=(-R_{x\mu}n_\mu,0,-R_{z\mu}n_\mu)/\sqrt{L} .$$

The matrix $T_{\mu_s'\nu}$, which rotates from xyz to $x_sy_sz_s$ is $T_{\mu_s\nu}=\hat{\boldsymbol{\mu}}_s\cdot\hat{\boldsymbol{\nu}}$. Thus

$$\begin{aligned}R_{i_s\mu}n_\mu&=T_{i_sj}R_{j\mu}n_\mu\\&=(-[1-(R_{y\mu}n_\mu)^2]^{1/2},R_{y\mu}n_\mu,0)\\&=(-(1-n_z^2)^{1/2},n_z,0)\end{aligned} \quad \text{(B2)}$$

using (B1). The gap matrix, with spin in new coordinates and momentum still in old coordinates, is therefore, for the A phase, as $\hat{\mathbf{d}}=\hat{\mathbf{y}}_s$, $\tilde{\phi}^{(1)}=\hat{\mathbf{z}}$, $\tilde{\phi}^{(2)}=\hat{\mathbf{x}}$,

$$\Delta_A^{\hat{n}}=\Delta_n\begin{bmatrix} i & 0 \\ 0 & i\end{bmatrix}(\hat{\mathbf{z}}+i\hat{\mathbf{x}})\cdot\hat{\mathbf{n}} ,$$

while for the B phase

$$\Delta_B^{\hat{n}}=\Delta_B\begin{bmatrix}(1-n_z^2)^{1/2}+in_z & 0\\ 0 & -(1-n_z^2)^{1/2}+in_z\end{bmatrix} .$$

Note both Δ_A^n,Δ_B^n are diagonalized as claimed. The "gaps" are then, as a function of z, given as in (58).

For the second possibility in (51b), we choose then $d^R=\tilde{\phi}^{(1)}=-\hat{\mathbf{z}}$, $\tilde{\phi}^{(2)}=-\hat{\mathbf{x}}$, so $\hat{\mathbf{l}}=\hat{\mathbf{y}}$. Then new spin system is still chosen with $\hat{\mathbf{z}}_s(n)=(d^R\times n)^R$, $\hat{\mathbf{y}}_s=\hat{\mathbf{y}}$. Then now $R_{y_sz}=R_{yz}=-1$. The matrices R are

$$R=\begin{bmatrix} \frac{1}{2} & \mp\sqrt{3}/2 & 0 \\ 0 & 0 & -1 \\ \pm\sqrt{3}/2 & \frac{1}{2} & 0 \end{bmatrix}, \quad \text{(B3)}$$

so

$$\Delta_A^n=\Delta_A\begin{bmatrix} i & 0 \\ 0 & i\end{bmatrix}(-\hat{\mathbf{z}}-i\hat{\mathbf{x}})\cdot\hat{\mathbf{n}} , \quad \text{(B4)}$$

$$\Delta_B^n=\Delta_B\begin{bmatrix}(1-n_z^2)^{1/2}-in_z & \\ & -(1-n_z^2)^{1/2}-in_z\end{bmatrix} . \quad \text{(B5)}$$

Thus except for an overall negative sign, the present $\Delta_\pm^n$ is just the previous $\Delta_\mp^n$. As far as heat transport (and other transport independent of spin directions) is concerned, the results are identical.

---

[1]A. F. Andreev, Zh. Eksp. Teor. Fiz. **46**, 1823 (1964) [Sov. Phys.—JETP **19**, 1228 (1964)].

[2]N. A. Greaves and A. J. Leggett, J. Phys. C **6**, 4383 (1983).

[3]G. Kieselmann and D. Rainer, Z. Phys. B **52**, 267 (1983).

[4]J. C. Wheatley, Rev. Mod. Phys. **47**, 415 (1975).

[5]R. Kaul and H. Kleinert, J. Low Temp. Phys. **38**, 539 (1980).

[6]D. D. Osheroff and M. C. Cross, Phys. Rev. Lett. **38**, 905 (1977).

[7]R. Balian and N. R. Werthamer, Phys. Rev. **131**, 1553 (1963).

[8]For example, consider particles of spin along $\pm\hat{\mathbf{x}}$ moving along the $+z$ axis, where the order parameter changes continuously from an $A1$ phase with $\mathbf{d}=(1/\sqrt{2})(-\hat{\mathbf{z}}+i\hat{\mathbf{y}})$ (pairs polarized along $+\mathbf{x}$) for $z<0$ to a normal phase at $z=0$, then to $\mathbf{d}e=(1/\sqrt{2})(-\hat{\mathbf{z}}-i\hat{\mathbf{x}})$ (pairs polarized along $+\hat{\mathbf{y}}$) for $z>0$. An equation for the spin degree of freedom is needed to completely describe the excitation when $z>0$ since its spin would not be polarized along $\pm\hat{\mathbf{y}}$.

[9]We thus disagree with the statement of Ref. 3 that "In. . .long-wavelength textures the internal state of a propagating wave packet adjusts adiabatically to the "local frame" set by the local superfluid order parameter." See also Ref. 8, where an adiabatic adjustment cannot be possible.

[10]We shall confine ourselves to this form of order parameter in our demonstration of ballistics. The algebra would be much more complicated for a more general form, but we believe that the results are not qualitatively different.

[11]A. J. Leggett, Rev. Mod. Phys. **47**, 331 (1975).

[12]In the remaining part of Sec. III E we shall put spin and space indices as superscripts and subscripts, respectively, to avoid confusion. We also drop the vector notation for momentum vectors when no ambiguity arises.

[13]Note our notation: for a vector in real space, e.g., $\hat{\mathbf{n}}$, $n_i^R\equiv R_{i\mu}n_\mu$ is a vector in spin space, whereas for a vector in spin space, e.g., $\mathbf{d}$, $(d^R)_\mu\equiv d_iR_{i\mu}$ is a vector in real space.

[14]Strictly speaking, at the boundaries between A, B, and C, the order parameter is rapidly varying and so (60), and even the more general (14), fail. We shall ignore these complications. Hence the results are qualitative and for comparison with the numerical calculations below only.

[15]We find the result that $\tilde{T}$, for given $\hat{\mathbf{n}}$ and energy, is independent of whether the excitation is incident from A or B is generally true; however, that $\tilde{T}$ for $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$ are equal is special to the piecewise-constant case and the KK case. The details and consequences of this will appear in future work.

[16]This symmetry is also special to the piecewise constant and KK order parameters, cf. 15.

[17]In the calculation we have assumed that the magnitudes of the energy gaps Δ_A,Δ_B are given by the zero-temperature values

[since $(1-T/T_c)^{1/2} \approx 1$] in the weak coupling limit, $\Delta_A^2 = 1.32\Delta_B^2$.

[18] J. P. Harrison, J. Low Temp. Phys. **37**, 467 (1979).
[19] C. J. Pethick, H. Smith, and P. Bhattacharyya, Phys. Rev. B **15**, 3384 (1977).
[20] O. Valls and A. Houghton, Phys. Lett. **54A**, 143 (1975).
[21] P. Wölfle, Rep. Prog. Phys. **42**, 269 (1979).
[22] R. T. Johnson, R. L. Kleinberg, R. A. Webb, and J. C. Wheatley, J. Low Temp. Phys. **18**, 501 (1974).