PHYSICAL REVIEW B VOLUME 32, NUMBER 4 15 AUGUST 1985

Droplet wave functions for the fractional quantum Hall effect

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(Received 5 February 1985)

Trial wave functions which, in the thermodynamic limit, describe states relevant to the fractional quantum Hall effect are explicitly constructed for up to eight electrons. Estimates are given for the ground-state energies and pair-correlation functions and for the low-lying excitation energies for states associated with the filling factors $\nu=\frac{1}{3}$, $\frac{2}{7}$, and $\frac{2}{5}$. The results help explain why the filling factors $\nu=\nu_m=m/(2m+1)$, $m=1,2,\ldots$, and $\nu=1-\nu_m$ show strong quantum Hall effects.

Any N-electron wave function of a two-dimensional electron gas in a strong perpendicular magnetic field may be written in the form[1]

$$\psi(z,z^*)=P(z,z^*)\prod_{k=1}^{N}\exp(-|z_k|^2/4)\ , \tag{1}$$

where $z_k=x_k+iy_k$ is the electron coordinate in complex notation. [All lengths will be in units of the magnetic length, related to the density of a single full Landau level n_1, by $n_1=(2\pi a_L^2)^{-1}$).] When all electrons are restricted to the lowest Landau level (the extreme quantum limit) $P(z,z^*)$ is a polynomial in the z_k's alone and, in addition, if $\psi(z)$ represents an isotropic state this polynomial is homogeneous. Furthermore, it follows from the antisymmetry of $\psi(z)$ that $P(z)$ has the Vandermonde determinant

$$P_v(z)\equiv\prod_{i<j}(z_j-z_i) \tag{2}$$

as a factor. Thus, any state can be represented by an antisymmetric polynomial $P(z)$ or equivalently by a symmetric polynomial $Q(z)$. We take the view[2-5] that the fractional Hall ground states occur in a hierarchy starting from those proposed in Laughlin's classic paper,[1]

$$P(z)=[P_v(z)]^{2p+1}\ , \tag{3}$$

which describe states with inverse filling factor $\nu^{-1}=2p+1$. The low-lying neutral excitations from the Laughlin states, at least at large momentum[6-8] are weakly interacting quasiparticle-quasihole pairs. The state describing a quasihole at the origin is[1]

$$P^{(h)}(z)=\left(\prod_{k=1}^{N}z_k\right)[P_v(z)]^{2p+1}\ , \tag{4a}$$

that describing a quasiparticle at the origin is[1]

$$P^{(e)}(z)=\left(\prod_{k=1}^{N}2\partial/\partial z_k\right)[P_v(z)]^{2p+1}\ , \tag{4b}$$

and the quasiparticle-quasihole excitation energy is

$$\Delta\equiv\langle\psi^{(h)}|U|\psi^{(h)}\rangle+\langle\psi^{(e)}|U|\psi^{(e)}\rangle-2\langle\psi_0|U|\psi_0\rangle\ , \tag{4c}$$

where U is the interaction part of the Hamiltonian and $|\psi^{(e)}\rangle$, $|\psi^{(h)}\rangle$, and $|\psi_0\rangle$ are, respectively, the particle, hole, and the $\nu^{-1}=2p+1$ Laughlin states.

The hierarchy of wave functions is defined by the following equations:[5]

$$Q_{i+1}(z)=C(Q_i(z))[P_v(z)]^{2p_i+1}\ , \tag{5a}$$

or

$$Q_{i+1}(z)=C(Q_i(z))^{\dagger}[P_v(z)]^{2p_i+1}\ , \tag{5b}$$

where for an antisymmetric polynomial $C(P(z))$ denotes the particle-hole conjugate polynomial associated with $P(z)$ and for a symmetric polynomial

$$C(Q(z))\equiv C(Q(z)P_v(z))/P_v(z)\ . \tag{6}$$

In Eq. (5b) $Q^{\dagger}(z)$ is the adjoint of $Q(z)$ (Refs. 2 and 9) and is obtained from $Q(z)$ by the replacement of $z_k^{m_k}$ by $(2\partial/\partial z_k)^{m_k}$. In the case of Eqs. (5a) and (5b), $Q_{i+1}(z)$ represents a state at $\nu_{i+1}^{-1}=2p_{i+1}+1\pm1/(\nu_i^{-1}-1)$ in which the quasiholes (quasielectrons) of the $\nu^{-1}=2p_{i+1}+1$ Laughlin state have formed the state associated with $Q_i(z)$. In particular, for $Q_0(z)=[P_v(z)]^2$ and $p_1=1$ Eqs. (5a) and (5b), respectively, give the $\nu=\frac{2}{7}$ and $\frac{2}{5}$ states proposed by Laughlin.[2] These statements are justified at greater length elsewhere[5] but we should emphasize that the hierarchy has been described in somewhat different ways by other workers.[3,4,6] At the nth stage of the hierarchy, we can expect n branches of low-lying excitations, corresponding to the formation of quasiparticle-quasihole pairs at each possible level of the hierarchy. Since the quasiparticle-quasihole pairs at the deepest level of the hierarchy have the smallest-magnitude fractional charges, these excitations should be lowest in energy. Obtaining estimates of these excitation energies for $\nu=\frac{2}{5}$ and $\frac{2}{7}$ has been the main motivation for the calculations described below. Such estimates are of critical importance, in view of the increasingly convincing experimental evidence[10,11] that the states with $\nu=\nu_m=m/(2m+1)$ $m=2,3,\ldots$ and their particle-hole conjugates show the strongest quantum Hall effects of states of those at nonprimitive higher levels of the hierarchy. These states correspond to a sequence, starting from the $\nu=\frac{1}{3}$ state, in which the ground state at one step is formed in the *quasiparticles* of the $\nu=\frac{1}{3}$ state at the next step.

The three ground-state wave functions we are interested in are[12]

$$P_{1/3}(z)=[P_v(z)]^3\ , \tag{7a}$$

$$P_{2/7}(z)=C[P_v^2(z)][P_v(z)]^3\ , \tag{7b}$$

and

$$P_{2/5}(z)=C[P_v^2(z)]^{\dagger}[P_v(z)]^3\ . \tag{7c}$$

The quasihole states we are interested in can be written compactly by defining

$$H(z)\equiv\prod_{k=1}^{N} z_k \ . \tag{8}$$

The polynomials for states where the quasiholes are formed in the parent fluid are $H(z)P_\nu(z)$ $(\nu=\frac{1}{3},\frac{2}{5},\frac{2}{7})$ and these can be shown to have fractional charge νe in each case. The polynomials for the states with the quasiholes formed in the daughter fluid are

$$P_{2/7}^{(dh)}(z)=C(H(z)Q_{1/3}(z))[P_\nu(z)]^3 \ , \tag{9a}$$

and

$$P_{2/5}^{(dh)}(z)=C(H(z)Q_{1/3}(z))^\dagger[P_\nu(z)]^3 \ , \tag{9b}$$

and these can be shown to have charge $\nu e/2$. The quasielectron counterparts for each of these states are obtained by replacing $H(z)$ by $H^\dagger(z)=\prod_k(2\partial/\partial z_k)$.

For a given number of electrons (N) and homogenous degree (M) any antisymmetric polynomial can be expanded in terms of polynomials of the form

$$P(m_1,m_2,\ldots,m_N)=\sum_P \phi(P)z_1^{m_{P(1)}}z_2^{m_{P(2)}}\cdots z_N^{m_{P(N)}}, \tag{10}$$

where m_1 to m_N are the powers in ascending order $(m_i<m_{i+1})$, the sum over P is over all permutations $[P(1),P(2),\ldots,P(N)]$ of $(1,2,\ldots,N)$ and $\phi(P)$ is the sign of the permutation. Thus, apart from the common factor $\prod_{k=1}^N\exp(-|z_k|^2/4)$ and the normalization factor $(2\pi)^{-N/2}2^{-M/2}(m_1!m_2!\cdots m_N!)^{-1/2}$, $P(m_1,m_2,\ldots,m_N)$ is a Slater determinant formed from the normalized single-particle eigenstates

$$\phi_m=z^m\exp(-|z|^2/4)/(\pi m!2^{m+1})^{1/2}$$

for the lowest Landau level in the symmetric gauge. Symmetric polynomials can be expanded in the same way except that $\phi(P)$ does not appear and the restriction for power ordering becomes $m_i\le m_i+1$. All the results discussed below were obtained by starting from the polynomial $P_\nu(z)$, and sometimes $H(z)$, and then performing an appropriate sequence of symbolic manipulations involving particle-hole conjugation and the following operations.

(i) Multiplication of an antisymmetric polynomial by an antisymmetric polynomial.

Here we use

$$P(k_1,k_2,\ldots,k_N)P(m_1,m_2,\ldots,m_N)=\sum_P\sum_{P'}\phi(P')z_1^{e_{P(1)}}\cdots z_N^{e_{P(N)}} \ , \tag{11a}$$

where

$$e_i=k+m_{P'(i)} \ . \tag{11b}$$

The sum over P in Eq. (11a) merely generates the $N!$ terms in the symmetric polynomial defined by Eq. (10) [without the $\phi(P)$]. Thus, it is necessary to sum over P' only and to add $\phi(P')$ to the coefficient in the product of the symmetric polynomial labeled by $(e_1,e_2,\ldots,e_N)$ permuted to ascending order.

(ii) Multiplication of a symmetric polynomial by an antisymmetric polynomial.

In this case

$$Q(k_1,k_2,\ldots,k_N)^\dagger P(m_1,m_2,\ldots,m_N)=\sum_P\phi(P)\sum_{P'}\phi(P')z_i^{e_1}\cdots z_N^{e_N} \ . \tag{12}$$

For a given P' if $e_i=e_j$ for any i or j we can drop the term since its contribution will vanish when P is summed over. Again we need to sum over P' only but the coefficient of the term in the product polynomial labeled by $(e_1,e_2,\ldots,e_N)$ permuted to (strictly) ascending order is incremented by $\phi(P')\phi(P^*)$, where P^* is the permutation which results in the ascending order.

(iii) "Differentiation" of an antisymmetric polynomial by a symmetric polynomial.

$$Q(k_1,k_2,\ldots,k_N)^\dagger P(m_1,m_2,\ldots,m_N)=2^M\sum_P\phi(P)\sum_{P'}\phi(P')\prod_{i=1}^N f_i z_i^{e_i} \ , \tag{13}$$

where

$$f_i=\begin{cases} m_{P'(i)}!/(m_{P'(i)}-k_i)! & k_i\le m_{P'(i)} \ ,\\ 0 & k_i>m_{P'(i)}\end{cases} \tag{14a}$$

and

$$e_i=m_{P'(i)}-k_i \ . \tag{14b}$$

This is like (ii) except that the coefficient of the appropriate term in the result polynomial is incremented by $2^M\phi(P')\phi(P^*)\prod_{k=1}^N f_i$.

(iv) Division of an antisymmetric polynomial by $P_\nu(z)$.

Given $N,M=\sum_{i=1}^N m_i$ there will be a finite number, say I, of distinct antisymmetric polynomials

$$P(m_1^{(i)},m_2^{(i)},\ldots,m_N^{(i)})\equiv P^{(i)}(i=1,\ldots,I) \ .$$

Any antisymmetric polynomial $\tilde{P}(z)$ in this class can be expanded

$$\tilde{P}(z)=\sum_{i=1}^I c_iP^{(i)}(z) \ . \tag{15}$$

$\tilde{P}(z)/P_\nu(z)\equiv\tilde{Q}(z)$ is a symmetrical polynomial of homogeneous degree $M_q=M-N(N-1)/2$. It is readily established that the number of symmetric polynomials $[Q^{(i)}(z)]$ in this class is also I. Using the methods described above we can determine the elements of the $I\times I$ matrix, T_{ij}, defined by

$$Q^{(i)}(z)P_\nu(z)=\sum_j T_{ij}P^{(j)}(z) \ . \tag{16}$$

Thus,

$$\tilde{Q}(z)=\sum_{i=1}^I c_iP^{(i)}(z)/P_\nu(z)=\sum_j d_jQ^{(j)}(z) \ , \tag{17a}$$

where

$$d_j=\sum_{i=1}^I c_iT_{i,j}^{-1} \ . \tag{17b}$$

We are now ready to present our results, starting with those for the ground-state energies and pair-correlation functions. These we estimate from expressions which emphasize the center of the droplets in order to minimize finite-size corrections. For the pair-correlation function we take[13]

$$g(r)\simeq\frac{(2\pi)^2}{n(0)n(r)}\sum_{s\ne0}\langle\psi|n_sn_0|\psi\rangle\frac{r^{2s}}{2^ss!}\exp\left(\frac{-r^2}{2}\right) \ ; \tag{18}$$

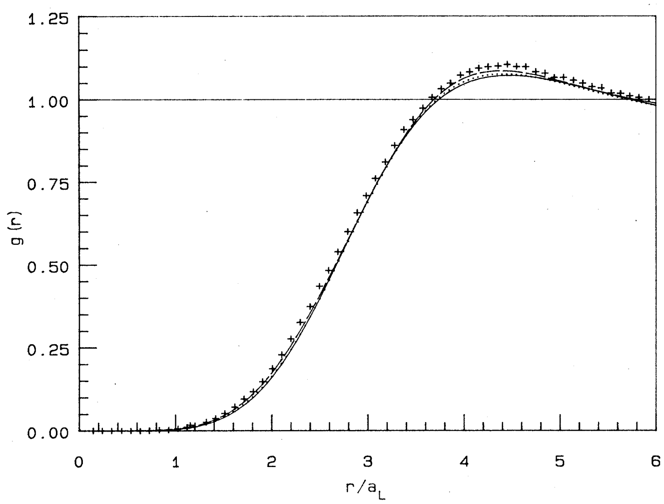

FIG. 1. Pair-correlation functions for the $\nu=\frac{1}{3}$ Laughlin state estimated from droplet wave functions as described in the text. The solid line is for $N_e=6$, the dotted line for $N_e=7$, and the dashed line for $N_e=8$. The crosses are from the 2DOCP calculations of Ref. 15 and represent the $N_e=\infty$ limit.

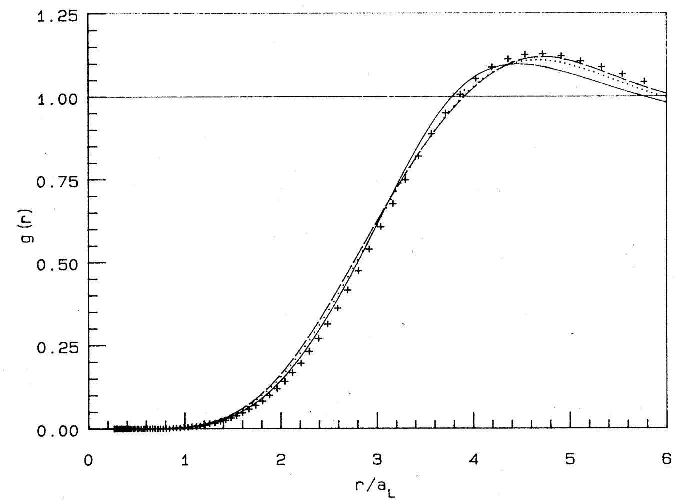

FIG. 2. Pair-correlation functions for the $\nu=\frac{2}{7}$ hierarchy state estimated from droplet wave functions with five (solid line), six (dotted line), and seven (dashed line) electrons. The crosses give the $N_e=\infty$ limit as calculated in Ref. 5.

i.e., we approximate $g(r)$ by the correlation between two points in the finite droplet, one of which is located at the center. In Eq. (18) n_m is an occupation-number operator. In Fig. 1 we compare pair-correlation functions calculated from Eq. (18) for $P_{1/3}(z)$ and $N=6$, 7, and 8 with the $N=\infty$ limit which is available from Monte Carlo calculation for the two-dimensional one-component plasma.[1,14,15] We see that even for $N=6$ the agreement is acceptable. This gives us confidence that our droplets are large enough for their centers to be similar to the infinite system. To obtain $P_{2/7}(z)$ and $P_{2/5}(z)$ we must first determine $Q_{1/2}(z)\equiv C(P_\nu^2(z))$. The number of electrons in the particle-hole conjugate of $[P_\nu(z_1,\ldots,z_{N_e})]^3$ is $N_h=2(N_e-1)$. Results for $Q_{1/2}(z)$ for $N_e=2$, 3, and 4 ($N_h=2$, 4, and 6) are summarized in Table I. These four polynomials have homogeneous degree $N_h(N_h+2)/4$ and among polynomials of this degree, are optimal in the sense of having small-magnitude values when pairs of coordinates are nearly equal. In Fig. 2 we show curves for $g(r)$ calculated for $\nu=\frac{2}{7}$ from droplet wave functions of 5, 6, and 7 electrons using Eq. (19). These are compared with their infinite-system limit as calculated in Ref. 5.

The energy per electron is estimated from $g(r)$ using

$$\frac{E}{N}\simeq\int\frac{d\mathbf{r}\,e^2n(r)[1-g(r)]}{r}\ . \qquad (19)$$

Results are given in Table II. For $\nu=\frac{1}{3}$ and $\frac{2}{7}$ the values obtained are quite close to their $N=\infty$ limits even for $N=4$. This is in contrast with results presented in Rezayi and Haldane's related recent study of electrons on the surface of a sphere.[8] They find a large systematic overestimate of the energy-per-electron magnitude which decreases like N^{-1} as N increases. The difference, as well a difference in the location of the first peak in the pair-correlation function for $\nu=\frac{1}{3}$, may be traced to their use of chord rather than geodesic distances. The energies calculated for the $\nu=\frac{2}{5}$ state are farther from their $N=\infty$ limit and, in fact, it has not been rigorously proven that the $N=\infty$ limit yields the assumed state of uniform-electron density. However, the main conclusion we wish to draw below follows from a qualitative consideration and the present trial wave functions are, at least, adequate for illustrating the relevant feature.

The quasiparticle-quasihole excitation energies we have calculated are listed in Table III. For $\nu=\frac{1}{3}$ our estimates

TABLE I. $Q_{1/2}(z_1,\ldots,z_{N_h})$ for $N_h=2,4,6$. The normalization was chosen so that the coefficients become integers. Polynomials for the larger values of N_h are available from the authors.

N_h	$Q_{1/2}(z_1,\ldots,z_{N_h})$
2	$Q(0,2)$
4	$3Q(0,0,3,3)+2Q(1,1,1,3)$
6	$10Q(0,0,0,4,4,4)-9Q(0,0,2,2,4,4,)$
	$+54Q(0,1,1,2,4,4)+3Q(0,2,2,2,2,4)$
	$-15Q(1,1,1,1,4,4)+12Q(1,1,2,2,2,4)$
	$+Q(2,2,2,2,2,2)$

TABLE II. Energies per electron in units of e^2/a_L as calculated from Eq. (19) for various-droplet wave functions. The $N=\infty$ numbers are from Ref. 5.

		$-E/N(e^2/a_L)$	
N	$P_{1/3}(z)$	$P_{2/7}(z)$	$P_{2/5}(z)$
4	0.373	0.398	0.574
5	0.389		
6	0.405	0.394	0.398
7	0.415		
8	0.418	. . .	. . .
∞	0.410	0.382	0.433

TABLE III. Quasiparticle-quasihole excitation energies in units of e^2/a_L. For the $\nu=\frac{2}{5}$ and $\frac{2}{7}$ states, values are given for excitations in both the parent fluid and for the lower-lying excitation in the daughter fluid as discussed in the text.

		$\Delta/(e^2/a_L)$	
N	$P_{1/3}(z)$	$P_{2/7}(z)$	$P_{2/5}(z)$
	Excitations in parent fluid		
3	0.0782		
4	0.0656	0.0413	0.2264
5	0.0711		
6	0.0872	0.0626	0.1345
7	0.1020		
8	0.1101		
	Excitations in daughter fluid		
4		0.0159	0.0770
6		0.0110	0.0527

are consistent with previous results.[9,16] When the excitations are made in the parent fluid our estimates for $\nu=\frac{2}{5}$ and $\frac{2}{7}$ are consistent with the approximate $\nu^{5/2}$ behavior expected on the basis of the charges of the quasiparticles and their sizes. However, as discussed above, for the hierarchy states at $\nu=\frac{2}{5}$ and $\frac{2}{7}$ it is possible to create lower-energy excitations by altering only the part of the wave function associated with the daughter Laughlin fluid. The excitation energy should be particularly low when it occurs in the daughter fluid of the $\nu=\frac{2}{7}$ state since the wave function vanishes as $(z_i-z_j)^3$ when z_i approaches z_j even for the excited state. The numerical results given in Table III estimate the excitation gap for $\nu=\frac{2}{7}$ to be about five times smaller than for $\nu=\frac{2}{5}$. We believe this difference is the reason that the fractional Hall-effect anomalies are much stronger for ν near the latter fraction.

One of the authors (A.H.M) would like to acknowledge stimulating discussions with S. M. Girvin.

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