Dilute instanton gas as the precursor to the integral quantum Hall effect

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An explicit expression is proposed for the renormalization-group equations for σ_{xx}, σ_{xy}, i.e., the magnetoconductance tensor for the two-dimensional disordered electronic system subject to a magnetic background field. The formulas are derived from the "dilute instanton gas" approximation, which is given a clear meaning within the previously derived nonlinear σ-model description for the transport properties of the system.

It is widely recognized by now that electron-localization effects due to random impurities are responsible for the high precision (1 part in 10^7) of the measurements of the quantized Hall conductance observed in GaAs heterojunctions and Si metal-oxide–semiconductor field-effect transistors (MOSFET's).[1]

Nevertheless, when attempting to explain the observed flat steps in the Hall conductance, starting from a microscopic theory, one is faced with a basic difficulty. This difficulty stems from the fact that the standard machinery for dealing perturbatively (diagramatically) with electron-impurity scattering does not lead to an explanation of the surprising and interesting phenomenon of the (integral) quantum Hall effect.

This insight has been presented in a recent series of papers,[2–4] in which the impurity scattering problem has been studied field theoretically. The effective field theoretic description in terms of the unitary nonlinear σ model contains a novel, topological term (θ term) which cannot be detected in the perturbative renormalization-group approach. The essential aspects of the quantum Hall effect, namely, the plateaus of quantized Hall conductance and the existence of extended electronic levels, which are necessary to carry the Hall current, have been shown to be direct consequences of the field theory and, furthermore, topological in origin.

A way of understanding the nature of the theory is via the notion of nonperturbative vacuum states (instantons) which are meaningful in the region of weak coupling ($1/\sigma_{xx}$) or weak localization and which are responsible for the breakdown of perturbation theory.[3,4]

In this Brief Report I reconsider the meaning of instantons for the electronic system. In particular, the dilute instanton gas approximation is found to be well justified in this case and is furthermore exploited in studying the renormalization-group equations for the conductance tensor. The results qualitatively describe the whole region of weak and intermediately strong coupling or localization. It is concluded that the dilute instanton gas serves as the precursor to the quantum Hall effect.

The central issue here is the understanding of the effective Lagrangian that describes the transport of the two-dimensional free-electron system in a random potential, subject to a constant magnetic background field, given by[2]

$$L = \tfrac{1}{4}\sigma_{xx}^{0}\int d^2r\, \mathrm{tr}(\nabla_\mu Q \nabla_\mu Q) + \tfrac{1}{8}\sigma_{xy}^{0}\int d^2r\, \mathrm{tr}(Q[\nabla_x Q, \nabla_y Q]) \ , \qquad (1)$$

where $Q = Q_{ab}^{pp'}(r)$ belongs to the coset space $U(n+m)/U(n)\times U(m)$.[5] The trace stands for the sum over a,b (replica indices) which run from 1 to n,m for $p=1,2$ (advanced, retarded indices). The physics of the disordered electronic system follows after the analytic continuation $n,m \rightarrow 0$.

Among the many aspects which relate this field theory to the disordered system, we mention the following.

(i) The first term is well known to apply to localization problems in which time-reversal symmetry is broken.[6] The connection with renormalization-group improved diagrammatic impurity-scattering techniques has been analyzed and discussed in detail in Ref. 6. The basic result is contained in the renormalization-group equation:

$$\frac{\partial \sigma_{xx}}{\partial \ln L} = -\frac{n+m}{4\pi} - \frac{nm+1}{8\pi^2\sigma_{xx}} + O(\sigma_{xx}^{-2}) \ . \qquad (2)$$

For all n,m (including the case of interest $n,m \rightarrow 0$) the theory is asymptotically free, and in terms of the disordered electronic system, this implies that all states in two dimensions are localized.

(ii) The second term L is a new feature and was discovered in a derivation presented in Ref. 2. The expression multiplying σ_{xy}^{0} is topologically invariant and equals the Jacobian for the mapping of the two-dimensional space into the Grassmann manifold $U(n+m)/U(n)\times U(m)$. The appearance of long-ranged behavior in the underlying one-particle Green's function theory near the sample boundary (boundary currents) imposes boundary conditions on the critical fluctuations $Q(r)$, which are such that

$$\tfrac{1}{8}\int d^2r\, \mathrm{tr}(Q[\nabla_x Q, \nabla_y Q]) = 2\pi i q \ , \qquad (3)$$

with integer q (topological charge). The occurrence of boundary currents therefore leads to the statement that the two-dimensional space ought to be thought of as being compactified to the sphere S^2.

(iii) The parameters σ_{xx}^{0}, σ_{xy}^{0} entering L are dimensionless numbers, measuring the bare conductance in units of e^2/h. They are determined by an underlying massive field theory which sets the length scale in L (phase-coherence length μ^{-1}). They are furthermore given in terms of complicated Kubo-like expressions,[2] which to a good approximation can be evaluated in mean-field theory.[7] This turns out to reduce Ando's self-consistent Born approximation.[8]

(iv) The Lagrangian forms a generalization of the more familiar CP^{N-1} model, which has been intensively studied because of its similarity with the four-dimensional Yang-

Mills theory.[9] As in four dimensions, the field theory exhibits finite-action solutions to the classical equations of motion (instantons). The exact role of instantons in general remains somewhat unclear due to infrared divergences in the semiclassical theory[9,10] (i.e., saddle-point plus one-loop corrections). In the so-called dilute gas of an instanton[11] one replaces the instanton with charge q by $|q|$ widely separated instantons with unit charge. In this way one gains an entropy factor and this variety is supposedly making the dominant contribution at large σ_{xx}^0 at which the density of instantons is small.

The single instanton is furthermore characterized by its position or orientation within the $U(n+m)/U(n)\times U(m)$ degress of freedom and an arbitrary scale size ρ. The arbitrary scale size of the single instanton causes the free energy of the dilute gas to diverge like[10,11]

$$\int \frac{d\rho}{\rho^3} D(\rho),\quad D(\rho)\propto \exp[-4\pi\sigma_{xx}(\rho)] \;,$$
$$\sigma_{xx}(\rho)=\sigma_{xx}^0-\frac{n+m}{4\pi}\ln\mu\rho \;, \tag{4}$$

where $D(\rho)$ is the density of instantons.

This infrared divergence indicates that the average size of the instantons is much larger than the average distance between them, and this then invalidates the original diluteness ansatz. For this reason one has to resort to rigorous, semiclassical methods by summing over all saddle-point (instanton) configurations and then taking the thermodynamic limit.[10]

In this latter respect, the situation is different for the electronic system. In the limit $n,m\to 0$ the one-loop correction to σ_{xx}^0 vanishes and the result shows an ultraviolet divergence; hence, the thermodynamic limit of the dilute gas of instantons does exist in this case.

The ultraviolet divergence is caused by instantons with size on the order of μ^{-1} (magnetic length); the average separation d_i follows from the density [Eq. (4)] and is given by $d_i\sim\exp(2\pi\sigma_{xx}^0)$. On the other hand, the perturbative β function gives rise to a correlation or localization length (a two-loop result):[12]

$$\xi=\xi_0\exp\left[\int_{\sigma_{xx}^0}^{\sigma_{xx}}\beta^{-1}(\sigma)\,d\sigma\right]\propto\exp(2\pi\sigma_{xx})^2 \;. \tag{5}$$

Hence, within the physical volume ξ^2 one can accommodate a large number of widely separated single instantons over a rather large region of σ_{xx}^0. We therefore may expect the methods of Callan, Dashen, and Gross[11] to lead to an improved renormalization-group analysis, beyond what is given by asymptotic freedom alone.

In order to discuss the physical parameters σ_{xx},σ_{xy}, we make use of the source-term formalism of Ref. 4. This amounts to evaluating the shift in the free energy due to the insertion of a slowly varying background field in L:[13]

$$\sigma_{xx}=\sigma_{xx}^0+c_0+\sum_{n=1}^{\infty}c_n\cos(2\pi n\sigma_{xy}^0) \;,$$
$$\sigma_{xy}=\sigma_{xy}^0+\sum_{n=1}^{\infty}\theta_n\sin(2\pi n\sigma_{xy}^0) \;, \tag{6}$$

where c_n,θ_n are functions of σ_{xx}^0 only. We can immediately conclude from these series that the space of physical parameters σ_{xx},σ_{xy} is periodic in σ_{xy} with period 1. Furthermore, on the lines $\sigma_{xy}=\sigma_{xy}^0=\frac{1}{2}+$integer, the Hall conductance is unrenormalized. This observation by itself can be considered as evidence for the earlier conjecture that a phase transition occurs at precisely those Fermi energies for which the bare Hall conductance is half-integer; for integer values the second term in Eq. (1) is immaterial and the usual conclusion of asymptotic freedom (localization in two dimensions) holds.

Furthermore, Eqs. (6) make explicit what we mean by the renormalization of the parameters $\sigma_{\mu\nu}$. The evaluation of these parameters with the aid of the renormalization group consists of eliminating the short-wavelength fluctuations in each topological section q of the functional integral. Such a program can be accomplished by evaluating the partition function of the dilute instanton gas in the background of a large instantion,[11] while accounting for the fact that instantons of opposite charge interact such as to lower the energy, whereas instantons of equal sign do not interact to the same level of approximation. This leads to the result

$$\sigma_{xx}=\sigma_{xx}^0-\cos(2\pi\sigma_{xy}^0)(\sigma_{xx}^0)^2\int\frac{d\rho}{\rho}D(\rho) \;,$$
$$\sigma_{xy}=\sigma_{xy}^0-\sin(2\pi\sigma_{xy}^0)(\sigma_{xx}^0)^2\int\frac{d\rho}{\rho}D(\rho) \;, \tag{7}$$

where the density of instantons $D(\rho)$ to one-loop order is independent of the scale size ρ and is given by[4]

$$D(\rho)=\tilde{D}_0\sigma_{xx}^0\exp(-4\pi\sigma_{xx}^0) \;,$$

where $\tilde{D}_0$ is a positive constant. We can next incorportate the two-loop order result of the perturbative β function [Eq. (2)] by replacing the first term in the expression for σ_{xx} by

$$\sigma_{xx}^0\to\sigma_{xx}^0-(8\pi^2\sigma_{xx}^0)^{-1}\ln\mu L \;.$$

The renormalization-group functions

$$\beta_{xx}=\partial\sigma_{xx}/\partial\ln L,\quad \beta_{xy}=\partial\sigma_{xy}/\partial\ln L$$

are then obtained as[14]

$$\beta_{xx}=-\frac{1}{8\pi^2\sigma_{xx}}-\sigma_{xx}^3\cos(2\pi\sigma_{xy})\tilde{D}_0e^{-4\pi\sigma_{xx}} \;,$$
$$\beta_{xy}=-\sigma_{xx}^3\sin(2\pi\sigma_{xy})\tilde{D}_0e^{-4\pi\sigma_{xx}} \;. \tag{8}$$

A convenient quantity to study is the ratio

$$\frac{\beta_{xy}}{\beta_{xx}}=\frac{\delta\sigma_{xy}}{\delta\sigma_{xx}}=\Gamma\frac{\sin(2\pi\sigma_{xy})}{\Lambda+\cos(2\pi\sigma_{xy})} \;, \tag{9}$$

which is the slope of the renormalization-group flow through the point $(\sigma_{xx},\sigma_{xy})$. The quantities Γ,Λ are given, within the dilute gas approximation, by

$$\Gamma\to 1,\quad \Lambda\to(\tilde{D}_0\sigma_{xx}^4 8\pi^2)^{-1}\exp(4\pi\sigma_{xx}) \;;$$

more generally, they will be a function of both σ_{xx},σ_{xy}. From Eq. (9) one can deduce the renormalization-group flow as illustrated in Fig. 1. The basic assumption is that the fixed point on the lines $\sigma_{xy}=\frac{1}{2}+n$ does exist; this fixed point is given by

$$\Lambda=\Lambda(\sigma_{xx},\sigma_{xy}=\tfrac{1}{2}+n)=1 \;.$$

For very small values of σ_{xx} (indicated by dotted lines in Fig. 1) the flow diagram is extrapolated from the dilute gas

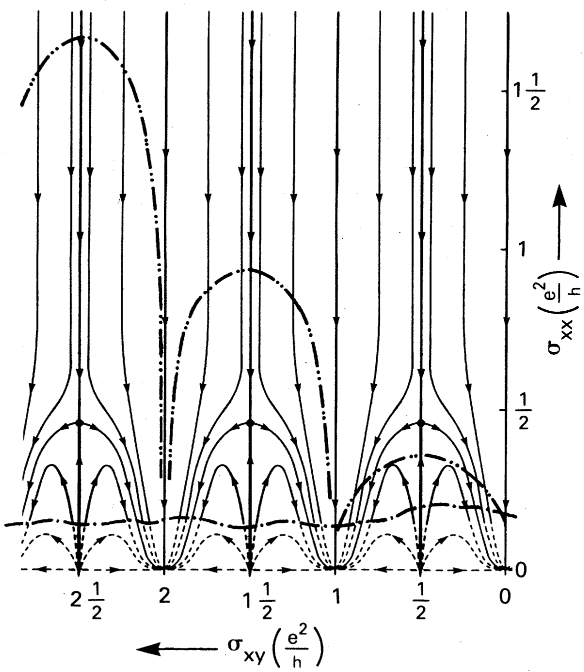

FIG. 1. The renormalization-group flow diagram as obtained from the dilute instanton gas method.

result (the full lines in Fig. 1), keeping in mind the aforementioned global properties of the flow diagram.

The result shows in an intriguing manner what the formal analysis of Refs. 2–4 was all about; upon scaling the disordered system from microscopic length scales (on the order of the cyclotron radius for strong magnetic fields) to macroscopic length scales, the transport properties of the system scale from whatever was microscopically determined, and hence nonuniversal values of the magnetoconductance tensor, to the quantized values $\sigma_{xx}=0$, $\sigma_{xy}=$integer in units of e^2/h. This aspect of universality is synonymous with the experimental observation of the (integral) quantum Hall effect.[15]

We conclude with a series of comments.

(i) The double-dotted lines (– · · – · · –) represent the location of the bare parameters in L for the strong magnetic field case. The result for Landau-level index n are located in between the lines $\sigma_{xy}=n$ and $n+1$.

The maximum of σ_{xx}^0 at the center of the Landau level increases linearly with Landau-level index n and is independent of the details of the random potential and magnetic field.[7] This allows for an estimate for the localization length for the states near the band center ($\sigma_{xy}=\frac{1}{2}+n$) in the higher Landau levels which is given by Eq. (5). The variation in localization length as one moves from the band center [Eq. (5)] to the band tails ($\xi\cong$cyclotron radius) becomes more and more pronounced in the higher Landau levels and this very strong effect might explain the occurrence of mobility edges and bands of extended states in the finite-temperature experiments. For the lower Landau levels, on the other hand, the bare coupling becomes large and the corresponding electronic levels are all strongly localized ($\xi\cong$cyclotron radius) with the exception of the band center.

(ii) The existence of the intermediate coupling fixed points at $\sigma_{xy}=\frac{1}{2}+n$ has general attractive consequences.[16] One expects a singular part of the free energy

$$f_s(\theta,\sigma)=b^{-d}f_s(b^{y}\theta,b^{-\tilde{y}}\sigma) \quad ,$$

where σ and θ are small deviations from the fixed points at σ_{xx}^* and $\sigma_{xy}^*=\frac{1}{2}+n$. Dependent upon the value of $y,\tilde{y}>0$, this singularity is of first order, second order, or a combination of these. For instance, the results of the large-N CP^{N-1} model[17] are translated in exactly this type of flow diagram with $\sigma_{xx}^*=0$; the first-order transition in this large-N limit is represented by taking $(y,\tilde{y})=(2,\infty)$.

On the other hand, it is expected on physical grounds that the localization length diverges for the electronic system as one approaches the singular point (i.e., a second-order transition). In this way, one can associate an infinite number of extended electron levels with this singularity, which are necessary to carry the Hall current.

(iii) In order to understand the above-mentioned experiments one would naively substitute an effective, temperature-dependent length for L, namely, the inelastic scattering length determined by the Coulomb interaction between the electrons.[18] The flow lines of Fig. 1 are then temperature driven (high temperatures to top, low temperatures to bottom) and the results then describe the experiments qualitatively well. In order to make a more detailed contact with experiment, it is desirable to plot the data for the conductance in a way as illustrated in Fig. 1. This will serve as a valuable check on the validity of the ideas on two-parameter scaling as suggested by the noninteracting problem.

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[13]This is accomplished by the replacement $Q\rightarrow U^{-1}QU$ in L with $U\in \mathrm{U}(n+m)$ and given in Ref. 4 (Appendix). The resulting expression leads to Eq. (6) by studying the symmetry properties (i.e., odd and even, respectively) under the transformation $Q(x,y)\rightarrow Q(y,x)$. The functions c_n, θ_n in Eq. (6) originate from the topological sectors $q=n$ and contribute, at a classical level, with factors $e^{-4\pi n\sigma^0_{xx}}$.

[14]The flow diagram of Fig. 1 has been conjectured in D. E. Khmel-

[14]In contrast to the one-loop result, the two-loop result for $D(\rho)$ in Eqs. (7) does not yield an uv-divergent contribution to the conductance parameters. Hence the β functions of Eqs. (8) are not universal. Indeed, the value of $\tilde{D}_0$ depends on the scheme of regularization, which provides a definition of the bare parameter σ^0_{xx} (Ref. 11). Its dependence follows from the fact that the perturbative β function [up to four-loop order (Ref. 6)] defines σ_{xx} up to a constant c: $\sigma^2_{xx}\rightarrow\sigma^2_{xx}+c$. Different choices for the bare parameter result in a different $\tilde{D}_0$ according to

$$\sigma_{xx}\rightarrow\sqrt{\sigma^2_{xx}+c}\sim\sigma_{xx}+\tfrac{1}{2}\sigma^{-1}_{xx}c\ \ldots\ ,$$

$$\tilde{D}_0\rightarrow\tilde{D}_0\exp\left(\frac{2\pi}{\sigma_{xx}}c\ \ldots\right)\ ,$$

such that Eqs. (8) represent the correct asymptotic behavior.

[15]The existence of the uv-stable fixed points at $\sigma_{xy}=n$ have been associated (Ref. 4) with the occurrence of a mass gap (localization), such that the continuous symmetry $\mathrm{U}(n+m)/\mathrm{U}(n)\times\mathrm{U}(m)$ is dynamically restored at length scales larger than the one given by the localization length.

[16]The flow diagram of Fig. 1 has been conjectured in D. E. Khmelnitskii, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 454 (1983) [JETP Lett. **38**, 552 (1983)] as a pedagogical guide to Refs. 2–4.

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