Higher-order fractional quantum Hall effect

W. P. Su
Department of Physics, University of Illinois at Urbana–Champaign,
1110 West Green Street, Urbana, Illinois 61801
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The Hamiltonian describing interacting two-dimensional electrons in a high magnetic field is diagonalized numerically for a small number of particles to obtain the low-lying excitation spectra. The results include estimates of energy gaps for values of ν (the lowest-Landau-level filling factor) equal to certain multiples of $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, and $\frac{1}{11}$. These ν's are characterized by the existence of periodic rigid parent states which generate maximum phase space. The even-denominator cases are markedly different.

In a previous paper[1] we have tried to extract essential physics of the fractional quantum Hall effect from some finite-system calculations. The main conclusion is that for $\nu=p/q$ and q odd, there are precisely q-equivalent ground states. Each ground state is characterized by a periodic parent state with period q. The parent states are rigid in the sense that the only way to change them without raising the energy is to shift them by an integral number of steps. The lowest-energy excitations then consist of kink-antikink pairs. The kinks interpolating between different ground states carry a charge $\pm e/q$. The precise quantization of the conductance relies on a geometrical derivation[1,2] of the Hall conductance. Numerical evidence was presented there for the $\nu=\frac{1}{3}$ and $\nu=\frac{1}{2}$ cases. In this paper we present results for higher fractions. Besides being of theoretical significance, those results should be of interest in view of the recent experiment by Chang *et al.*[3] in which more higher-order fractions are resolved.

Apart from a constant kinetic-energy term, the Hamiltonian[4] is essentially an electron-electron interaction truncated to the lowest Landau level

$$H=\sum_j \epsilon_M C_j^\dagger C_j+\sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4} A_{j_1j_2j_3j_4}C_{j_1}^\dagger C_{j_2}^\dagger C_{j_3}C_{j_4} , \qquad (1)$$

where

$$\epsilon_M=-\frac{3.9e^2}{2L} , \qquad (2)$$

$$A_{j_1j_2j_3j_4}=\frac{e^2}{2l^2N_s}\sum_q{}' \frac{1}{|q|}\delta'_{j_1+j_2,j_3+j_4} \times\exp\left[-\frac{l^2q^2}{2}-iq_x(j_1-j_3)\frac{L}{N_s}\right] , \qquad (3)$$

$$q_x=\frac{2\pi j_x}{L} , \quad j_x=0,\ \pm1,\ \pm2,\ \ldots , \qquad (4)$$

$$q_y=\frac{2\pi j_y}{L} , \quad j_y\equiv j_1-j_4\ (\mathrm{mod}N_s) . \qquad (5)$$

We adopt a square geometry (area $= L^2$) and periodic boundary conditions so that the integer indices labeling the Landau orbitals are between one and N_s (the degeneracy of the Landau level). l is the magnetic length. The filling factor ν is equal to N_e/N_s, N_e being the total number of electrons. Zero momentum is excluded in the sum (3). δ' means momentum conservation $j_1+j_2\equiv j_3+j_4(\mathrm{mod}N_s)$. One important feature of the above Hamiltonian is that it depends only on the differences of the j's. Because of the momentum conservation the total momentum $J=\sum_{i=1}^{N_e} j_i$ is a good quantum number. The energy eigenstates can be classified according to their total momentum. As in Ref. 1 we use

$$|j_1,j_2,\ \ldots\rangle=(C_{j_1}^\dagger C_{j_2}^\dagger\ \cdots)|0\rangle$$

to denote a single-particle Slater state in which the j_1th, j_2th, etc., orbitals are occupied. Because of the symmetry property $E(J)=E(J+N_s/q)$ we can restrict J to a number smaller than or equal to (N_s/q) $(\nu=p/q)$.

Figure 1 displays the low-lying spectra for $\nu=\frac{1}{5}$. From the figure we estimate the $\nu=\frac{1}{5}$ gap to be about 0.01 (the unit of energy is e_2/l), which is certainly smaller than the $\nu=\frac{1}{3}$ gap and the $\nu=\frac{2}{5}$ gap which is about 0.05 from Fig. 2. It is interesting to note that in Figs. 1 and 2 the gap size is fairly independent of the number of particles despite the smallness of the systems. Also notice that the optimal values of J correspond to periodic parent state with period 5 in both Figs. 1 and 2. For example, the parent state $|1,5,6,10,11,15,16,20\rangle$, which has a total J equal to $84\equiv4(\mathrm{mod}20)$, nicely characterizes the ground state at $J=4$, for $N_e=8$ and $N_s=20$. It is worth mentioning that the amplitude of the oscillation of the mean occupation number $\rho(j)=\langle C_j^\dagger C_j\rangle$ (Fig. 4 in Ref. 1) has decreased by a factor of 2 in going from $N_e=6$ to $N_e=8$ so it is only a

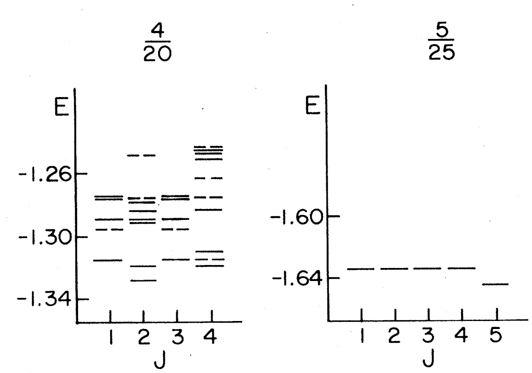

FIG. 1. Low-lying energy (in units of e^2/l) spectra for $\nu=\frac{1}{5}$ (4 and 5 electrons).

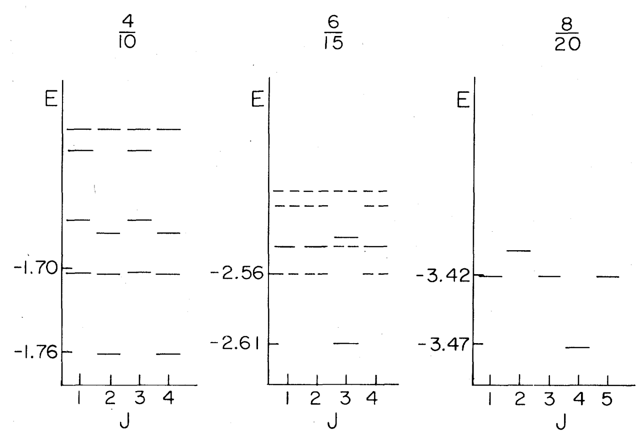

FIG. 2. Low-lying energy (in units of e^2/l) spectra for $\nu=\frac{2}{5}$ (4, 6, and 8 electrons).

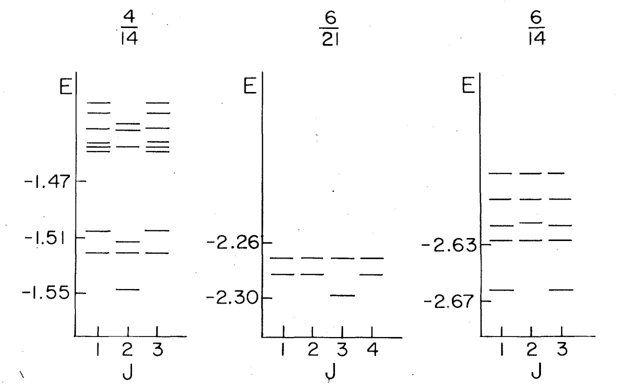

FIG. 3. Low-lying energy (in units of e^2/l) spectra for $\nu=\frac{2}{7}$ (4 and 6 electrons) and for $\nu=\frac{3}{7}$ (6 electrons).

finite-size effect. The fact that the $\frac{2}{5}$ gap is larger than the $\nu=\frac{1}{5}$ gap suggests that $\nu=\frac{1}{5}$ is no more fundamental than $\nu=\frac{2}{5}$.

In Fig. 3 the $N_e=4$ and $N_e=6$ cases for $\nu=\frac{2}{7}$ and the $N_e=6$ case for $\nu=\frac{3}{7}$ are shown. Due to large dispersion of the $\nu=\frac{6}{21}$ spectrum an estimate of the $\nu=\frac{2}{7}$ gap of about 0.02 is not as reliable as the $\nu=\frac{1}{5}$ or $\nu=\frac{2}{5}$ cases. But it is certainly much larger than the $\nu=\frac{1}{7}$ gap which is estimated to be 0.003 and smaller than the $\nu=\frac{3}{7}$ gap which is about 0.035. The results reveal a trend that for ν equal to an irreducible multiple of $1/q$, the closer it is to $\frac{1}{2}$ the larger the energy gap. This trend seems to be there in Fig. 4 as well. We also note that the energy gaps at $\nu=\frac{2}{5}$ and $\nu=\frac{3}{7}$ are comparable to the $\nu=\frac{1}{3}$ gap. These two features are in agreement with experimental findings of Chang *et al.*[3] If one can take the estimates of the $\nu=\frac{3}{7}$ and $\nu=\frac{4}{9}$ gaps from Figs. 3 and 4 seriously, then the energy gap at $\nu=(n/2n+1)$ $(n=1,2,3,\ldots)$ seems to decrease linearly with $\frac{1}{2}-\nu=1/[(2(2n+1)]$. The energy gaps for fractions we have studied so far are tabulated in Table I.

Some even-denominator cases have also been examined. The $\nu=\frac{1}{4}$ case shares with $\nu=\frac{1}{2}$ the accidental extra degeneracy. For $\nu=\frac{3}{8}$ and $N_e=6$, if the parent state had a period 8 then the optimal J should be even, but it does not turn out to be the case. The same occurs for $\nu=\frac{3}{10}$ and $N_e=6$.

We would like to point out that while the periodic parent state is a useful construct in that it predicts the right total momentum of the ground states, it is in general not the lowest-energy single-particle Slater state and its overlap with the true ground state decreases with the size of the system. For example, in the $\nu=\frac{6}{18}$ case the probability of the parent state $|1,4,7,10,13,16\rangle$ in the ground state is only 0.8%. This is of course consistent with the very small amplitude oscillation of the average occupation number $\rho(j)$. In the $\nu=\frac{6}{15}$ case the probability of the parent state $|1,5,6,10,11,15\rangle$ is quite significant ($=0.24$); this is, however, only a finite-size effect much like the amplitude of the oscillation in $\rho(j)$.

Another useful aspect of the parent state is the number of states N one can generate from it by switching on momentum-conserving interactions. For a fairly complicat-

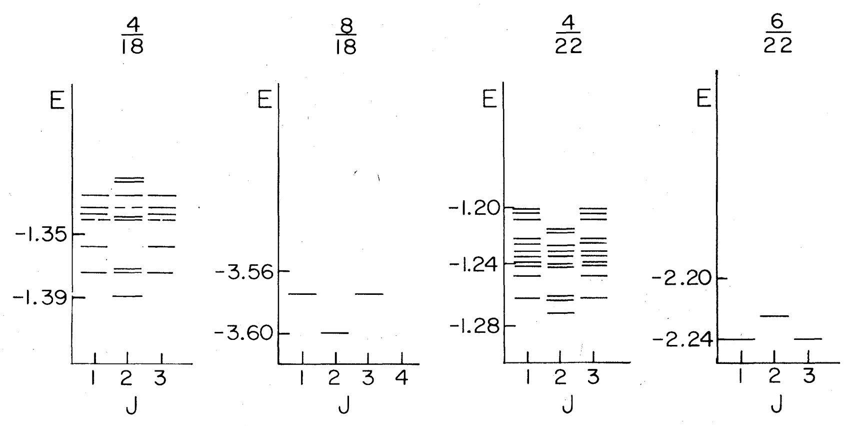

FIG. 4. Low-lying energy (in units of e^2/l) spectra for $\nu=\frac{2}{9}$, $\frac{4}{9}$, $\frac{2}{11}$, and $\frac{3}{11}$.

TABLE I. Energy gap (in units of e^2/l) for some odd-denominator filling factors ν.

ν	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{2}{11}$	$\frac{3}{11}$
Energy gap	0.08	0.01	0.05	0.003	0.02	0.035	0.018	0.025	0.01	0.018

TABLE II. Dimension of phase space for various odd-denominator filling factors ν, and total momenta J. Numbers corresponding to parent states with period q are underlined. $\nu=p/q$; p and q are relatively prime.

ν \ J	1	2	3	4	5	6	7	8
$\frac{4}{12}$	40	$\underline{43}$	40	42				
$\frac{5}{15}$	200	200	200	200	$\underline{201}$			
$\frac{6}{18}$	1035	1026	$\underline{1038}$	1026	1035	1028		
$\frac{7}{21}$	5537	5537	5537	5537	5537	5537	$\underline{5538}$	
$\frac{8}{24}$	30 624	30 664	30 624	$\underline{30\,667}$	30 624	30 664	30 624	30 666
$\frac{4}{20}$	240	$\underline{245}$	240	244				
$\frac{5}{25}$	2125	2125	2125	2125	$\underline{2126}$			
$\frac{4}{10}$	20	$\underline{22}$						
$\frac{6}{15}$	333	333	$\underline{335}$					
$\frac{8}{20}$	6288	6308	6288	$\underline{6310}$				
$\frac{4}{14}$	70	$\underline{73}$						
$\frac{6}{21}$	2583	2583	$\underline{2586}$	2583				
$\frac{6}{14}$	$\underline{217}$	212						
$\frac{4}{18}$	168	$\underline{172}$						
$\frac{8}{18}$	2424	$\underline{2438}$						
$\frac{4}{22}$	330	$\underline{335}$						
$\frac{6}{22}$	$\underline{3399}$	3384						

TABLE III. Dimension of phase space for various even denominator filling factors ν, and total momenta J. Numbers corresponding to parent states with period q ($2q$) are underlined singly (doubly). $\nu=p/q$; p and q are relatively prime.

ν \ J	1	2	3	4	5	6	7	8	9	10
$\frac{4}{8}$	8	$\underline{\underline{10}}$	8	$\underline{9}$						
$\frac{5}{10}$	25	25	25	25	$\underline{26}$					
$\frac{6}{12}$	78	75	$\underline{\underline{80}}$	75	78	$\underline{76}$				
$\frac{7}{14}$	245	245	245	245	245	245	$\underline{246}$			
$\frac{8}{16}$	800	808	800	$\underline{\underline{810}}$	800	808	800	$\underline{809}$		
$\frac{9}{18}$	2700	2700	2703	2700	2700	2703	2700	2700	$\underline{2704}$	
$\frac{10}{20}$	9250	9225	9250	9225	$\underline{\underline{9252}}$	9225	9250	9225	9250	$\underline{9226}$
$\frac{4}{16}$	112	$\underline{\underline{116}}$	112	$\underline{115}$						
$\frac{5}{20}$	775	775	775	775	$\underline{776}$					
$\frac{6}{24}$	5616	5598	$\underline{\underline{5620}}$	5598	5616	$\underline{5601}$				
$\frac{6}{16}$	$\underline{\underline{504}}$	$\underline{497}$								
$\frac{9}{24}$	54 477	54 477	$\underline{54\,484}$							
$\frac{6}{20}$	$\underline{\underline{1944}}$	$\underline{1932}$								

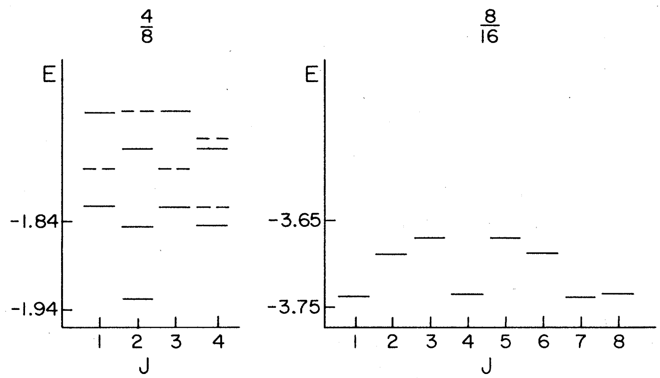

FIG. 5. Low-lying energy (in units of e^2/l) spectra for $\nu = \frac{1}{2}$ (4 and 8 electrons).

ed Hamiltonian such as (1) we expect N to be equal to the total number of Slater states with the same total momentum J as the parent state. The function $N(J)$ [as well as the energy $E(J)$] possesses an inversion symmetry $N(-J) = N(J)$. The inversion symmetry combined with the translational symmetry $N(J) = N(J + N_s/q)$ give rise to a reflection symmetry of the functions $N(J)$ and $E(J)$ about values of $J = J_*$ corresponding to periodic parent states. The slight deviation from such an exact symmetry of Fig. 2 in Ref. 1 is due to the approximate nature of the calculation (the $\nu = \frac{6}{18}$ spectrum) and due to an error (the $J = 1$ spectrum for $\nu = \frac{5}{15}$ should be the same as the $J = 4$ spectrum). Therefore, $N(J_*)$ is always a local extremum. For odd-denominator fractions $N(J_*)$ is always a true maximum as shown in Table II. We thus see that there is a strong correlation between energy and phase space. A maximum phase space invariably leads to a lowest-energy state. This is plausible if we note that $N(J)$ is also the total number of eigenstates with momentum J. The more eigenstates there are with momentum J, the more likely it is to find an eigenstate with momentum J split off from all the excited states with any momentum.

For even-denominator fractions the situation is entirely different. We examine the case $\nu = \frac{1}{2}$ first. From Table III it seems that whenever the total number of particles N_e is even, a parent state with period 4 always generates the largest phase space. This is intimately related to the exclusion principle as one can see in a simple example. Take $\nu = \frac{2}{4}$, $\Phi = |1,3\rangle$ has a period 2. Due to the exclusion principle the two particles cannot scatter, therefore $N(4) = 1$. On the other hand, the period 4 parent state $\Psi' = |1,2\rangle$ can generate two states $N(3) = 2 > N(4)$. If we had Bose statistics then $N(3) = 2$, and $N(4) = 3$ would be the other way around. Apparently due to the same reason $|3,4,7,8,11,12,\ldots\rangle$ generates a larger phase space than $|2,4,6,8,10,12,\ldots\rangle$ does. Because of this enlargement of phase space it seems a period 4 parent state always generates a true ground state as evidenced in Fig. 5, and Fig. 3 in Ref. 1. For an odd N_e a period 4 parent state is clearly impossible. The existence of two types of competing period parent states accounts for the oscillating behavior of the energy per particle as a function of N_e (as reported in Ref. 1 for $\nu = \frac{1}{2}$) and is responsible for the proliferation of equivalent ground states. From Table III it seems the unexpected doubling of the period of the parent state does occur for other even-denominator fractions as well.

With the evidence presented here and in Ref. 1 we feel we have a reasonable understanding of the fractional quantum Hall effect in terms of the Hamiltonian (1) and the concepts of ground-state degeneracy[5] and associated kink excitations. Despite the lack of a completely analytic solution, the energy gap for various ν can be estimated from small system calculations. We have also gained considerable insight into the absence of even-denominator fractional quantum Hall effect through such calculations.

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[5]It should be noted that the degeneracy analysis applies only for toroidal geometry. See F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. **54**, 237 (1985) for numerical results on spherical geometry. While we believe that the former geometry is more realistic and that ground-state degeneracy is an important aspect of the fractional quantum Hall effect, for some purposes other geometries might be more convenient for treating two-dimensional electron gas in a strong magnetic field.