Wetting transitions: A functional renormalization-group approach

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(Received 20 December 1984)

A linear functional renormalization group is introduced as a framework in which to treat various wetting transitions of films on substrates. A unified treatment of the wetting transition in three dimensions with short-range interactions is given. The results of Brézin, Halperin, and Leibler in their three different regimes are reproduced along with new results on the multicritical behavior connecting the various regimes. In addition, the critical behavior as the coexistence curve is approached at complete wetting is analyzed. Wetting in the presence of long-range substrate-film interactions that fall off as power laws is also studied. The possible effects of the nonlinear terms in the renormalization group are examined briefly and it appears that they do not alter the critical behavior found using the truncated linear renormalization group.

I. INTRODUCTION

At the interface between one bulk phase α of a system and a wall or substrate, a layer of a second phase, β, may form if the wall preferentially adsorbs it (Fig. 1). At a *wetting transition* the equilibrium thickness of this layer of β phase diverges.[1] Such a transition may be viewed as the unbinding of the α-β interface from the wall. This view of the transition has been developed by Lipowsky, Kroll, and Zia[2] and by Brézin, Halperin, and Leibler[3] (BHL), who consider an effective Hamiltonian for the interface-substrate separation, $z(\boldsymbol{\rho})$,

$$H=\int d^{d-1}\rho\left[\frac{\sigma}{2}\,|\nabla_\rho z(\boldsymbol{\rho})|^2+V(z(\boldsymbol{\rho}))\right], \tag{1.1}$$

where, for a d-dimensional system, the $(d-1)$-component vector $\boldsymbol{\rho}$ specifies a point on the substrate. The first term in Eq. (1.1) is the α-β interfacial tension, σ, times the excess area of the interface due to its fluctuations in position, while $V(z)$ is an interface potential which derives from the interactions between the substrate and the α and β phases and the relative free energies of these phases. In principle, this effective Hamiltonian is obtained by integrating out all fluctuations other than those of the interface position. Implicit in this Hamiltonian is a short distance cutoff at, say, length Λ^{-1}. This cutoff must satisfy $\sigma\Lambda^{-(d-1)}\gg k_BT$ (equivalent to Λ^{-1} being large compared to the bulk correlation length) in order to justify ignoring possible overhangs in the interface and higher-order terms in $|\nabla z|$.[4]

If the potential $V(z)$ in (1.1) were simply a low-order polynomial in z, then we would have the familiar Landau-Ginzburg-Wilson Hamiltonian for a continuous-spin Ising system. A renormalization-group treatment of such a Hamiltonian can be carried out systematically for $(d-1)$ near or above four dimensions by expanding $V(z)$ perturbatively for small z and keeping only a few terms.[5] However, for the wetting problem we have a very different sort of potential. Because the α-β interface cannot be in the substrate (see Fig. 1), the potential $V(z)$ should be large or even infinite for $z<0$, thus preventing the interface from fluctuating into this unphysical region. At the bulk coexistence of the α and β phases $V(z)$ will go to a constant as $z\to\infty$; we may choose this constant to be zero. With this choice of zero the potential $V(z)$ is an effective interaction between the α-β interface and the wall. If this interaction contains an attractive component, then it may succeed in binding the α-β interface to the wall so that the expectation value $\langle z\rangle$ of the thickness of the film of the β phase is finite. A wetting transition occurs when, as the temperature, chemical potential, or another field is varied, the attractive part of the potential $V(z)$ is no longer able to bind the interface and $\langle z\rangle$ diverges. The total interfacial free energy between the bulk α phase and the wall may be written

$$\sigma_{\alpha w}=\sigma_{\beta w}+\sigma-\Sigma_B \tag{1.2}$$

(with $\sigma\equiv\sigma_{\alpha\beta}$) in terms of a binding energy per unit area, Σ_B, of the α-β interface to the wall. The wetting transition occurs when $\Sigma_B\to 0^+$ and thus Σ_B plays the role of the singular part of the free energy for wetting transitions. Concomitant with the vanishing of Σ_B, there is a diverging correlation length parallel to the interface $\xi_{\parallel}$, which is just the capillary length for the interface. At length scales shorter than $\xi_{\parallel}$, the interface fluctuations will be controlled by the surface tension while at longer length scales

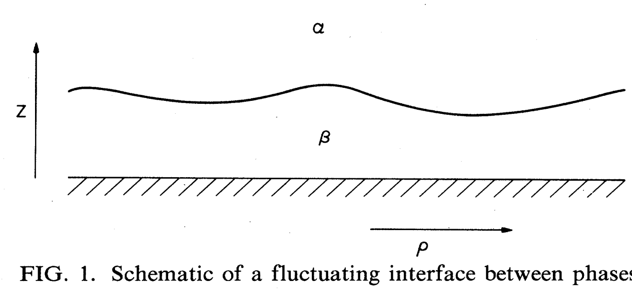

FIG. 1. Schematic of a fluctuating interface between phases α and β near to a wall or substrate, shown hatched. The distance, z, of the interface from the wall varies as function of the coordinates $\boldsymbol{\rho}$ parallel to the wall.

they will be restricted by interactions with the wall.

BHL (Ref. 3) have considered the potential

$$V(z) = -ae^{-z/\xi_b} + be^{-2z/\xi_b} , \tag{1.3}$$

appropriate for a system at bulk coexistence with only short-range interactions,[2,6] where $b>0$ and ξ_b is the bulk correlation length in the phase (β in Fig. 1) attracted to the wall. For bulk dimensionality $d=3$, they find that the critical behavior at the wetting transition with this potential in Eq. (1.1) has a very complicated dependence on the dimensionless parameter

$$\omega \equiv k_B T_w / 4\pi \xi_b^2 \sigma , \tag{1.4}$$

where T_w is the temperature at the wetting transition. This analysis is based on a renormalization-group treatment of the interface Hamiltonian, Eq. (1.1), using a different form of the potential, $V(z)$, for each of three regimes, $0<\omega<\frac{1}{2}$, $\frac{1}{2}<\omega<2$, and $\omega>2$. In the first regime they simply use the potential, Eq. (1.3), but in the other regimes they replace one or both of the exponentials in (1.3) with Gaussians. We do not find the reasons for making such substitutions very easy to understand in their formulation. One purpose of the present paper is to give a more unified treatment of these three regimes in which the different forms of the renormalized potential do not have to be put in by hand. The formulation will turn out to be useful for analyzing various wetting problems (see Secs. VII and VIII). To this end we write down a *functional* renormalization-group equation for general d and arbitrary potential $V(z)$ for the interface Hamiltonian, Eq. (1.1). This functional approach can be used for any potential to determine if there are fluctuation corrections to the mean-field critical behavior at wetting which is obtained simply by minimizing $V(z)$. In some cases nontrivial critical behavior at wetting can be obtained from our functional renormalization group restricted to the perturbative regime about $V(z)=0$. The example of the potential given by Eq. (1.3) in $d=3$ is worked out in detail below; we obtain some results beyond those of BHL (Ref. 3) and give what we believe is a more unified treatment of their rather surprising results. New, nontrivial critical wetting behavior is also obtained in $d\leq 3$ for certain potentials $V(z)$ that vary as a power of z for $z\to\infty$.

A summary of this paper is as follows. In Sec. II a functional differential renormalization-group (RG) equation is derived for a general interface Hamiltonian and in Sec. III it is used to derive the upper critical dimension for the critical wetting problem with short-range interactions. In Secs. IV and V the marginal three-dimensional case is analyzed in detail using a linear truncated RG. The results of these sections are best summarized by referring to Fig. 2, which is a phase diagram for critical wetting in three dimensions, as a function of the dimensionless parameter ω defined by BHL [Eq. (1.4)]. For $\omega<2$, the wetting transition occurs when the strength, a, of the attractive tail of the interface-wall potential in Eq. (1.3) goes to zero. In both regime I, $0<\omega<\frac{1}{2}$, and regime II, $\frac{1}{2}<\omega<2$, the film thickness, $\langle z\rangle$, diverges logarithmically as the wetting transition is approached (as $a\to a_c=0$) and the capillary length, $\xi_{\parallel}$, diverges as a power of the deviation from wetting, $\tau\equiv(T-T_w)\sim(a-a_c)$ (with logarithmic corrections in regime II). The critical exponents vary with ω and are nonanalytic at the multicritical point P at $\omega=\frac{1}{2}$. For $\omega>2$, regime III, the wetting transition occurs at a *finite* strength, $a_c\neq 0$, of the wall attraction. The film thickness diverges as $1/\tau$ and the capillary length diverges as $e^{1/\tau}$ in this regime. Near to the multicritical point M, at $\omega=2$, a_c vanishes as $\omega-2$ and for a large range of τ, the behavior will be dominated by the multicritical point, at which $\langle z\rangle\sim 1/\tau^2$ and $\xi_{\parallel}\sim e^{1/\tau^2}$.

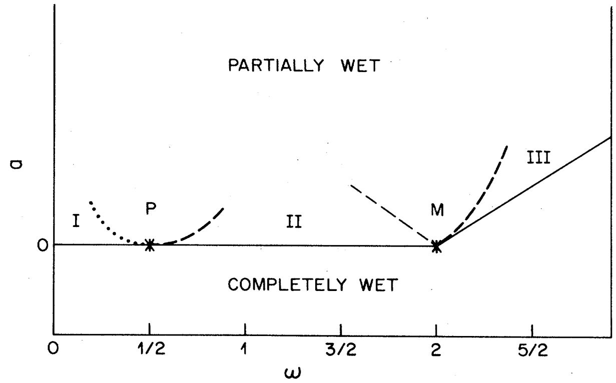

FIG. 2. Phase diagram for critical wetting in three bulk dimensions as a function of the strength, a, of the attractive tail of the exponentially decaying interface-wall potential and the dimensionless parameter ω which is inversely proportional to the interfacial tension. As the wetting transition is approached from the partially wet side, the critical behavior will be different in each of the regimes I, II, and III, which are separated by multicritical points denoted by stars. Near to the multicritical points, indicated by P and M, the behavior over a range of potential strength a will be controlled by the multicritical points. Crossover to the asymptotic behavior occurs in the vicinity of the dashed and dotted lines. However, across the dotted line the crossover from P to I will be manifested only as a change in the amplitude of singularities.

In Sec. VI, which is somewhat speculative, the validity of the truncated linear RG, which ignores various nonlinear terms actually present in the full RG, is examined. It appears that including the nonlinear terms will not alter the critical behavior found using the linear RG. Sections VII and VIII are devoted, respectively, to an analysis of the singularity for complete wetting as the coexistence curve is approached in three dimensions, and to a discussion of the wetting transitions in less than three dimensions in the presence of power-law interactions with the wall. Finally, Sec. IX summarizes and discusses the results of the paper.

II. RENORMALIZATION-GROUP EQUATIONS

In this section we derive renormalization-group equations for the interface Hamiltonian, Eq. (1.1), to first order in the potential V. This has been done for three bulk dimensions ($d=3$) in Ref. 7; here we will consider general dimension d. To avoid ultraviolet divergences we impose a sharp cutoff in momentum space. When the potential V

in (1.1) vanishes, the Hamiltonian reduces to a Gaussian model, which may be written in terms of the Fourier transformed interfacial coordinates,

$$\hat{z}(\mathbf{k})=\int d^{d-1}\rho\, e^{i\mathbf{k}\cdot\boldsymbol{\rho}} z(\boldsymbol{\rho}) , \tag{2.1}$$

as

$$H_0(\Lambda,\sigma)=\frac{\sigma}{2}\int_{|\mathbf{k}|<\Lambda} d^{d-1}|\mathbf{k}|^2|\hat{z}(\mathbf{k})|^2 , \tag{2.2}$$

where Λ is the cutoff momentum. This Gaussian Hamiltonian is a fixed point of a renormalization-group transformation that integrates out those degrees of freedom, $\hat{z}(\mathbf{k})$, with $\Lambda/b<|\mathbf{k}|<\Lambda$, and then rescales the system to new coordinates

$$\boldsymbol{\rho}'=\boldsymbol{\rho}/b,\quad z'(\boldsymbol{\rho}')=b^{(d-3)/2}z(\boldsymbol{\rho}) . \tag{2.3}$$

We will expand the renormalization group about this line of Gaussian fixed points (parametrized by σ) to lowest order in the interaction part of the Hamiltonian,

$$H_I=\int d^{d-1}\rho\, V(z(\boldsymbol{\rho})) . \tag{2.4}$$

In order to generate continuous renormalization-group flow equations we rescale by a factor $b=1+l$, with l infinitesimal. Let us divide $z(\boldsymbol{\rho})$ into the part to be integrated out (the "fast" part),

$$z_f(\boldsymbol{\rho})=\int_{\Lambda'<|\mathbf{k}|<\Lambda} d^{d-1}k\, e^{-i\mathbf{k}\cdot\boldsymbol{\rho}}\hat{z}(\mathbf{k}) , \tag{2.5}$$

where $\Lambda'=\Lambda/b\approx\Lambda(1-l)$ for $l\to 0^+$, and the remainder (the "slow" part)

$$z_s(\boldsymbol{\rho})=z(\boldsymbol{\rho})-z_f(\boldsymbol{\rho}) . \tag{2.6}$$

The interaction part of the Hamiltonian may then be expanded in powers of $z_f(\boldsymbol{\rho})$ (which is of order $\sqrt{l}$ and hence arbitrarily small), as

$$H_I=\int d^{d-1}\rho\left[V(z_s(\boldsymbol{\rho}))+z_f(\boldsymbol{\rho})\frac{dV(z_s(\boldsymbol{\rho}))}{dz_s(\boldsymbol{\rho})}+\frac{z_f^2(\boldsymbol{\rho})}{2}\frac{d^2V(z_s(\boldsymbol{\rho}))}{dz_s(\boldsymbol{\rho})^2}+\cdots\right] . \tag{2.7}$$

The momentum-shell integration is then straightforward at first order in $V(z)$ and rescaling according to Eq. (2.3) yields the renormalization-group flow equations for the interfacial tension and potential,[8]

$$\frac{d\sigma}{dl}=O(V^2/\sigma) ,$$
$$\frac{\partial V(z)}{\partial l}=(d-1)V(z)+\frac{3-d}{2}z\frac{\partial V(z)}{\partial z}+\frac{1}{\tilde{\sigma}}\frac{\partial^2 V(z)}{\partial z^2}+O(V^2/\sigma) , \tag{2.8}$$

where

$$\tilde{\sigma}=\frac{\sigma(4\pi)^{(d-1)/2}\Gamma((d-1)/2)}{k_BT\Lambda^{(d-3)}} , \tag{2.9}$$

and $\Gamma(x)$ is the usual gamma function obtained from doing the momentum-shell integral. When truncated at this order, the functional renormalization-group flow, Eq. (2.8), is simply a linear differential equation (a diffusion equation with rescaling) and can be explicitly integrated starting with an arbitrary bare potential $V_0(z)$ at $l=0$, yielding

$$V_l(z)=\frac{e^{(d-1)l}}{\sqrt{2\pi}\delta(l)}\int_{-\infty}^{\infty}dz'\,V_0(z')\times\exp\{-[z\zeta(l)-z']^2/2\delta^2(l)\} , \tag{2.10}$$

where the width of the convolution, $\delta(l)$, is given by

$$\delta^2(l)=2(e^{(3-d)l}-1)/(3-d)\tilde{\sigma} , \tag{2.11}$$

and the rescaling factor in the z direction is

$$\zeta(l)=\exp[l(3-d)/2] . \tag{2.12}$$

Note that the parameter l in these flow equations is the logarithm of the factor by which lengths *parallel* to the interface have been rescaled.

In order to investigate critical behavior at wetting transitions, we rescale until a scale e^{l^*} at which the curvature at the minimum of the renormalized potential V_{l^*} is of order one. At this scale the fluctuations will no longer be important and the renormalized correlation length parallel to the interface will be of order one. The original parallel correlation length, $\xi_{\parallel}$, is therefore of order e^{l^*}.

III. ABOVE THE UPPER CRITICAL DIMENSION, $d>3$

If the fluctuations of the interface are ignored, as is done in mean-field theory, then for the interfacial Hamiltonian, Eq. (1.1), the film thickness $\langle z\rangle$ is determined purely by the global minimum of the potential $V(z)$. Continuous or critical wetting occurs in mean-field theory when this minimum moves continuously out to infinite z as a parameter (temperature, chemical potential, etc.) is varied. In this paper we will consider two asymptotic (large z) forms of the interfacial potential. If the wall has only finite-range interactions with the bulk phases, then an interfacial potential of the asymptotic form

$$V(z)\approx -ae^{-z/\xi_b}+be^{-z/\xi_b} \tag{3.1}$$

is obtained from a Landau-Ginzburg mean-field free energy at two-phase coexistence, where ξ_b is the bulk correlation length in the phase nearest the wall.[2,6] If the wall has long-range interactions with the phases α or β (such as the van der Waals interactions between fluctuating dipoles) then a potential of the asymptotic form

$$V(z)\approx -az^{-n}+bz^{-m} \tag{3.2}$$

may occur at coexistence, with exponents $m>n>0$. Both of these potentials exhibit critical wetting in mean-field theory when $b>0$ and $a\to 0+$. It is also of interest to consider systems slightly away from bulk two-phase coexistence which we discuss in Secs. VII and VIII below. Then the excess bulk free energy per unit area of the film near the wall, which is linear in its thickness z, must be added to the above potentials.

It is straightforward to see within our formulation, that including the interfacial fluctuations does not alter the nature of the singularities at the critical wetting transition for dimensions $d>3$, as has been argued by various authors.[6,9] This becomes apparent within our functional renormalization-group scheme if we integrate out the fluctuations but do not rescale the distances either parallel or perpendicular to the interface. The unrescaled, renormalized potential is then

$$\widetilde{V}_l(z)=\frac{1}{\sqrt{2\pi}\delta(l)}\int_{-\infty}^{\infty}dz'\,V_0(z')\exp[-(z-z')^2/2\delta^2(l)]\ , \tag{3.3}$$

which is simply a convolution of the bare potential, $V_0(z)$, with a Gaussian of width $\delta(l)$. For $d>3$ this width, given by Eq. (2.11), remains finite as $l\rightarrow\infty$, so that renormalization only smears the potential over a finite range of z. This does not alter the nature of the singularities at wetting for the potentials Eq. (3.1) and (3.2) discussed above. In fact, for the power-law potential, Eq. (3.2), the mean-field critical behavior remains valid down to bulk dimensionality

$$d_c(m)=(3m+2)/(m+2) \tag{3.4}$$

as has been pointed out by Lipowsky.[9] Our functional renormalization-group approach is applied to the power-law potentials in Sec. VIII below. First we will treat the case of short-range potentials in the marginal dimensionality $d=3$.

IV. CRITICAL WETTING IN THREE DIMENSIONS

In three dimensions, the spatial rescaling factor $\zeta(l)$ for lengths perpendicular to the interface which is given by Eq. (2.12) is not needed at lowest order in the potential, and the differential flow equation (2.8) reduces to the simple form

$$\frac{\partial V}{\partial l}=2V+\frac{1}{\widetilde{\sigma}}\frac{\partial^2 V}{\partial z^2}\ . \tag{4.1}$$

In this section we will investigate the case of *short-range* interactions with the wall so that for large z the bare potential $V_0(z)$ has the form given by Eq. (3.1). It is convenient to measure distances in the z direction in terms of the bulk correlation length ξ_b which we henceforth set equal to one. The flow equation then involves a dimensionless "diffusion" constant

$$\omega=\frac{1}{\widetilde{\sigma}\xi_b^2}=\frac{k_BT}{4\pi\sigma\xi_b^2}\ , \tag{4.2}$$

in terms of which we have

$$\frac{\partial V}{\partial l}=2V+\omega\frac{\partial^2 V}{\partial z^2}\ . \tag{4.3}$$

The width $\delta(l)$ of the convolution in Eq. (2.10) which solves Eq. (4.3) is then simply

$$\delta^2(l)=2\omega l\ . \tag{4.4}$$

Since, in contrast to the case for $d>3$, δ diverges for large l, the short distance (small z) parts of the bare potential can affect the renormalized potential at long distances for large l. Thus we must generally consider the form of $V_0(z)$ for z near zero. A reasonable form to take is to assume that for negative z the potential is just a fixed positive constant c, representing a "soft" wall, while for z positive, $V_0(z)$ has the same form as for large z:

$$V_0(z)=\begin{cases}c, & z<0\\ be^{-2z}-ae^{-z}, & z>0\ .\end{cases} \tag{4.5}$$

This potential is illustrated schematically by the solid line in Fig. 3. As we will see, as long as $V_0(z)$ has only one minimum and does not diverge too rapidly for z large and negative, the critical behavior near the wetting transition will not depend on the details of the potential in the region $z\lesssim 0$. It would be quite reasonable physically to consider a potential with a *hard* wall, namely Eq. (4.5) with $c\rightarrow\infty$. However, our flow equation (4.3), which is truncated at linear order in $V(z)$, cannot properly handle a potential that diverges to infinity. A full renormalization group, on the other hand, could, in principle, handle such a potential and we argue in Sec. VI that the critical behavior obtained using the truncated flow equation (4.3) and a soft wall is probably correct even for the hard wall.

To analyze the behavior of the renormalized potential for large l, it is convenient to divide the potential into its constituent parts:

$$V(z)=W(z)+R(z)+A(z)\ , \tag{4.6}$$

where the bare parts are the following: the wall

$$W_0(z)=\begin{cases}c, & z<0\\ 0, & z>0\ ,\end{cases} \tag{4.7}$$

the repulsive tail

$$R_0(z)=\begin{cases}be^{-2z}, & z>0\\ 0, & z<0\ ,\end{cases} \tag{4.8}$$

and the attractive part

$$A_0(z)=\begin{cases}-ae^{-z}, & z>0\\ 0, & z<0\ .\end{cases} \tag{4.9}$$

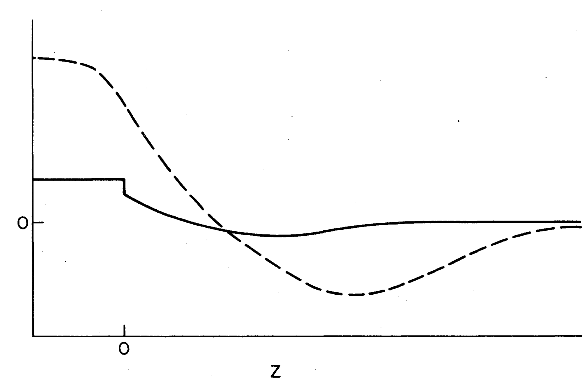

FIG. 3. Schematic plots of the interface-wall potential as a function of distance, z. The solid line is the bare potential V_0 and the dashed line the effective renormalized potential, V_l, for the longer-wavelength fluctuations of the interface at scale e^l.

In order to investigate the critical behavior near wetting, we renormalize the potential until the curvature at its minimum, which is at $z=z_m$, is of order one and monotonically increasing. This occurs at the scale e^{l^*} given by

$$\frac{\partial^2 V_{l^*}}{\partial z^2}(z_m)\sim 1 \ . \tag{4.10}$$

At this scale the fluctuations are of order one and so the parallel correlation length, $\xi_{\parallel}(l^*)$ will also be of order one. From the length rescaling Eq. (2.3) we then have that the original correlation length parallel to the interface is

$$\xi_{\parallel}\sim e^{l^*} \ . \tag{4.11}$$

At critical wetting this correlation length diverges, so $l^*\to\infty$. In the completely wet portion of the phase diagram the renormalized potential never gets a minimum with a curvature that is of order one and increasing.

The average position of the interface, $\langle z\rangle$, will similarly be given by the position, z_m^*, of the minimum in the renormalized potential at scale l^*; since there is no rescaling of perpendicular distances,

$$\langle z\rangle=z_m^* \ . \tag{4.12}$$

The behavior of the various parts of the renormalized potential can be obtained straightforwardly for large l by performing the convolution in Eq. (2.10) by steepest descents. This can most easily be done if the distances z are scaled by l, anticipating that the important distance scales will be of order l. We define

$$z=\mu l \ , \tag{4.13}$$

whence

$$V_l(\mu l)=\frac{e^{2l}}{\sqrt{4\pi\omega}}l^{1/2}\int_{-\infty}^{\infty}d\mu'\,V_0(\mu' l)e^{-l(\mu-\mu')^2/4\omega} \ , \tag{4.14}$$

and it is then straightforward to perform a steepest descent integration for large l.

Specifically, we have for the attractive part

$$A_l(\mu l)=-\frac{al^{1/2}}{\sqrt{4\pi\omega}}e^{2l}\int_0^{\infty}d\mu'\,e^{l[-\mu'-(\mu'-\mu)^2/4\omega]} \ , \tag{4.15}$$

so that the exponent in the integral is maximized at

$$\mu_s'=\mu-2\omega \ . \tag{4.16}$$

Thus for $\mu>2\omega$ and $l\to\infty$ the integral will be dominated by the saddle point at μ_s', yielding

$$A_l(\mu l)\approx -ae^{l(2+\omega-\mu)} \ \text{for}\ \mu>2\omega \ . \tag{4.17}$$

Conversely, for $\mu<2\omega$, the integral in Eq. (4.14) is dominated by μ' near zero, yielding for $\mu<2\omega$

$$A_l(\mu l)=-\frac{1}{\sqrt{4\pi\omega}}\frac{1}{\sqrt{l}}\frac{1}{1-\mu/2\omega}e^{l(2-\mu^2/4\omega)}[1+O(1/l)] \ . \tag{4.18}$$

Thus the renormalized $A(z)$ is a Gaussian at short distances $z<2\omega l$ and decays as an exponential at long distances $z>2\omega l$. Similarly, for $\mu>4\omega$

$$R_l(\mu l)\approx be^{l(2+4\omega-2\mu)} \ , \tag{4.19}$$

while for $\mu<4\omega$

$$R_l(\mu l)\approx\frac{b}{\sqrt{4\pi\omega}}\frac{1}{\sqrt{l}}\frac{1}{2-\mu/2\omega}e^{l(2-\mu^2/4\omega)} \ . \tag{4.20}$$

The renormalized wall potential for large l is dominated by the small negative z parts of W_0 for all $\mu>0$ so that

$$W_l(\mu l)\approx\frac{c}{\sqrt{4\pi\omega}}\frac{1}{\sqrt{l}}e^{l(2-\mu^2/4\omega)}\frac{2\omega}{\mu} \ . \tag{4.21}$$

The minimum in the full potential moves to higher z and its value at the minimum changes under renormalization, as is shown schematically in Fig. 3.

We now analyze the various possible regimes. It is clear by inspection that for $\mu>4\omega$, W is negligible relative to R for large l. If we assume the minimum of V_l is in this regime, then we have by simply combining Eqs. (4.17) and (4.19) the minimum at

$$z_m\approx 3\omega l+\ln(2b/a) \ . \tag{4.22}$$

The curvature of the potential at z_m is

$$\left.\frac{\partial^2 V_l(z)}{\partial z^2}\right|_{z_m}\approx\frac{a^2}{2b}e^{(2-2\omega)l} \ , \tag{4.23}$$

from which we obtain that

$$l^*\approx\frac{1}{1-\omega}\ln\frac{1}{a} \ . \tag{4.24}$$

Substituting into Eq. (4.22) yields

$$z_m^*\approx(1+2\omega)l^* \ , \tag{4.25}$$

so that the ansatz that the minimum at l^* occurs with $\mu>4\omega$ is justified only for $\omega<\frac{1}{2}$. In this regime, $\omega<\frac{1}{2}$, which BHL (Ref. 3) call *regime* I, the critical value of a is $a_c=0$, so that we may take $a\sim\tau=T-T_w$, yielding

$$\xi_{\parallel}\sim e^{l^*}\sim(1/\tau)^{1/(1-\omega)} \ , \tag{4.26}$$

and

$$\langle z\rangle=z_m^*\approx\frac{1+2\omega}{1-\omega}\ln\frac{1}{\tau}$$
$$\approx(1+2\omega)\ln\xi_{\parallel} \ , \tag{4.27}$$

in agreement with BHL.[3]

For $\omega>\frac{1}{2}$, the minimum in V_{l^*} occurs for $\mu<4\omega$ so that Eq. (4.20) for R_l must be used. However it is still possible that the minimum occurs for $\mu>2\omega$ so that Eq. (4.17) is still valid. We thus make the ansatz that at z_m^*, $2\omega<\mu<4\omega$ so that

$$V_l(z)\approx -ae^{l(2+\omega)-z}+\frac{be^{2l-z^2/4\omega l}}{\sqrt{4\pi\omega l}}\left[\frac{1}{2-z/2\omega l}\right]$$
$$+\frac{ce^{2l-z^2/4\omega l}}{\sqrt{4\pi\omega l}}\left[\frac{2\omega l}{z}\right] \ , \tag{4.28}$$

since we can no longer neglect W_l relative to R_l. In evaluating the derivatives of V_l with respect to z, it can be seen that derivatives of the parts in large parentheses in Eq. (4.28) will always yield terms which are smaller by powers of $1/l$ than those from the exponentials. Thus, to the desired accuracy for l large, we can replace the terms in large parentheses by their values at the minimum of V_l. The terms in V_l from R_l and W_l then have the same form and we can replace their sum by

$$R_l+W_l \rightarrow \frac{K}{\sqrt{l}} e^{2l-z^2/4\omega l} . \tag{4.29}$$

This is equivalent to the form used by BHL (Ref. 3) in their regimes II and III.

By minimizing $V_l(z)$ and setting the second derivative at the minimum to one, we find that the assumption $2\omega l^* < z_m^* < 4\omega l^*$ is justified for

$$\tfrac{1}{2} < \omega < 2 , \tag{4.30}$$

which defines *regime* II. Again, the critical value of a is

$$a_c = 0 , \tag{4.31}$$

but now we have

$$\xi_{\parallel} \sim \left[\tau \left[\ln \frac{1}{\tau} \right]^{\sqrt{\omega/8}} \right]^{-1/(2+\omega-\sqrt{8\omega})} , \tag{4.32}$$

and

$$\langle z \rangle \approx \sqrt{8\omega}(\ln\xi_{\parallel} - \tfrac{1}{8}\ln\ln\xi_{\parallel})$$

$$\approx \frac{\sqrt{8\omega}}{2+\omega-\sqrt{8\omega}} \left[\ln\frac{1}{\tau} - \tfrac{1}{8}(2+\omega)\ln\ln\frac{1}{\tau} \right] , \tag{4.33}$$

which agrees with BHL (Ref. 3) except for a minor algebraic error of theirs in the logarithmic part of Eq. (4.32). Note that for $\omega = \frac{1}{2}$, the leading exponents in Eq. (4.32) and (4.26) agree.

If $\mu < 2\omega$ at the minimum of V_{l^*}, then the situation is somewhat more complicated. This will happen for $\omega > 2$ which defines *regime* III. In this regime, A, R, and W will all have the same leading asymptotic behavior for large l and fixed μ up to prefactors which depend on μ:

$$V_l(\mu l) \approx \frac{K_l(\mu)}{\sqrt{4\pi\omega l}} e^{2l-\mu^2 l/4\omega} , \tag{4.34}$$

where

$$K_l(\mu) = \left[-\frac{a}{1-\mu/2\omega} + \frac{b}{2-\mu/2\omega} + \frac{c}{\mu/2\omega} + O(1/l) \right] . \tag{4.35}$$

It is clear that to leading order

$$\frac{\partial V_l}{\partial z} = \frac{1}{l}\frac{\partial V_l}{\partial \mu} \approx -\frac{\mu}{2\omega} V_l , \tag{4.36}$$

and

$$\frac{\partial^2 V_l}{\partial z^2} \approx \frac{\mu^2}{4\omega^2} V_l , \tag{4.37}$$

so that at the minimum, K_l must vanish to leading order in $1/l$. This implies that the critical value of a will be nonzero, in contrast to regimes I and II. In order to analyze the critical behavior we need to calculate the derivatives of V_l to next order for l large since the leading-order contribution to $\partial^2 V_l/\partial z^2$, Eq. (4.37), will vanish at the minimum. The first nonvanishing contribution to the curvature at the minimum, μ_m, is given by

$$\left. \frac{\partial^2 V_l(z)}{\partial z^2} \right|_{z=\mu_m l} \approx \frac{e^{2l-\mu_m^2 l/4\omega}}{\sqrt{4\pi\omega l}} \frac{1}{l} \left[\frac{-\mu_m}{2\omega} \right] \left. \frac{\partial K_l}{\partial \mu} \right|_{\mu=\mu_m} \tag{4.38}$$

Note that the order $1/l$ terms in K_l do not play a role and can be neglected. The curvature can only be of order one for large l if

$$\mu_m = \sqrt{8\omega} + O((\ln l)/l) . \tag{4.39}$$

Thus the critical value of a is given implicitly by

$$K_\infty(a_c, \mu=\sqrt{8\omega}) = 0 . \tag{4.40}$$

For $a < a_c$, the curvature at the minimum of V_l will decrease exponentially for large l while for $a > a_c$ it will increase exponentially. If we write $\tau \sim a - a_c$ and expand μ_m and $K_\infty(\mu)$ around $\mu = \sqrt{8\omega}$ and a_c, we obtain from $\partial V_l/\partial z = 0$,

$$\mu_m(l) - \sqrt{8\omega} \sim \tau + O(1/l) , \tag{4.41}$$

while from

$$\partial^2 V_{l^*}/\partial z^2 |_{z=l^*\mu_m(l^*)} = 1$$

we find

$$l^*[\mu_m(l^*) - \sqrt{8\omega}] + 3(\omega/8)^{1/2}\ln l^* = 0 . \tag{4.42}$$

This yields

$$\langle z \rangle \approx \sqrt{8\omega}(\ln\xi_{\parallel} - \tfrac{3}{8}\ln\ln\xi_{\parallel}) . \tag{4.43}$$

Since $\tau \sim (\ln l^*)/l^*$, we have

$$\xi_{\parallel} \sim \exp\left[\frac{1}{C\tau} \left[\ln\frac{1}{C\tau} + \ln\ln\frac{1}{C\tau} + O(1) \right] \right] , \tag{4.44}$$

with C a nonuniversal (ω dependent) constant so that

$$\langle z \rangle \approx \sqrt{8\omega}\frac{1}{C\tau} \left[\ln\frac{1}{C\tau} + \ln\ln\frac{1}{C\tau} + \cdots \right] . \tag{4.45}$$

Note that just above a_c, the curvature at the minimum of the renormalized potential starts off at order one, but first decreases as a power of l before it increases exponentially. We must thus be careful to stop integrating only when the curvature increases *back* to order one.

Although the form of the potential used by BHL (Ref. 3) in regime III does not yield a renormalized potential of the form Eq. (4.34), the differences do not affect the critical behavior. Equation (4.43) disagrees with BHL (Ref. 3) due to an algebraic error in their expression for $\langle z \rangle$. From Eqs. (4.35) and (4.40) it can be seen that the critical value of a approaches zero linearly as $\omega \rightarrow 2^+$. This behavior was not found by BHL.[3]

The binding free energy per unit area, Σ_B, is given by

$$\Sigma_B \approx -e^{-2l^*} V_{l^*}(z_m^*) . \tag{4.46}$$

It is straightforward to show that in each of regimes I—III,

$$V_{l^*}(z_m^*)\sim \left.\frac{\partial^2 V_{l^*}(z)}{\partial z^2}\right|_{z=z_m^*}\sim 1 \; . \tag{4.47}$$

Equation (4.46) then implies that

$$\Sigma_B \sim \xi_{\parallel}^2 \; , \tag{4.48}$$

so that hyperscaling is satisfied *without* the logarithmic corrections which one might have expected at this upper critical dimension $d=3$.

V. CROSSOVER AND MULTICRITICALITY FOR ω NEAR $\frac{1}{2}$ OR 2

For completeness, in this section we quote some results for the behavior near the multicritical points separating regimes I and II, and II and III, which we denote P and M, respectively (see Fig. 2). The results can be readily derived, as in the preceding section.

At $\omega=\frac{1}{2}$, the behavior is identical to regime I. As $\omega\rightarrow\frac{1}{2}^-$ there will be a crossover in the *amplitude* of, say, $\xi_{\parallel}$, for τ of order

$$\tau_c \sim e^{-B/\epsilon^2} \; , \tag{5.1}$$

where $\epsilon=\frac{1}{2}-\omega$ and B is a constant. As $\omega\rightarrow\frac{1}{2}^+$ the behavior for $\tau \gg \tau_c$ will be like that at P while the asymptotic critical behavior for regime II (which contains $\ln\tau$) will only be reached for $\tau \ll \tau_c$. This crossover is indicated by the dotted and dashed lines in Fig. 2.

Near the multicritical point M at $\omega=2$, the behavior is more interesting. At $\omega=2$,

$$\xi_{\parallel}\sim e^{A/\tau^2} \; , \tag{5.2}$$

and

$$\langle z \rangle \approx 4(\ln\xi_{\parallel}-\tfrac{1}{4}\ln\ln\xi_{\parallel})\sim\tau^{-2} \; , \tag{5.3}$$

so that both the correlation length and $\langle z \rangle$ as functions of $\tau\sim(T-T_w)$ diverge most rapidly at $\omega=2$, while $\langle z \rangle$ in terms of $\xi_{\parallel}$ is midway between the regime II and III results. For $\omega\rightarrow 2^-$ the regime II critical behavior will hold for

$$\tau \ll \tau_c \sim \epsilon \equiv 2-\omega \; , \tag{5.4}$$

which should naively be expected since

$$a_c \sim -\epsilon \tag{5.5}$$

for $\omega\rightarrow 2^+$.

However, in regime III, the asymptotic critical behavior will be valid only for

$$\tau \ll \epsilon^2 \; . \tag{5.6}$$

Surprisingly, the constant C in Eq. (4.44) for the correlation length does *not* diverge as $\omega\rightarrow 2^+$. Because of the anomalously narrow critical region on the regime III side, as shown in Fig. 2, the crossover scaling function for a and ϵ small will be very singular and *not* show the regime III critical behavior. For example, we can write $\xi_{\parallel}$ in terms of $\tau_M \sim a$ in the form

$$\xi_{\parallel}\sim \exp\left[\frac{8}{\epsilon^2}\Xi\left[\frac{\epsilon}{K\tau_M^{\phi}}\right]\right] , \tag{5.7}$$

with the crossover exponent $\phi=1$ and K a nonuniversal constant. The limits of $\Xi(x)$ are

$$\begin{aligned}
&\Xi(x)\sim \ln x-\tfrac{1}{2}\ln\ln x+D \quad \text{for } x\gg 1 \; ,\\
&\Xi(x)\sim Ex^2 \quad \text{for } |x| \ll 1 \; ,\\
&\Xi(x)\rightarrow F \quad \text{for } x\rightarrow -1^+ \; ,\\
&\Xi(x)=\infty \quad \text{for } x<-1 \; ,
\end{aligned} \tag{5.8}$$

with D, E, and F universal constants.

VI. NONLINEAR EFFECTS AND VALIDITY OF RESULTS

In the preceding two sections, we derived the critical behavior of a particular model potential $V_0(z)$ in three dimensions using a linear truncation of the full renormalization-group equations. In this section we will examine the validity of the results.

What we are really interested in is an interface which is constrained to have z positive, i.e., an infinite potential for negative z. As mentioned previously, the truncated linear RG used in this paper clearly cannot handle such a potential correctly and we must consider the effects of nonlinear terms in the RG flow equations. Before proceeding with a discussion of such effects, it is instructive to consider what class of potentials will yield the same critical behavior under the *linear* RG as the model analyzed in Secs. IV and V. If we restrict ourselves to potentials with a single minimum which is controlled by the balance between an attractive and repulsive exponential tail with the leading large z behavior given by Eqs. (4.8) and (4.9), then this is primarily a question of the behavior of $V_0(z)$ for negative z, i.e., the wall part $W(z)$. It is easy to see that for $z\sim\mu l$ with μ positive and fixed and l large, the renormalized wall potential $W_l(z)$ will have the same form as Eq. (4.21) as long as $W_0(z)$ increases more slowly than an exponential for large negative z. Physically, it would appear reasonable to replace a hard wall with a soft wall which grows rapidly for $z<0$ (say, as $e^{\sqrt{-z}}$) or perhaps even with the constant potential wall used in Sec. III. One might hope that general possible nonlinear terms in the RG flows could then be handled perturbatively about the linear solution. It can be shown, however, that this will not work even for the constant wall potential. This is because for large l, $V_l(z)$ will be large for a range of z and nonlinear terms in the regions where it is large can produce large effects on the behavior in the important region of $V_{l^*}(z)$, i.e., z near z_m^*. We thus cannot resort to general perturbative arguments to treat the nonlinear RG flows but must carefully consider the origin and form of the nonlinearities.

There are two natural ways that one might proceed. This first would be to consider a soft-wall potential (which is amenable to the momentum space RG approach used here) and analyze the effects of various nonlinear terms in an *exact* differential RG.[5,10] The second approach would be to consider an approximate RG which bounds the renormalized potential when it is large and reduces to the linear RG used in Sec. IV when the potential is small with errors which can be controlled. Neither of these approaches is easy to implement, the first because exact differential RG's (Refs. 5 and 10) are rather complicated and clumsy and the second because of the dual requirements placed on such an approximate RG. We will therefore here just discuss why we believe the linear RG results are likely to be correct and leave a more careful treatment for future work.

Although an exact differential renormalization group must contain many functions of several momenta (for example, the potential will not just be a function of the real space z), it is instructive to consider a "toy" RG which contains some of the potentially dangerous terms in order to investigate the reasonableness of the linear truncation. One of the potentially most dangerous terms involves contributions to the renormalization of the potential proportional to higher powers of V. Since a constant potential is only trivially renormalized, we know that nonlinear terms can involve only derivatives of V with respect to z. The lowest-order term has the form $(\partial V/\partial z)^2$. (As mentioned above such terms cannot be treated straightforwardly perturbatively.) Since the fluctuations of the interface which are integrated out will more often go into regions where V is smaller, the nonlinear terms will tend to *decrease* the renormalized V; this is in fact what is found in a perturbative functional RG derived by one of us.[7] The truncated flow equation

$$\frac{\partial V}{\partial l}=2V+\omega\left[\frac{\partial}{\partial z}\right]^2 V-\frac{1}{2}\left[\frac{\partial V}{\partial z}\right]^2 \tag{6.1}$$

should at least qualitatively describe the leading nonlinear behavior for small V.

We are thus led to ask what is the magnitude of $(\partial V_s/\partial z)^2$ for the parts of the potential at scales $0<s<l$ which affect V_l near its minimum. More precisely, in the linear approximation of Sec. IV we can write

$$V_l(z)=\frac{e^{2(l-s)}}{\sqrt{4\pi\omega(l-s)}}\int dz'\, V_s(z')e^{-(z'-z)^2/4\omega(l-s)} \tag{6.2}$$

and then ask how large $V_s(z')$ is for the z' which dominate the integral in Eq. (6.2) for $l=l^*$ and z near z_m^*. We will call this dominant z', z_D'. In regime I, it is straightforward to see that the dominant parts of $V_s(z')$ are exponentially small until $l-s$ is of order one. Therefore the nonlinear terms will only enter at the final stages of the renormalization and not modify the form of the singularities. For regimes II and III, on the other hand, the analysis is slightly messy. What is found is that the important parts of the repulsive and wall parts of V_s are where z' is near

$$z_D'=zs/l \tag{6.3}$$

for which in regime II

$$V_s(z_D')\sim\exp[(s/2l)\ln l-\tfrac{1}{2}\ln s] \tag{6.4}$$

and in regime III

$$V_s(z_D')\sim\exp[\tfrac{3}{2}(s/l)\ln l-\tfrac{3}{2}\ln s] \tag{6.5}$$

with $\partial V_s/\partial z'$ the same order as V_s in both regimes. Expressions (6.4) and (6.5) are smaller than order one except for s small and $(l-s)/s \lesssim 1/\ln l$. Thus, the *local* effects of the nonlinear terms on $\widetilde{V}_s(z')$ should be small in the region of z' which dominates the behavior near the minimum at scale l^*. However it is possible that the cumulative effects of the nonlinear terms in the regions further away (for smaller z') could (as mentioned previously) build up to affect the renormalized potential in the important region. The *sign* of the nonlinear term in Eq. (6.2) and the general argument that the nonlinear terms will suppress the growth of the potential suggest that this is not the case. The regions where $V_s(z')$ is large do not contribute significantly in the linear RG. If the nonlinear terms suppress V_s in these regions, their effects on the important region of $V_l(z)$ will *decrease* and thus still be negligible. On the other hand, the effects of the regions where $V_s(z')$ is small will be determined by the linearized RG flow. Thus it is natural to expect that nonlinear terms of the form in Eq. (6.1) will *not* affect the asymptotic critical behavior. Similar qualitative arguments can be made, for example, for the effects of renormalizations of the local surface tension which are of order V^2.

Since the full RG equations are very complicated, however, it would be more straightforward if one could argue that the effects of any reasonable form of wall, in particular an infinite hard wall, is bounded *above* at long distances and length scales by $e^{-z^2/4\omega l}e^{2l}/\sqrt{l}$ and has this limiting behavior (up to functions of z/l) for V_l smaller than one. At this stage, it is not clear how to demonstrate this. Some bounds can simply be derived, however, by noting that the e^{2l} factor in the renormalized potential is just the area of the interface at length scale e^l. It follows that even at a distance z of order one (i.e., the cutoff) from a hard wall, the renormalized potential will be bounded above by a number of order unity times e^{2l}. Hence the behavior *near* the wall is not underestimated by the linear RG. Hopefully, careful arguments can be made which yield the desired bounds convincingly; however, a careful definition of the renormalized potential is clearly necessary.

We note that in regime III, in addition to the effects of a repulsive wall, the effects of a short distance attraction must also be analyzed when it has a strength near the critical value of order one. This is likely to be rather more difficult. The exactly soluble two-dimensional problem which also has a transition at a critical strength of order one[11] should provide a useful testing ground for approximate methods.

To conclude this discussion, it is almost certain that the critical behavior in regime I is given correctly by the linearized RG, but in regimes II and III, at this stage is is only a reasonable speculation that the results of BHL and Secs. IV and V are correct.

VII. SINGULARITY AT COMPLETE WETTING

A related problem to critical wetting is the divergence in the film thickness for *repulsive* wall interactions as the coexistence curve is approached by varying, say, the chemical potential, μ. The effective wall potential V will again have a wall part which we take to be a constant for $z<0$, a *repulsive* exponential part ae^{-z} with a positive and of order one, and an attractive linear potential δz with slope proportional to the deviation, δ, of the chemical potential from coexistence. We thus take

$$V_0(z)=\begin{cases} c, & z<0 \\ ae^{-z}+\delta z, & z>0 \ . \end{cases} \tag{7.1}$$

In mean-field theory, we have the capillary length

$$\xi_{\parallel}\sim\delta^{-1/2} \tag{7.2}$$

and the mean film thickness

$$\langle z\rangle \approx \ln(1/\delta) \ . \tag{7.3}$$

It is straightforward to see that when we include fluctuations in three dimensions, there are two regimes, one corresponding to the minimum in the renormalized potential being in the exponential tail of the repulsive part of the other to it being in the short distance regime affected by the wall. We thus have, for $\omega<2$,

$$\xi_{\parallel}\sim\delta^{-1/2} \tag{7.4}$$

and

$$\langle z\rangle \approx \frac{2+\omega}{2}\ln\frac{1}{\delta}\approx(2+\omega)\ln\xi_{\parallel} \ , \tag{7.5}$$

while for $\omega>2$, Eq. (7.4) still holds but

$$\langle z\rangle \approx \sqrt{2\omega}\left[\ln\frac{1}{\delta}-\tfrac{1}{4}\ln\ln\frac{1}{\delta}\right]$$

$$\approx\sqrt{8\omega}(\ln\xi_{\parallel}-\tfrac{1}{8}\ln\ln\xi_{\parallel}) \ . \tag{7.6}$$

Thus the critical behavior of the capillary length remains mean-field-like for all ω while the coefficient of the $\ln(1/\delta)$ in $\langle z\rangle$ varies. This latter effect should be testable since the surface tension and ξ_b are measurable and hence ω can be determined.

VIII. CRITICAL WETTING WITH LONG-RANGE INTERACTIONS

Our functional renormalization group can also be applied to the power-law potential

$$V(z)=\begin{cases} c, & z<0 \\ -az^{-n}+bz^{-m}, & z>0 \ , \end{cases} \tag{8.1}$$

where we have $n<m$ and b and c positive. For this potential and dimensionalities $d<3$ we again find three regimes of critical behavior. However, for a given dimensionality and potential, the critical behavior is independent of the surface tension, unlike the case of $d=3$ and exponentially decaying $V(z)$ discussed above.

When the subdominant power law obeys

$$m<m_c(d)=\frac{2(d-1)}{3-d} \tag{8.2}$$

the interfacial fluctuations will not alter the mean-field critical behavior, which is[12]

$$\langle z\rangle\sim\tau^{-1/(m-n)} \ , \tag{8.3}$$

$$\xi_{\parallel}\sim\tau^{-(m+2)/2(m-n)} \ , \tag{8.4}$$

where

$$\tau\sim a\sim(T-T_w) \ . \tag{8.5}$$

For exponents

$$n<m_c(d)<m \tag{8.6}$$

in (8.1), the critical behavior is determined by the long-range tail of the attractive part of the potential and the wall or short-range limit of the repulsive part of the potential. In this regime (which is analogous to regime II) the critical behavior is

$$\langle z\rangle\sim\tau^{-1/(m_c-n)} \ , \tag{8.7}$$

$$\xi_{\parallel}\sim\tau^{-(m_c+2)/2(m_c-n)} \ , \tag{8.8}$$

with the critical point still given by (8.5). In the third regime,

$$n>m_c(d) \ , \tag{8.9}$$

the wetting critical behavior is determined, purely by the short-range part of the potential. This critical behavior is governed by a fixed point which involves a nonzero potential and therefore appears to be outside of the scope of our perturbative approach. In this regime, the critical value of a is greater than zero.

The power-law potential, Eq. (8.1), can also be used away from bulk coexistence. Then the linear term in the potential is given by $n=-1$. Thus the critical behavior on approach to complete wetting is as in Eqs. (8.3) and (8.4) for $m<m_c(d)$, and (8.7) and (8.8) for $m>m_c(d)$, where $n=-1$ must be used in both cases. Note that in this case there can be no regime with $a_c\neq 0$.

Kroll and Lipowsky[11] have examined the case of $m\to\infty$ in two dimensions, where the critical behavior can be calculated using a transfer operator technique. They do indeed find the critical behavior given by (8.7) and (8.8) for $n<m_c(2)=2$. For the marginal case $n=2$, they find that the wetting transition occurs when the attractive part of the potential is still finite and exhibits an essential singularity of the form $\langle z\rangle\sim\exp(\tau^{-1/2})$. This curious result is reminiscent of the Kosterlitz-Thouless[13] and roughening[14] transitions and therefore one might hope to rederive it using a renormalization group, but it is outside of the scope of our present linear functional approach.

IX. CONCLUSIONS

In this paper we have used a linearized functional renormalization group to analyze the critical behavior near second-order wetting transitions. The upper critical dimension falls out very simply in this formulation. In three dimensions, the rather complicated critical behavior found by BHL (Ref. 3) for short-range interactions is rederived using this functional RG. New information on the phase diagram and multicritical behavior is yielded, in particular the behavior near to the multicritical point $\omega=2$ below which the interface can be bound to the wall with an arbitrary weak potential and above which it cannot. We find that the critical strength of the attractive substrate potential goes to zero linearly with $\omega-2$. For $\omega<2$, the film thickness in units of the bulk correlation length diverges as $\ln\tau$ with a coefficient which is a *universal* function of ω. However for $\omega>2$, $\langle z\rangle\sim 1/\tau$. The marked difference between these behaviors, and the universality for $\omega<2$ should be verifiable experimentally. For ω near two, the behavior will be dominated by the multicritical point at which $\langle z\rangle\sim 1/\tau^2$ and there will only be a narrow critical region for $(\omega-2)$ small and positive, as shown in Fig. 2. This may complicate interpretation of experimental or numerical results.

So far, we have said very little about the behavior near wetting transitions for short-range interactions in dimensions less than three. In two dimensions, the critical behavior is known exactly by various methods:[11,15] $\langle z\rangle$ diverges as $1/\tau$ and $\xi_{\parallel}$ as $1/\tau^2$. As in regime III in $d=3$, a nonzero critical strength of the attraction to the wall is needed to bind the interface. The behavior between two and three dimensions should be similar to $d=2$ and regime III with the correlation length exponent ν diverging as $d\rightarrow 3^-$. It may be possible to perform a $3-\epsilon$ expansion about the behavior in regime III, although at present it is not clear how to do this. Controlling the effects of the nonlinearities in the RG flows will almost certainly be necessary to carry out such an expansion.

At this stage, it is not clear what is the regime of validity of the linear truncation of the RG flows used here in three dimensions, although, as discussed in Sec. VI, we speculate that the results of this paper should should survive a fuller treatment. This should be simplest to show for three dimensions in regime I where the effects of fluctuations are small and the nonlinear terms in the RG are most likely to be negligible. It is hoped that approximate variational RG's which reduce to our linear RG for small potentials should be useful both for checking the results discussed here and perhaps for investigating related problems.

There are many other applications of the linear functional RG introduced here. Several of them, including the behavior as the coexistence curve is approached in the region of complete wetting and the effect of long-range substrate-adsorbate potentials were investigated in the last two sections. Another problem which should be amenable to this approach is tricritical wetting where the wetting transition changes from first to second order. This is likely to yield very rich behavior.

Finally, we note that the systems investigated here provide examples of problems which are more simply treated in terms of the full Wilson RG scheme[5] of a many parameter Hamiltonian space (here a function) than in terms of a few parameters.

ACKNOWLEDGMENTS

The authors would like to thank E. Brézin and R. Lipowsky for discussions which stimulated this work and R. Lipowsky in particular for raising questions concerning potential difficulties arising from nonlinear terms in the renormalization group. The hospitality of the Aspen Center for Physics where much of this work was carried out is gratefully acknowledged.

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