Integral formulation of transport equations: Quantum theory versus Boltzmann equation

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Mathematical tools needed in applying the integral formulation of quantum transport theory are developed and analyzed. Particular attention is paid to the sequence of approximations reducing the quantum equations to the corresponding integral formulation of the Boltzmann equation. The role played by the concepts of intracollisional field effect and finite collision duration time in the approximation scheme is discussed.

I. INTRODUCTION

The electric fields encountered in the smallest (submicrometer) semiconductor components are so strong that Boltzmann transport theory, which is conventionally used for device modeling, becomes suspect (see, e.g., Ref. 1). Thus the need for high-field quantum transport equations has become pressing and, indeed, many independent and different formulations have appeared recently.[2–10] However, the application of these quantum transport equations to problems of practical interest has remained a notoriously difficult problem with very few reported results. In conventional Boltzmann theory the iterative technique (see, e.g., Ref. 11) often provides a good starting point for further calculations. The iterative technique is based on transforming the Boltzmann equation (BE) into an integral equation (IBE) (in contrast to the customary integro-differential form) which is then iterated until convergence has been achieved. Recently, Sarker[12] has suggested using the integral form of quantum transport equations as a starting point for studying high-field transport phenomena in small systems. By making certain approximations Sarker simplifies the integral quantum transport equation (IQT) to a form which appears tractable. The purpose of the present investigation is to gain deeper insight into the nature of the IQT by contrasting it against the IBE. In particular, we state precisely the sequence of approximations that are needed in order to reduce the IQT to IBE. This task might appear as a mere technical exercise, but it turns out to be surprisingly nontrivial, requiring physical arguments as well as careful algebraic manipulations. Of course, it is a necessary prerequisite to understand precisely where and how the IQT goes beyond the IBE if the IQT is to be applied successfully to realistic transport problems.

The organization of this paper is the following. In order to make the presentation self-contained, we briefly discuss the differential form of the quantum transport equations[8,13,14] and indicate how the conventional Boltzmann equation is recovered from it. Next we state the IBE and introduce the IQT (Refs. 12 and 15–17) and analyze some of its properties. Finally, we derive the IBE from IQT.

II. GENERALIZED KADANOFF-BAYM EQUATIONS VERSUS BOLTZMANN EQUATION

The generalized Kadanoff-Baym (GKB) equations are[8,14]

$$[(G^0)^{-1}-U-\Sigma,G^{>,<}]-[\Sigma^{>,<},G] = -\{\Sigma^{<},G^{>}\}/2+\{\Sigma^{>},G^{<}\}/2 \ . \quad (1)$$

Equation (1) has to be supplemented with the Dyson equation for the retarded and advanced Green functions $G^{r,a}$. The Wigner distribution function $f(\mathbf{p},\mathbf{R},T)$ is obtained from the correlation function $G^{<}$ with the prescription

$$f(\mathbf{p},\mathbf{R},T)=-iG^{<}(\mathbf{p},\tau=0,\mathbf{R},T) = -i\int\frac{d\omega}{2\pi}G^{<}(\mathbf{p},\omega,\mathbf{R},T) \ . \quad (2)$$

For further discussion of (1), see Ref. 8, whose notation we will follow. Equation (1) can be transformed into a number of (still exact) forms, of which we will make use of the following:

$$\frac{\partial f(\mathbf{p},T)}{\partial T}=\int_{-\infty}^{\infty}dt'(\Sigma^{r}G^{<}+\Sigma^{<}G^{a}-G^{r}\Sigma^{<}-G^{<}\Sigma^{a}) \quad (3a)$$

$$=-\int_{-\infty}^{T}dt'(\Sigma^{>}G^{<}+G^{<}\Sigma^{>} -\Sigma^{<}G^{>}-G^{>}\Sigma^{<}) \ , \quad (3b)$$

where the time labels of all the terms in the integrand have the structure

$$AB=A(T-t',(t'+T)/2)B(t'-T,(t'+T)/2) \ .$$

Here we consider uniform systems and the diagonal momentum label is often suppressed. Note that there is no ordinary Boltzmann driving term present in (3): this can be recovered by changing the variables from the canonical momentum $\mathbf{p}$ to the kinematical momentum $\mathbf{k}=\mathbf{p}-\mathbf{A}(\mathbf{R},T)$, where $\mathbf{A}(\mathbf{R},T)$ is the vector potential.[8,17,18]

The right-hand side of (3) involves the correlation functions $G^{>,<}$ which depend on four variables $(\omega,\mathbf{p},T,\mathbf{R})$

rather than the Wigner function f which depends only on three variables $(\mathbf{p},\mathbf{R},T)$. To overcome this problem one often makes the ansatz

$$G^<(\mathbf{p},\omega,\mathbf{R},T)=iA(\mathbf{p},\omega,\mathbf{R},T)f(\mathbf{p},\mathbf{R},T)\ , \tag{4}$$

where A is the full nonequilibrium spectral density, $iA=G^a-G^r$. As in equilibrium A satisfies

$$\int \frac{d\omega}{2\pi}A(\mathbf{p},\omega,\mathbf{R},T)=1\ , \tag{5}$$

but it is not a positive semidefinite quantity. The question of the range of the applicability of the ansatz (4) is left open: We note here that the transport equations (3) together with the ansatz (4) have proven to be extremely efficient in giving simple and transparent derivations of results otherwise requiring long calculations (for Holstein's electron-phonon transport equations,[19] see Ref. 20; for high-field electron-phonon transport equations, see Ref. 8).

Methods of recovering the Boltzmann equation from the GKB are well known.[14] Here we will discuss the electron-dilute impurity system, but similar arguments can be applied to the weakly coupled electron-phonon system as well. The self-energy for the elastic impurity problem is given by

$$\Sigma(\mathbf{p},t,t')=c\sum_{\mathbf{p}'}|V(\mathbf{p}-\mathbf{p}')|^2G(\mathbf{p}',t,t')\ , \tag{6}$$

where c is the (small) concentration of impurities. To recover the Boltzmann equation the following steps are employed. (i) Lowest-order gradient expansion is used throughout. The rules for gradient expansion can be compactly summarized as follows (here we state the result for temporal variables, a similar result holds for spatial variables). We define

$$(A\odot B)_{tt'}=\int_{-\infty}^{\infty}ds\,A(t,s)B(s,t')\ . \tag{7}$$

Then the Fourier transform of $A\odot B$ with respect to the difference variable $\tau=t-t'$ is given by $[T=(t+t')/2]$

$$(A\odot B)_{\omega,T}=A(\omega,T)\exp\left[\frac{1}{2i}\left[\frac{\overleftarrow{\partial}}{\partial T}\frac{\vec{\partial}}{\partial\omega}-\frac{\overleftarrow{\partial}}{\partial\omega}\frac{\vec{\partial}}{\partial T}\right]\right]B(\omega,T)\ . \tag{8}$$

Thus, the lowest-order gradient approximation consists of taking

$$(A\odot B)_{\omega T}=A(\omega,T)B(\omega,T)\ .$$

(ii) Terms of second order in small quantities are neglected. These include terms like $[\Sigma,G^{><}]$ because they are proportional to $\partial/\partial T$ and the impurity concentration c. (iii) The spectral density is approximated by the corresponding free-particle spectral density,

$$A_0(\mathbf{p},\omega,\mathbf{R},T)=2\pi\delta(\omega-\epsilon(\mathbf{p}))\ .$$

This amounts to neglecting quasiparticle renormalization effects and the intracollisional field effect (ICFE). Application of steps (i)–(iii) then yields the familiar electron-impurity Boltzmann equation (BE).

To recover the BE from (3) involves a slightly different kind of reasoning. We will discuss the analysis in some detail because similar arguments are used in the integral formulation of transport theory. Focusing on (3a), the first step is to use equilibrium retarded and advanced Green's functions and self-energies. Next, the ansatz (4) is employed with the free-particle spectral density. These two steps amount to neglecting the intracollisional field effect which describes the interference between acceleration under the external field and the scattering due to the impurity potential. Now recall that BE assumes pointlike collisions, both in time and space. Another way of expressing this property of the BE is to note that one uses Fermi golden-rule transition rates in the BE, and these transition rates are derived by considering an asymptotic time limit, i.e., completed collisions. Yet another way of saying this is that the BE assumes zero collision duration. All of the above implies that the distribution function in the collision integral of (3) must be decoupled from the internal dynamics of the microscopic collision processes in order to bring (3) into accordance with the Boltzmann picture. Mathematically, this is achieved by making the replacement

$$f(\mathbf{p},(t'+T)/2)\rightarrow f(\mathbf{p},T)\ .$$

Having done this, the time integrations in (3) collapse to energy-conserving δ functions, and the familar BE has been recovered. We note in passing that letting the upper bound of the t' integration in (3b) approach infinity does not lead to the BE (as suggested recently by Barker[2]): This procedure, in addition to being slightly ambiguous (what to do with the T inside the integration?), leads to an incorrect overall multiplicative factor.

Finally, a procedure corresponding to the gradient expansion can be constructed by Taylor-expanding

$$f(\mathbf{p},(t'+T)/2)=f(\mathbf{p},T)+[(t'-T)/2]\frac{\partial f(\mathbf{p},T)}{\partial T}+\cdots\ . \tag{9}$$

It is an easy exercise to convince oneself that the gradient expansion [when applied to (1)] and the procedure (9) [when applied to (3)] lead to identical results.

III. INTREGRAL TRANSPORT EQUATIONS

The Boltzmann equation

$$\left[\frac{\partial}{\partial t}+\mathbf{E}(t)\cdot\frac{\partial}{\partial\mathbf{k}}\right]f(\mathbf{k},t)=-\sum_{\mathbf{k}'}W(\mathbf{k},\mathbf{k}')[f(\mathbf{k},t)-f(\mathbf{k}',t)] \tag{10}$$

can be expressed as an integral equation (see, e.g., Jacoboni and Reggiani in Ref. 11):

$$f(\mathbf{k},t)=\exp\left[-\int_0^t ds\,\lambda(\mathbf{k}_t(s))\right]f(\mathbf{k}_t(s=0),t=0)+\int_0^t dt'\exp\left[-\int_{t'}^t ds\,\lambda(\mathbf{k}_t(s))\right]\times\sum_{\mathbf{k}'}W(\mathbf{k}_t(t'),\mathbf{k}')f(\mathbf{k}',t')\ , \tag{11}$$

where

$$\mathbf{k}_t(s)=\mathbf{k}-\int_s^t ds'\mathbf{E}(s') . \tag{12}$$

The IBE (11) is well suited for studying a system which is in thermal equilibrium until $t=0$ when the external perturbation is turned on. Another possible form of the IBE is

$$f(\mathbf{k},t)=\int_0^\infty dt' \exp\left[-\int_{t-t'}^t ds\,\lambda(\mathbf{k}_t(s))\right] \times\sum_{\mathbf{k}'} W(\mathbf{k}_t(t-t'),\mathbf{k}')f(\mathbf{k}',t-t') , \tag{13}$$

which is useful when the external field is turned on in the infinite past and one does not want to refer to the initial distribution (or does not have information about it). In (11) $\lambda(\mathbf{k})$ is given by

$$\lambda(\mathbf{k})=\sum_{\mathbf{k}'} W(\mathbf{k},\mathbf{k}')=2\pi\sum_{\mathbf{k}'} c\,|\,V(\mathbf{k}-\mathbf{k}')\,|^2\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}')) , \tag{14}$$

where the second equality is valid for the elastic impurity problem. The equivalence of (10) and (11) is easily demonstrated by operating by $[\partial/\partial T+\mathbf{E}(t)\cdot\partial/\partial\mathbf{k}]$ on the left-hand side of (11) or (13), and verifying that the result is, indeed, the collision term on the right-hand side of (10). We note that in equilibrium, $\mathbf{E}(t)=0$, both (11) and (13) reduce to

$$\lambda(\mathbf{k})f(\mathbf{k})=\sum_{\mathbf{k}'} W(\mathbf{k},\mathbf{k}')f(\mathbf{k}') , \tag{15}$$

which is a statement of a vanishing collision term, as one would expect in equilibrium.

The quantum-mechanical analog of (11) is[15–17]

$$G^<=(1+G^r\Sigma^r)G_0^<(1+\Sigma^aG^a)+G^r\Sigma^<G^a . \tag{16}$$

The structure of (16) is very suggestive when compared to the IBE (11): the first term corresponds to the decay of the initial state whereas the second term is the "scattering-in" term. Further, it is intuitively clear that in the second term the advanced and retarded Green functions will combine to give the exponential term in (11), and the self-energy term $\Sigma^<$ will generate the piece proportional to the distribution function. Precisely how this will happen is the central issue of the present work.

An important preliminary ingredient to investigate is to see how thermal equilibrium is contained in (16). In thermal equilibrium there is no T dependence and the zeroth-order gradient expansion is, in fact, exact. Recalling the relation between f and $G^<$ we write (16) as [here we focus on the second term in (16) only, i.e., the analog of (13)]

$$if(\mathbf{p})=\int\frac{d\omega}{2\pi}G^r(\mathbf{p},\omega)\Sigma^<(\mathbf{p},\omega)G^a(\mathbf{p},\omega) . \tag{17}$$

The equilibrium Green functions $G^{r,a}$ are given by

$$G^{r,a}(\mathbf{p},\omega)=[\omega-\epsilon(\mathbf{p})-\Sigma(\mathbf{p},\omega)\pm i\Gamma(\mathbf{p},\omega)/2]^{-1} , \tag{18}$$

where Σ and $\mp\Gamma/2$ are the real and imaginary parts of the self-energy $\Sigma^{r,a}$,

$$\Sigma^{r,a}(\mathbf{p},\omega)=\sum_{\mathbf{p}'}\frac{c\,|\,V(\mathbf{p}-\mathbf{p}')\,|^2}{\omega-\epsilon(\mathbf{p}')\pm i\eta} . \tag{19}$$

In equilibrium the ansatz (4) is exact.[21] We can replace the full spectral density in (4) by the free-particle spectral density,

$$A_0(\mathbf{p},\omega)=2\pi\delta(\omega-\epsilon(\mathbf{p})) , \tag{20}$$

because we are considering a dilute concentration of impurities. As a result, (17) can be written as

$$f(\epsilon(\mathbf{p}))=\sum_{\mathbf{p}'}\frac{2\pi}{\Gamma(\mathbf{p},\epsilon(\mathbf{p}'))} \times\left[\frac{1}{\pi}\frac{\Gamma(\mathbf{p},\epsilon(\mathbf{p}'))/2}{[\epsilon(\mathbf{p}')-\epsilon(\mathbf{p})-\Sigma]^2+(\Gamma/2)^2}\right] \times c\,|\,V(\mathbf{p}-\mathbf{p}')\,|^2 f(\epsilon(\mathbf{p}')) . \tag{21}$$

Since we are working in the low-concentration limit it is consistent to replace the quantity in braces by a δ function. This final step brings (21) into precise agreement with the equilibrium result obtained from the IBE, Eq. (13) [note that $\Gamma(\mathbf{p},\omega=\epsilon(\mathbf{p}))=\lambda(\mathbf{p})$].[22]

It might now appear that recovering the IBE from (16) is a straightforward task: the only thing one has to do is to go "slightly beyond" the zeroth-order gradient expansion and the final result should emerge automatically. However, this is not the case and there are several complications. First, the IBE is obtained from the BE by an exact mathematical transformation. If one now tries to evaluate the time integrals in (16) (two in the case of the second term, on which we will focus hereforth) in an approximate way without paying attention to the physics underlying the BE, it is not likely that the correct result can be obtained. Second, when deriving the BE from GKB (see Sec. II) it was sufficient to use zeroth-order $G^{r,a}$ both in scattering and in driving fields. As we shall see below, this is not the case with the IQT.

We begin our analysis of the IQT by constructing expressions for the advanced and retarded Green's functions. These functions satisfy (for a brief moment we drop the labels r and a)

$$\left[i\frac{\partial}{\partial t}-\epsilon(\mathbf{p}-\mathbf{A}(t))\right]G(\mathbf{p},t,t') -\int dt_1\Sigma(\mathbf{p},t,t_1)G(\mathbf{p},t_1,t')=\delta(t-t') , \tag{22}$$

$$\left[-i\frac{\partial}{\partial t'}-\epsilon(\mathbf{p}-\mathbf{A}(t'))\right]G(\mathbf{p},t,t') -\int dt_1 G(\mathbf{p},t,t_1)\Sigma(\mathbf{p},t_1,t')=\delta(t-t') .$$

Now we have to solve these equations in the "Boltzmann spirit." To understand what this notion means we first solve (22) within the gradient approximation. Though the result turns out to be only qualitatively correct, nevertheless it gives us a useful hint as how one should proceed.

Applying the gradient approximation on (22) yields

$$[\omega-\epsilon(\mathbf{p}-\mathbf{A}(T))-\Sigma(\mathbf{p},\omega,T)]G(\mathbf{p},\omega,T)=1\ . \tag{23}$$

Transforming back to ω space we get

$$G^r(\mathbf{p},\tau,T)=\int\frac{d\omega}{2\pi}\exp(-i\omega\tau)G^r(\mathbf{p},\omega,T)=-i\Theta(\tau)\exp\{-i\tau[\epsilon(\mathbf{p}-\mathbf{A}(T))+\Sigma^r(\mathbf{p},\omega=\epsilon(\mathbf{p}-\mathbf{A}(T)),T)]\}\ , \tag{24}$$

where we made use of the fact that Σ is small (either dilute concentration of impurities or a weak electron-phonon coupling) to replace the frequency argument in the self-energy by its value at the pole of the free Green's function. Sarker[12] has made use of similar arguments in his treatment of IQT's. Equation (24) is, however, not sufficient to yield IBE's. Recalling that the gradient approximation implies slow temporal variations, one can guess that the correct solution might be

$$G^r(\mathbf{p},\tau,T)=-i\Theta(\tau)\exp\left[-i\int_{T-\tau/2}^{T+\tau/2}ds[\epsilon(\mathbf{p}-\mathbf{A}(s))+\Sigma^r(\mathbf{p},s)]\right]\ . \tag{25}$$

The guess (25) appears quite plausible because (i) it reproduces correctly the free field-dependent Green function whose exact expression is known,[8] and (ii) an approximate evaluation of the time integral in (25) gives (24). The time argument of the self-energy is slightly ambiguous in (25), however, and a more careful analysis is needed.

Before trying to find the consistent solution to (22) one must transform it into a gauge-invariant form.[17,18] This step is crucial: approximating a gauge-dependent equation leads to an incorrect result. The transformation we need is

$$\tau=t-t',\quad T=(t+t')/2,\quad \mathbf{k}=\mathbf{p}-\mathbf{A}((t+t')/2)\ , \tag{26}$$

and we introduce a tilde to distinguish the transformed functions

$$F(\mathbf{p},t,t')=F(\mathbf{k}+\mathbf{A}(T),T+\tau/2,T-\tau/2)=\tilde{F}(\mathbf{k},\tau,T)\ . \tag{27}$$

Adding the two equations (22) we find that the transformed functions obey

$$\left[\frac{i\partial}{\partial\tau}-\tfrac{1}{2}[\epsilon(\mathbf{k}_+)+\epsilon(\mathbf{k}_-)]\right]\tilde{G}(\mathbf{k},\tau,T)-\tfrac{1}{2}\int dt_1[\tilde{\Sigma}(\mathbf{k}_T(T_+),\tau_+,T_+)\tilde{G}(\mathbf{k}_T(T_-),\tau_-,T_-)+\tilde{G}(\mathbf{k}_T(T_+),\tau_+,T_+)\tilde{\Sigma}(\mathbf{k}_T(T_-),\tau_-,T_-)]=\delta(\tau)\ , \tag{28}$$

where we introduced, following Sarker,[12]

$$\mathbf{k}_\pm=\mathbf{k}_T(T\pm\tau/2)\ ,$$
$$T_\pm=\tfrac{1}{2}(T\pm\tau/2+t_1),\quad \tau_\pm=\pm(T-t_1)+\tau/2\ . \tag{29}$$

Equation (28) is still exact and is cast in a manifestly gauge-invariant form (no reference is made to the vector potential and the physical $\mathbf{E}$ field appears throughout). Now is the correct point to find an approximate solution to (28). We proceed in three steps. (i) The self-energies are replaced by the corresponding equilibrium quantities. This is consistent with the neglect of the intracollisional field effect in the Boltzmann picture and it was also employed while deriving the BE in the differential formulation. Consequently, the self-energies in (28) depend only on one time variable, $\tau_\pm$. (ii) BE describes completed collisions. The self-energies in (28) can be interpreted as generalized scattering rates, and in order to transform (28) into the Boltzmann picture the Green's functions must be moved outside the time integration. Further, the integration variable t_1 should be set equal to $T+\tau/2$ in the first term of (28) and equal to $T-\tau/2$ in the second term. This procedure is entirely analogous to the one used in connection with the differential form of quantum-transport equations (3). (iii) The self-energies are evaluated at the pole of the Green's function multiplying them, parallel to the argument leading to Eq. (24). Completion of these steps results in

$$\left[i\frac{\partial}{\partial\tau} - \tfrac{1}{2}[\epsilon(\mathbf{k}_+)+\epsilon(\mathbf{k}_-)+\tilde{\Sigma}(\mathbf{k}_+,\omega=\epsilon(\mathbf{k}_+))+\tilde{\Sigma}(\mathbf{k}_-,\omega=\epsilon(\mathbf{k}_-))] \right] \tilde{G}(\mathbf{k},\tau,T)=\delta(\tau) , \tag{30}$$

whose solution is

$$\tilde{G}^{r,a}(\mathbf{k},\tau,T)=\mp i\Theta(\pm\tau)\exp\left[-i\int_{T-\tau/2}^{T+\tau/2} dt_1[\epsilon(\mathbf{k}_T(t_1))+\tilde{\Sigma}^{r,a}(\mathbf{k}_T(t_1),\omega=\epsilon(\mathbf{k}_T(t_1)))] \right] , \tag{31}$$

where the advanced and retarded self-energies are given in (19).

We now turn to the IQT, Eq. (16). Transforming once again to the gauge-invariant variables and setting $\tau=0$, we get

$$\tilde{f}(\mathbf{k},T)=-i\int d\tau' dT' \tilde{G}^{r}(\mathbf{k}_T(T'_+),\tau'_+,T'_+)\tilde{\Sigma}^{<}(\mathbf{k}_T(T'),\tau',T')\tilde{G}^{a}(\mathbf{k}_T(T'_-),\tau'_-,T'_-) , \tag{32}$$

where

$$T'_\pm=\tfrac{1}{2}(T\pm\tau'/2+T'),\quad \tau'_\pm=\pm(T-T')+\tau'/2 , \tag{33}$$

and the retarded and advanced functions are given by (31). The prescription of how to evaluate the double integral in (32) is clear from the arguments leading to $\tilde{G}^{r,a}$. The self-energy must be decoupled from $\tilde{G}^{r,a}$: this is achieved by moving the $\tilde{G}$'s outside τ' integration and evaluating them at $\tau'=0$. The result is

$$\begin{aligned}\tilde{f}(\mathbf{k},T)=-i\int_0^\infty d\bar{T}\,\tilde{G}^{r}(\mathbf{k}_T(T-\bar{T}/2),\bar{T},T-\bar{T}/2)\tilde{G}^{a}(\mathbf{k}_T(T-\bar{T}/2),-\bar{T},T-\bar{T}/2)\\ \times\int d\tau'\sum_{\mathbf{k}'} c\,|\,V(\mathbf{k}_T(T-\bar{T})-\mathbf{k}')\,|^2\tilde{G}^{<}(\mathbf{k}',\tau',T-\bar{T}) .\end{aligned} \tag{34}$$

Consider now the product $\tilde{G}^{r}\tilde{G}^{a}$ occurring in (34). To begin with, the momentum label needed in (34) is evaluated as [see also (31)]

$$[\mathbf{k}_T(T-\bar{T}/2)]_{T-\bar{T}/2}(t_1)=\mathbf{k}-\int_{T-\bar{T}/2}^{T} ds\,\mathbf{E}(s)-\int_{t_1}^{T-\bar{T}/2} ds\,\mathbf{E}(s)=\mathbf{k}_T(t_1) . \tag{35}$$

The single-particle energies and the real parts of the self-energies cancel while the imaginary parts add with the result

$$\tilde{G}^{r}\tilde{G}^{a}=\Theta(T)\exp\left[-\int_{T-\bar{T}}^{T} dt_1\Gamma(\mathbf{k}_T(t_1),\omega=\epsilon(\mathbf{k}_T(t_1))) \right] , \tag{36}$$

in precise agreement with the exponential prefactor in the IBE.

The final step is to evaluate the τ' integral in (34). This is done in the same way as it was done while solving for $\tilde{G}^{r,a}$: $\int d\tau'\tilde{G}^{<}(\tau')$ is approximated by $\tilde{G}^{<}[\omega=\epsilon(\mathbf{k}_T(T-T'))]$. Finally, we apply the ansatz (4) with the free-electron spectral density (20). These steps reduce (34) to the integral Boltzmann equation (IBE), Eq. (13).

Summarizing, we have shown that IBE's can be recovered from IQT's with an approximate way of evaluating two time integrals: one in connection with solving for the advanced and retarded Green's functions and the other in simplifying the IQT itself. The precise form of the approximation is dictated by physical assumptions implied by the Boltzmann equation: (i) neglect of the intracollisional field effect, (ii) completed collisions (a zero collision duration) which allows a decoupling of Green's functions and self-energies under time integrations, and (iii) the smallness of the self-energy, which is due to either a dilute concentration of impurities or the weakness of the electron-phonon coupling.

IV. DISCUSSION

Our analysis shows that the application of IQT's to problems which require a treatment going beyond the Boltzmann picture is quite tricky. The first difficulty one encounters is the need for a "good" solution for $\tilde{G}^{r,a}$, while in differential language it is often possible to circumvent this problem. This is illustrated by the ease with which BE's can be recovered from GKB equations. The second problem is that the approximate way of treating the time integrals employed in this work seems difficult to be generalized to a situation where the BE picture does not act as a physical guideline. In other words, we would like to stress that extreme care should be exercised when approximating IQT's: it is all too easy to introduce unwanted Boltzmannian features by approximations not chosen judiciously.

Note added in proof. The first term in the IQT, Eq. (16), is analyzed in detail in Ref. 23.

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