Effective two-body interaction in Coulomb Fermi liquids

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A simple expression for the effective two-body interaction in a Fermi liquid whose constituents interact *via* a Coulomb force is derived on the basis of a diagrammatic analysis. The interaction, which is spin dependent, is expressed in terms of the local-field factors and the Lindhard polarizability. It includes the contribution of density fluctuations and spin fluctuations, both longitudinal and transverse. A comparison with other interactions known in the literature is made. A generalization to the two-component Coulomb Fermi liquid is also given. An interesting application concerning the possibility of superconductivity in an electron-hole liquid is pointed out.

I. INTRODUCTION

In this paper we shall address ourselves to the following question: what is the effective interaction between two quasiparticles in a degenerate Fermi liquid? We shall be particularly interested in the interaction between two electrons in electron and electron-hole liquids. In the simplest approximation this is given by the expression

$$U_{\text{diel}}(q,\omega)=\frac{4\pi e^2}{q^2\epsilon(q,\omega)}\ , \tag{1}$$

where $\epsilon(q,\omega)$ is the frequency- and wave-number-dependent dielectric function. Equation (1), although correct for "external" test particles, is not correct for electrons belonging to the system, since the electrons in question can exchange themselves or have multiple scattering with any other electron of the system. In the case of spin-$\frac{1}{2}$ particles, new mechanisms of interaction appear which follow from the possibility of exciting spin-density fluctuations. As a consequence the effective interaction becomes spin dependent even if the bare interaction is not. None of these effects are contained in Eq. (1).

Any attempt to improve upon Eq. (1) by the methods of formal many-body theory must face at the outset the formidable task of evaluating an infinite series of Feynman diagrams contributing to the scattering amplitude. Even if such a calculation were possible, the resulting interaction would still be a complicated function of not only the energy and momentum transfers but also of the total momentum and energy of the interacting electrons. Such an expression is not of much practical use. Our aim in this paper is to derive a very simple approximate formula for the effective interaction which, as we shall see, can be expressed in terms of the same "local-field factors" that are widely used in microscopic theories of the polarizability and spin susceptibility of an interacting Fermi liquid. Essentially, we arrive at our result by examining the formal structure of the effective interaction diagrams and assuming that the irreducible particle-hole interactions depend only on the momentum transfer along the particle-hole channel. In this way the evaluation of an infinite series of Feynman diagrams is reduced to the summation of a geometric series. The final expression for the effective interaction is easy to use and to understand on physical grounds. In the limit of small momentum and energy transfer its singular part (i.e., the one that depends on the value of the ratio $\underline{r}=\underline{q}/\omega$) reduces to the singular part of the Landau scattering amplitude with $l=0$. At large q and ω it reduces to the bare Coulomb interaction. We shall discuss the relationship of this effective interaction with other forms existing in the literature. We shall also give its generalization to a two-component Fermi liquid, such as an electron-hole liquid (EHL).

In another paper we make use of our expression for the effective interaction between two electrons in an EHL to discuss the interesting question of the possibility of superconductivity in an EHL.

II. DEFINITION OF THE TWO-BODY EFFECTIVE INTERACTION

In this section we shall explain the meaning of the "two-body effective interaction" and define the terminology to be used in the rest of the paper. Let us introduce the irreducible particle-particle interaction $J_{\sigma_1,\sigma_2}(p_1,p_2,q)$ defined as the sum of all the interaction diagrams which are irreducible in the particle-particle channel. These diagrams have two incoming lines with four-momenta[1] and spins p_1,σ_1 and p_2,σ_2, respectively, two outgoing lines with four-momenta and spins p_1-q,σ_1 and p_2+q,σ_2, respectively, and cannot be decomposed into two parts connected to each other by two particle lines. In a system consisting of only two particles interacting *via* a potential $v(q)$, the irreducible particle-particle interaction would coincide with the antisymmetrized potential:

$$J_{\sigma_1,\sigma_2}(p_1,p_2,q)=v(q)-v(p_1-p_2-q)\delta_{\sigma_1,\sigma_2}\ . \tag{2}$$

It is, therefore, natural to assume that in the many-body system J_{σ_1,σ_2} gives the antisymmetrized effective interaction that we are seeking. Here, however, one must also account for the many-body renormalization of the external lines. Assuming a quasiparticle form of the Green's function

$$G(p)=\frac{z_p}{p_0-\epsilon_p+i\,\delta p_0}\,, \tag{3}$$

where z_p is the renormalization constant and $\epsilon_p=p^2/2m^*-\mu$ the quasiparticle energy relative to the chemical potential, we are led to define the quantity

$$\tilde{J}_{\sigma_1,\sigma_2}(p_1,p_2,q)=(z_{p_1}z_{p_2}z_{p_1-q}z_{p_2+q})^{1/2}J_{\sigma_1,\sigma_2}(p_1,p_2,q) \tag{4}$$

as the antisymmetric two-body effective interaction.

We shall consider a paramagnetic state, in which $\tilde{J}$ has only two independent components, $\tilde{J}_{\uparrow\uparrow}$ and $\tilde{J}_{\uparrow\downarrow}$. We shall show by means of a diagrammatic analysis how these two components can be approximately expressed as

$$\tilde{J}_{\uparrow\uparrow}(p_1,p_2,q)\cong U_{\uparrow\uparrow}(q)-U_{\uparrow\uparrow}(p_1-p_2-q) \tag{5a}$$

and

$$\tilde{J}_{\uparrow\downarrow}(p_1,p_2,q)\cong U_{\uparrow\downarrow}(q)-U^{\mathrm{T}}_{\uparrow\downarrow}(p_1-p_2-q)\,, \tag{5b}$$

where $U_{\uparrow\uparrow}$, $U_{\uparrow\downarrow}$, and $U^{\mathrm{T}}_{\uparrow\downarrow}$ can be calculated from the knowledge of the microscopic polarizability and spin susceptibility of the system. The name of effective interaction is usually reserved in the literature to the local parts of Eqs. (5a) and (5b), namely, $U_{\uparrow\uparrow}(q)$ and $U_{\uparrow\downarrow}(q)$. The complete effective interaction is then constructed by antisymmetrizing $U_{\uparrow\uparrow}(q)$ and leaving $U_{\uparrow\downarrow}(q)$ unchanged. Our treatment shows that a nonlocal term $U^{\mathrm{T}}_{\uparrow\downarrow}(p_1-p_2-q)$ must also enter the $\uparrow\downarrow$ interaction on the same footing as the nonlocal term in the $\uparrow\uparrow$ interaction. This new term will be identified as the interaction *via* transverse spin fluctuations.

The diagrammatic expansion of $\tilde{J}$ is very similar to that of J with the difference that to any internal line we must associate the free-particle Green's function G_0 (with effective mass m^*), and any symbolic block entering the expansion must be consistently renormalized: an interaction block by incorporating a factor $(z_1z_2z_3z_4)^{1/2}$ and a vertex block by incorporating a factor $(z_1z_2)^{1/2}$.

Generally speaking, the diagrammatic expansion of $\tilde{J}$ consists of three classes of diagrams: (i) diagrams that are reducible in the direct particle-hole channel (i.e., the one with particle-hole momentum q), (ii) diagrams that are reducible in the exchange particle-hole channel (i.e., the one with particle-hole momentum p_1-p_2-q), and (iii) diagrams that are totally irreducible. Reducibility in a particle-hole channel means that the diagram *can* be decomposed into two parts connected to each other by a particle-hole pair of lines. It is easy to see that such a diagram must be necessarily irreducible in the particle-particle channel. "Totally irreducible" means irreducible in both particle-hole channels as well as in the particle-particle channel. There is one more definition that we should introduce before proceeding with the diagrammatic analysis of the next section. Following Nozières,[2] we shall call "proper" any diagram that cannot be decomposed into two parts connected by a single interaction line carrying momentum q.

III. DIAGRAMMATIC ANALYSIS OF THE IRREDUCIBLE PARTICLE-PARTICLE INTERACTION

Let us consider the class of diagrams which are reducible in the direct particle-hole channel. These are sometimes referred to as the "dangerous" diagrams because in the small-q limit they give rise to the singular part of the Landau scattering amplitude. The "dangerous" diagrams can be either proper or improper. The improper ones are shown in Fig. 1(a). Here $\tilde{\Lambda}$ represents the proper, renormalized vertex part and the ∿∿ is the screened interaction $v(q)/\epsilon(q,\omega)$. The diagram containing the bare interaction line has been subtracted out because it does not belong to the class of diagrams under consideration. The expansion of $\tilde{\Lambda}$ in terms of $\tilde{I}$, the proper particle-hole interaction irreducible in the particle-hole channel, is shown in Fig. 1(b) (see, for example, Ref. 2). The expansion of v/ϵ in terms of $\tilde{\Lambda}$ is shown in Fig. 1(c). From these expansions it immediately appears that Fig. 1(a) contains only and all improper "dangerous" diagrams. The "proper" dangerous diagrams are shown in Fig. 2(a). Notice that, due to spin conservation, the intermediate particle-hole pairs can have either $\uparrow\uparrow$ or $\downarrow\downarrow$ spins in all diagrams of Figs. 1(a) and 2(a). These are the kinds of excitations that contribute to the polarizability and to the longitudinal spin susceptibility.

Let us now consider the diagrams reducible in the other particle-hole channel. In the case of the $\uparrow\uparrow$ interaction they can be obtained from the previous ones simply by interchanging the momenta of the final states and are shown in Figs. 2(b) and 2(c). In the case of the $\uparrow\downarrow$ interaction, on the other hand, the only diagrams of this class are those shown in Fig. 2(c), and they correspond to an essentially new physical process. The clue is that the intermediate particle-hole pairs now have opposite spins (they are in a triplet state with $S_z=1$). These are the kinds of excitations that contribute to the transverse spin susceptibility.

In Fig. 3 we show some of the diagrams that are totally irreducible. Obviously they include the direct and exchange scattering with bare interaction $v(q)$. Besides these elementary processes there are also some higher-

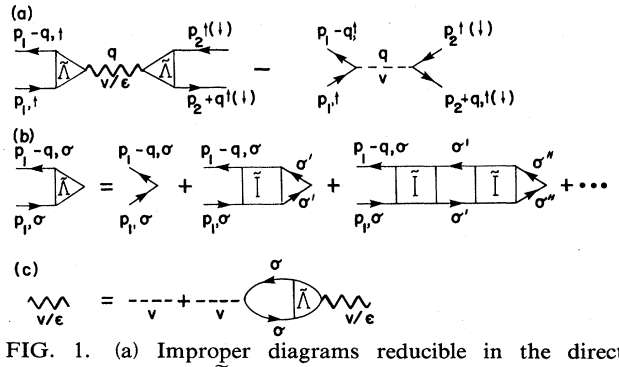

FIG. 1. (a) Improper diagrams reducible in the direct particle-hole channel. $\tilde{\Lambda}$ is the proper renormalized vertex and v/ϵ is the screened interaction. (b) Diagrammatic expansion of $\tilde{\Lambda}$ in terms of the renormalized irreducible particle-hole interaction $\tilde{I}$. (c) Diagrammatic equation for the screened interaction.

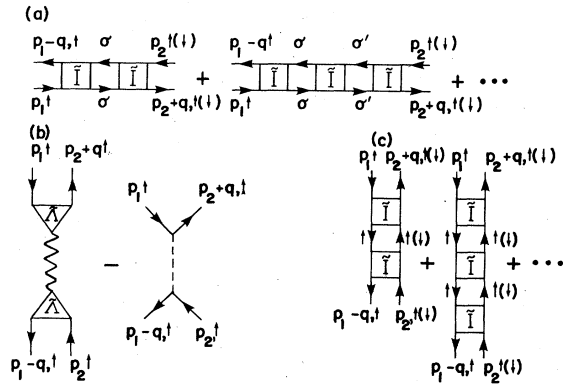

FIG. 2. (a) Proper diagrams reducible in the direct particle-hole channel. (b) and (c) Diagrams reducible in the exchange particle-hole channel.

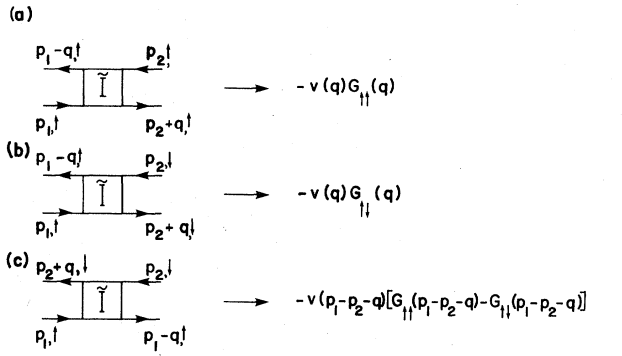

FIG. 4. Replacement of the particle-hole interaction block by local functions of the particle-hole momentum. (a) The incoming particle-hole pair has $S_z=0$ and the spins are not changed in the scattering. (b) The incoming particle-hole pair has $S_z=0$ and the spins are reversed in the scattering. (c) The incoming particle-hole pair has $S_z=1$.

order processes that can be viewed as a consequence of nonlinear coupling between density fluctuations. At present we have no grasp whatsoever on such processes and we shall disregard them in the following work.

IV. EVALUATION OF DIAGRAMS

In order to evaluate the diagrams introduced in the previous section, we resort to a drastic approximation in that we assume that the particle-hole interaction $\tilde{I}$ depends only on the momentum transfer along the particle-hole channel ("local" approximation). There are three cases of interest:

(i) The incoming particle-hole pair has $S_z=0$ and the spins are not changed by the scattering event. In this case we set $\tilde{I}=-v(q)G_{\uparrow\uparrow}(q)$, where q is the momentum of the pair [see Fig. 4(a)].

(ii) The incoming particle-hole pair has $S_z=0$ and the spins are reversed by the scattering event. In this case we set $\tilde{I}=-v(q)G_{\uparrow\downarrow}(q)$, where q is the momentum of the pair [see Fig. 4(b)].

(iii) The incoming particle-hole pair has $S_z=1$. In this case isotropy in spin space demands that the scattering amplitude be equal to that of the triplet state with $S_z=0$. This state is $(|\uparrow\uparrow\rangle-|\downarrow\downarrow\rangle)/\sqrt{2}$ where the first spin is for the electron and the second one for the hole. Thus, we must set $\tilde{I}=-v(q)G_{\uparrow\uparrow}(q)+v(q)G_{\uparrow\downarrow}(q)$, where q is the momentum of the pair.

The functions $G_{\uparrow\uparrow}(q)$ and $G_{\uparrow\downarrow}(q)$ are called "local-field factors" for reasons that will become evident in the following. In the Coulomb case they vanish as q^2 when $q\to 0$ so that $\tilde{I}$ has the character of a short-range interaction. It is useful to introduce their spin-symmetric and spin-antisymmetric combinations:

$$G^s(q)=\frac{G_{\uparrow\uparrow}(q)+G_{\uparrow\downarrow}(q)}{2} \tag{6a}$$

and

$$G^a(q)=\frac{G_{\uparrow\uparrow}(q)-G_{\uparrow\downarrow}(q)}{2}\ . \tag{6b}$$

Let us begin by evaluating the proper renormalized vertex $\tilde{\Lambda}$ according to the expansion of Fig. 1(b). Integration over the internal momenta can be immediately carried out and gives a factor $\chi_0(q)/2$ for each pair of particle-hole lines. χ_0 is the Lindhard polarizability corresponding to the effective mass m^*. We have already seen that the spins of the intermediate particle-hole pairs can be either $\uparrow\uparrow$ or $\downarrow\downarrow$. Thus, the diagrams of Fig. 1(b) correspond to the series

$$\tilde{\Lambda}(q)=1+\sum_{n=1}^{\infty}[\chi_0(q)/2]^n[-v(q)G_{\uparrow\uparrow}(q)-v(q)G_{\uparrow\downarrow}(q)]^n$$
$$=\frac{1}{1+v(q)G^s(q)\chi_0(q)}\ , \tag{7}$$

which is our local approximation to the proper vertex. In terms of $\tilde{\Lambda}$ we can now evaluate the screened interaction v/ϵ, given by the diagrammatic equation of Fig. 1(c), and hence the density-density response function

$$\chi_s=(1/\epsilon-1)/v(q)\ .$$

The latter turns out to be

$$\chi_s(q)=\frac{\chi_0(q)}{1-v(q)[1-G^s(q)]\chi_0(q)}\ . \tag{8}$$

Let us now evaluate the ladder diagrams of Fig. 2(a), whose sums we denote by $J_{\text{ladder}}^{\uparrow\uparrow\,(\uparrow\downarrow)}$ for the $\uparrow\uparrow$ ($\uparrow\downarrow$) case. Each interaction block can be regarded as the sum of two

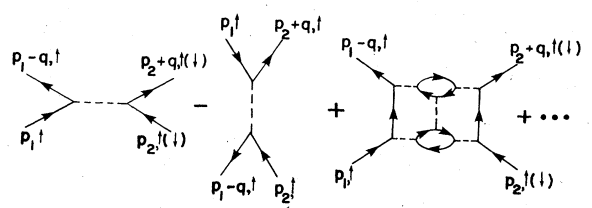

FIG. 3. Some totally irreducible diagrams, e.g., irreducible in the particle-particle and particle-hole channels. Only the first two are included in our expression for the effective interaction.

blocks, one flipping the spins as in Fig. 4(b) and one leaving them unchanged as in Fig. 4(a). The fact that the incoming and outgoing spins are fixed puts constraints on the number of spin flips that can take place in the intermediate steps. Specifically, we must have an even number of spin-flipping factors in the ↑↑ case and an odd number of them in the ↑↓ case. The two restricted sums are easily evaluated and we find

$$J^{\uparrow\uparrow\,(\uparrow\downarrow)}_{\text{ladder}}=\sum_{n=2}^{\infty}\left[\frac{\chi_0(q)}{2}\right]^{n-1}\left[\frac{[-vG^{\uparrow\uparrow}(q)-vG^{\uparrow\downarrow}(q)]^n\mp[-vG^{\uparrow\uparrow}(q)+vG^{\uparrow\downarrow}(q)]^n}{2}\right]$$
$$=(-vG^s)^2\frac{\chi_0(q)}{1+vG^s(q)\chi_0(q)}\mp(-vG^a)^2\frac{\chi_0(q)}{1+vG^a(q)\chi_0(q)}\,, \tag{9}$$

where the upper sign is the for ↑↓ case and the lower sign for the ↑↑ case. Having thus evaluated the diagrams of Fig. 2(a) we are in a position to calculate the longitudinal spin susceptibility, whose diagrammatic expansion is shown in Fig. 5(a). This is easily done noting that the terms of the series, beginning with the third one, sum up to $(\chi_0/2)^2 J^{\sigma\sigma'}_{\text{ladder}}$ and that the Pauli matrices σ_z at the vertices introduce a relative minus sign between terms with $\sigma=\sigma'$ and terms with $\sigma=-\sigma'$ [see Fig. 5(a)]. Thus, we find

$$\chi^{\text{L}}_a(q)=\frac{\chi_0(q)}{1+v(q)G^a(q)\chi_0(q)}\,. \tag{10}$$

Equations (8) and (10) are very important because they show how the local-field factors are related to the polarizability and spin susceptibility of the system. Many schemes have been proposed in the past to calculate these two quantities in an interacting Fermi liquid. Most of them assume that they can be written in the form of Eqs. (8) and (10), respectively, where the local-field factors describe the modification of the average Hartree field due to exchange and correlation effects. In the small-q limit the two response functions must satisfy the compressibility and susceptibility sum rules[3] which, applied to Eqs. (8) and (10), give

$$\frac{\kappa_f}{\kappa}=\frac{m}{m^*}\left[1-\frac{m^*p_f}{\pi^2\hbar^3}\lim_{q\to 0}[v(q)G^s(q)]\right] \tag{11a}$$

and

$$\frac{\chi_f}{\chi}=\frac{m}{m^*}\left[1-\frac{m^*p_f}{\pi^2\hbar^3}\lim_{q\to 0}[v(q)G^a(q)]\right]\,. \tag{11b}$$

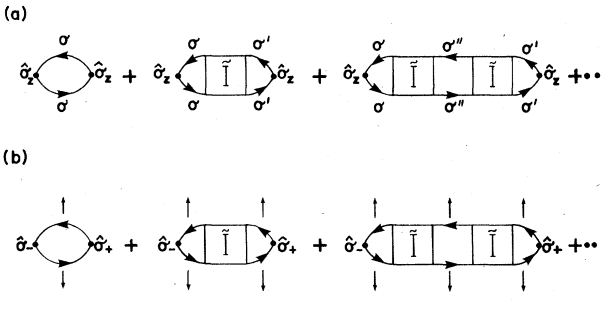

FIG. 5. (a) Diagrammatic expansion of the longitudinal spin susceptibility. (b) Diagrammatic expansion of the transverse spin susceptibility.

Here, κ,κ_f and χ,χ_f are the compressibility and spin susceptibility in the interacting and noninteracting systems, respectively. It is thus seen that in the small-q limit the local-field factors are related to the Landau interaction parameters, namely,

$$-\lim_{q\to 0}v(q)G^s(q)=f^s_0\,, \tag{12a}$$

$$-\lim_{q\to 0}v(q)G^a(q)=f^a_0\,. \tag{12b}$$

In ^{3}He and, recently, in the electron liquid, Pines[4] has used Eq. (12) as the starting point for a phenomenological theory of the local-field factor. In electron and electron-hole liquids a fully microscopic scheme proposed by Singwi *et al.*[5] allows a first-principle calculation of the local-field factors and has proven successful in predicting many equilibrium properties. Once the local fields are known the effective interaction can be calculated very easily.

Let us finally evaluate the diagrams of Figs. 2(b) and (c). In the case of the ↑↑ interaction there is really nothing new and the two classes of diagrams sum up to $(v-v\tilde{\Lambda}^2/\epsilon)_{p_1-p_2-q}$ and $-J^{\uparrow\uparrow}_{\text{ladder}}(p_1-p_2-q)$, respectively, the minus sign arising from the interchange of the final states. In the case of the ↑↓ interaction only the diagrams of Fig. 2(c) are present and, using the expression for the particle-hole interaction in the triplet state [Fig. 4(c)], they are found to sum up to

$$U^{\text{T}}_{\uparrow\downarrow}(p_1-p_2-q)=2(vG^a)^2\frac{\chi_0}{1+vG^a\chi_0}\bigg|_{p_1-p_2-q}\,. \tag{13}$$

These diagrams are related to the transverse spin susceptibility whose expansion is shown in Fig. 5(b). For the latter a simple calculation gives

$$\chi^{\text{T}}_a=\frac{1}{2}\frac{\chi_0(q)}{1+v(q)G^a(q)\chi_0(q)}\,. \tag{14}$$

This completes the evaluation of the diagrams needed to construct the effective interaction.

V. TWO-BODY EFFECTIVE INTERACTION

The results obtained in the preceding section can be combined to give an expression for the two-body effective interaction. The "direct" interactions $U_{\uparrow\uparrow}$ and $U_{\uparrow\downarrow}$ in Eqs. (5) are simply the sums of the improper diagrams shown in Fig. 1(a) plus the ladder diagrams shown in Fig. 2(a) plus the bare Coulomb interaction. Thus,

$$U_{\uparrow\uparrow\,(\uparrow\downarrow)}(q)=\frac{v\tilde{\Lambda}^2}{\epsilon}(q)+J^{\uparrow\uparrow\,(\uparrow\downarrow)}_{\text{ladder}}(q)\ . \tag{15}$$

The exchangelike term in the $\uparrow\downarrow$ interaction arises from the diagrams of Fig. 2(c) whose sum has already been calculated and is given, for an isotropic system, by Eq. (13). The vertex function, the dielectric function, and the sums of the ladder diagrams are given in Eqs. (7), (8), and (9), respectively. Using the algebraic identity,

$$\frac{v\tilde{\Lambda}^2}{\epsilon}+(-vG^s)^2\frac{\chi_0}{1+vG^s\chi_0}=v+[v(1-G^s)]^2\frac{\chi_0}{1-v(1-G^s)\chi_0}$$

and introducing the polarization fields

$$\psi_s(q)=v(q)[1-G^s(q)]\ , \tag{16a}$$

$$\psi_a(q)=-v(q)G^a(q)\ , \tag{16b}$$

the three functions $U_{\uparrow\uparrow}$, $U_{\uparrow\downarrow}$, and $U^{\mathrm{T}}_{\uparrow\downarrow}$ in Eqs. (5) can be finally written as

$$U_{\uparrow\uparrow}(q)=v(q)+[\psi_s(q)]^2\chi_s(q)+[\psi_a(q)]^2\chi^{\mathrm{L}}_a(q)\ , \tag{17a}$$

$$U_{\uparrow\downarrow}(q)=v(q)+[\psi_s(q)]^2\chi_s(q)-[\psi_a(q)]^2\chi^{\mathrm{L}}_a(q)\ , \tag{17b}$$

$$U^{\mathrm{T}}_{\uparrow\downarrow}(p_1-p_2-q)=[2\psi_a(p_1-p_2-q)]^2\chi^{\mathrm{T}}_a(p_1-p_2-q)\ , \tag{17c}$$

where $\chi_s(q)$, $\chi^{\mathrm{L}}_a(q)$, and $\chi^{\mathrm{T}}_a(q)$ are given in Eqs. (8), (10), and (14).

The structure of these expressions is quite transparent. Besides the bare interaction $v(q)$, there is an interaction *via* density fluctuations ($\psi_s^2\chi_s$), an interaction *via* longitudinal spin fluctuations ($\psi_a^2\chi^{\mathrm{L}}_a$), and, in the $\uparrow\downarrow$ case, an interaction *via* transverse spin-density fluctuations ($4\psi_a^2\chi^{\mathrm{T}}_a$). In each case we can think of a particle as the source of a pseudopotential ψ_s (or ψ_a) which *linearly* polarizes the medium inducing a density (or spin density) $\chi^{s(a)}\psi^{s(a)}$. The induced perturbation acts back on the other particle with the same pseudopotential $\psi^{s(a)}$. This is the origin of the squares in Eqs. (17). It should be noted that this linear structure, with no interference between the various mechanisms of interaction, arises diagrammatically from neglecting the higher-order totally irreducible diagrams of Fig. 3.

Let us now consider the momentum dependence of the interaction. In the case of $\uparrow\uparrow$ spins a nonlocal term (i.e., one dependent on p_1-p_2-q) is trivially introduced by the antisymmetrization prescribed in Eq. (5a) but, in the case of $\uparrow\downarrow$ spins the nonlocal term is associated with a special physical process, namely, the exchange of transverse spin fluctuations. These transverse spin fluctuations can flip the spin of the electrons and thus allow an exchange even in the $\uparrow\downarrow$ case.

A few comments on the earlier theories by Kukkonen and Overhauser[6] (KO) and Kukkonen and Wilkins[7] (KW) are now in order. The KO approach was based on a self-consistent perturbation theory that reduced the problem of finding the effective interaction to the solution of coupled algebraic equations. This approach gives the same results as ours for the local parts of the interactions $U_{\uparrow\uparrow}$ and $U_{\uparrow\downarrow}$ but is incapable of predicting the new nonlocal term in the $\uparrow\downarrow$ interaction. In any case, the local-field factors appearing in their theory should be regarded as phenomenological parameters, whereas in the present paper, they are clearly shown to be connected to the microscopic susceptibilities of the system. The KW theory, which was developed to calculate the contribution of electron-electron scattering to the thermal resistivity of a simple metal, assumes the following effective interaction:

$$U^{\sigma\sigma'}_{\mathrm{KW}}(q)=\left.\frac{v\tilde{\Lambda}^2}{\epsilon}\right|_q-\left.\frac{v\tilde{\Lambda}^2}{\epsilon}\right|_{p_1-p_2-q}\delta_{\sigma\sigma'}\ . \tag{18}$$

The vertex function is determined by extrapolating to finite values of q a relation that follows from the Ward identities[2] and holds in the small-q limit:

$$\lim_{q\to 0}\tilde{\Lambda}(q)\cong\frac{\kappa}{\kappa_f}=\lim_{q\to 0}\frac{\tilde{\chi}(q)}{\chi_0(q)}\ ,$$

where $\tilde{\chi}$ is the *proper* density-density response function. The resulting expression for $\tilde{\Lambda}$ coincides with that of Eq. (7). From the analysis of Sec. III we see that Eq. (18) includes only the "improper" diagrams and their exchange counterparts. Although KW recognized the existence of ladder-diagram contributions to the interaction they did not realize that they could be summed at essentially the same level of approximation already involved in their evaluation of Λ. The constraint in the small-q limit is provided in the general case by the requirement that the singular part of the interaction (the one depending on the ratio q/ω) reduce to the singular part of the Landau scattering amplitude with $l=0$ (higher-order components not being too important in a simple Coulomb system). It is necessary to include all the dangerous diagrams (and not only the improper ones) to satisfy this constraint. Furthermore, it is necessary to introduce *two* independent local fields G^s and G^a, which is another way of saying that the effective interaction must be determined using the information contained in the dielectric function *and* in the spin susceptibility on equal footing. Our scheme satisfies these requirements as detailed in Appendix A. The local approximation provides a definite way of extending the Landau scattering amplitude to finite wave vectors.

VI. COMPARISON AND DISCUSSION

In this section we shall consider the effective interaction between two electrons in an electron liquid in the presence of a rigid charge-neutralizing background. In this case, the mass renormalization is negligible and we can put $m^*=m$. We shall compare various forms of the effective interaction and discuss the physical origin of the differences between them. The effect of lattice polarizability will be examined in the following section.

In Fig. 6 we plot the *static* effective interactions $U_{\uparrow\uparrow}(q,0)$, $U_{\uparrow\downarrow}(q,0)$ [Eqs. (17)] together with the Thomas-Fermi (TF) interaction

$$U_{\mathrm{TF}}(q)=4\pi e^2/(q^2+q^2_{\mathrm{TF}})\ ,$$

the dielectric interaction $U_{\mathrm{diel}}(q,0)$ of Eq. (1), and the KW

interaction of Eq. (18). All the interactions are in units of $4\pi e^2/q_F^2$. For the local-field factors we have taken, for simplicity, the parametrized form recently proposed by Iwamoto and Pines:[4]

$$G^{\uparrow\uparrow}(q)=\frac{q^2}{q^2+q_{\uparrow\uparrow}^2}\ , \tag{19a}$$

$$G^{\uparrow\downarrow}(q)=\frac{q^2}{q^2+q_{\uparrow\downarrow}^2}\ , \tag{19b}$$

where $q_{\uparrow\uparrow}$ and $q_{\uparrow\downarrow}$ are r_s-dependent parameters given in Ref. 3 and are such that the compressibility and susceptibility sum rules are satisfied. At large q all the above-mentioned interactions reduce to the bare interaction $v(q)$ but in the small-q region, which is important for practical applications, large differences exist, as is evident in Fig. 6.

Let us take the Thomas-Fermi interaction for reference, since it describes the simplest situation, namely, that of two test particles in an electron liquid with no exchange and correlation between the electrons. In the small-q limit this tends to

$$U_{\rm TF}(0)=1/2\gamma\ , \tag{20}$$

where $\gamma=2\alpha r_s/\pi$ and $\alpha=(4/9\pi)^{1/3}$. Note that this limit is always positive.

Let us now include exchange and correlation *within the medium*. This leads immediately to the interaction $U_{\rm diel}(q,0)$ of Eq. (1) which can also be written as

$$U_{\rm diel}(q,0)=v(q)+v^2(q)\frac{\chi_0(q,0)}{1-v(q)[1-G^s(q)]\chi_0(q,0)}\ . \tag{21}$$

The effect of exchange and correlation as incorporated in the local-field factor $G^s(q)$ is to decrease the value of the denominator in Eq. (21) [recall that $\chi_0(q,0)<0$]. In terms of the spectrum of density fluctuation excitations this implies that the density of low-energy pair excitations is increased by the local-field factor and the strength of the plasmon peak is correspondingly decreased. Thus, the static screening becomes more effective in reducing the bare interaction as is clearly seen in Fig. 6. The small-q limit of $U_{\rm diel}(q,0)$ is

$$U_{\rm diel}(0,0)=1/2\gamma-g^s\ , \tag{22}$$

where

$$g^{s(a)}\equiv\lim_{q\to 0}G^{s(a)}(q)/q^2\ .$$

It becomes negative when $2\gamma g^s>1$. The compressibility sum rule [Eq. (11a)] gives $\kappa_f/\kappa=1-2\gamma g^s$. Thus, the interaction becomes negative at small wave vectors for r_s greater than the critical value $r_{sc}\sim 5.2$ at which the compressibility of an electron gas in the deformable jellium model would diverge. In spite of this, the interaction remains regular in q space. This is because the background is rigid in our case; therefore, no singularity is expected at $r_s\sim r_{sc}$.

Let us now come to our form of the effective interaction equations (17). Here, as explained in the Introduction, we include exchange and correlation not only within the medium but also between the two interacting particles and the medium. Comparing Eq. (17) with Eq. (21) we note that (i) the coupling to density fluctuations is not given by the bare potential $v(q)$ but by the weaker pseudopotential $\psi^s(q)$, and (ii) a new term associated with exchange-induced spin-density fluctuations appears. Because of the first fact, density fluctuations are now less effective in reducing the bare potential $v(q)$ so that both $U^{\uparrow\uparrow}(q,0)$ and $U^{\uparrow\downarrow}(q,0)$ turn out to be considerably larger than $U_{\rm diel}(q,0)$ and $U_{\rm TF}(q)$. And because of the second fact a difference arises between $U_{\uparrow\uparrow}$ and $U_{\uparrow\downarrow}$, the former being weaker than the latter. One can say that longitudinal spin-density fluctuations mediate an attraction between electrons with parallel spins and a repulsion between electrons with antiparallel spins, as is evident from Eqs. (17). The small-q limits of these interactions are

$$U^{\uparrow\uparrow\,(\uparrow\downarrow)}(q,0)=\frac{1}{2\gamma}+g^s\mp\frac{2\gamma(g^a)^2}{1-2\gamma g^a}\ , \tag{23}$$

where the lower sign is for $\uparrow\downarrow$ and the upper sign for $\uparrow\uparrow$ spins. The quantity $1-2\gamma g^a$ is related to the magnetic susceptibility and vanishes at very large r_s ($\sim$70–80) where one expects an instability towards the ferromagnetic state. This instability exists both in the rigid and in the deformable jellium model. Correspondingly, the effective interactions would have singularities in q space at such large values of r_s.

Finally, let us consider the KW interaction, Eq. (18). In the previous section, we have seen that it differs from ours because it does not include the class of proper "dangerous" diagrams. It is interesting to carry this comparison further in a more quantitative way. The quantities to be compared are $U_{\rm KW}$ and $U_{\uparrow\downarrow}$ and the difference between them is large, as shown in Fig. 6. The reason for this large difference is easy to understand if we rewrite the direct term of Eq. (18) as

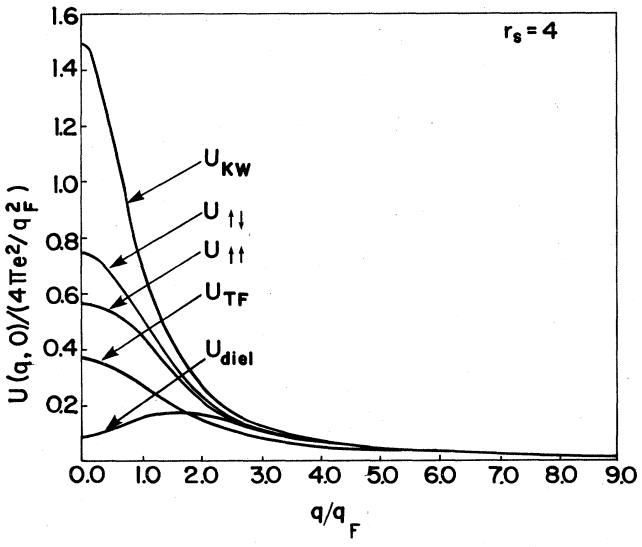

FIG. 6. Static effective interactions $U_{\uparrow\uparrow}$ and $U_{\uparrow\downarrow}$ from Eqs. (17), $U_{\rm diel}$ from Eq. (1), $U_{\rm KW}$ from Eq. (18), and $U_{\rm TF}$ for Thomas-Fermi versus q at $r_s=4$. q is in units of Fermi momentum and $U(q)$ in units of $4\pi e^2/q_F^2$. Local-field factors are taken from Eqs. (19).

$$U_{\mathrm{KW}}(q,\omega)=v(q)+\{v(q)[1-G^s(q)]\}^2\frac{\chi_0(q,\omega)}{1-v(q)[1-G^s(q)]\chi_0(q,\omega)}-[v(q)G^s(q)]^2\frac{\chi_0(q,\omega)}{1+v(q)G^s(q)\chi_0(q,\omega)}\ . \tag{24}$$

Comparing now Eqs. (24) and (17b), we see that the former can be obtained from the latter if one puts $G^a(q)=G^s(q)$. It appear as though the electrons were coupled to spin-density fluctuation *via* the strong potential $-vG^s$ rather than $-vG^a$. Also, these spin-density fluctuations should have higher spectral density at low energy because G^s replaces G^a in the denominator of the last term of Eq. (24). Hence, a large repulsive contribution is present, which makes the KW interaction stronger than all the others. Equation (24) is not satisfactory from a physical point of view. There is no reason why the pseudopotential vG^a appearing in the spin-density-fluctuation terms should be set equal to vG^s appearing in the density-fluctuation term. It is true, instead, that $g^a<G^s$ whenever $\uparrow\downarrow$ correlations exist. The situation becomes particularly serious when r_s is near to r_{sc}, where the compressibility of the deformable jellium model diverges. The small-q limit of U_{KW} is

$$U_{\mathrm{KW}}(0,0)=\frac{1}{2\gamma}\frac{1}{1-2\gamma g^s} \tag{25}$$

and diverges at the critical r_s. This is, in fact, the divergence associated with the ferromagnetic instability but, because of the assumption $g^s\sim g^a$, it is brought down to r_{sc}, well within the metallic density range. Physically, the purely electronic part of the interaction should remain regular in this range. The effect of the softening of the electron-lattice system should appear only when the term of interaction *via* the lattice is added. Our form of the interaction does not suffer from this difficulty.

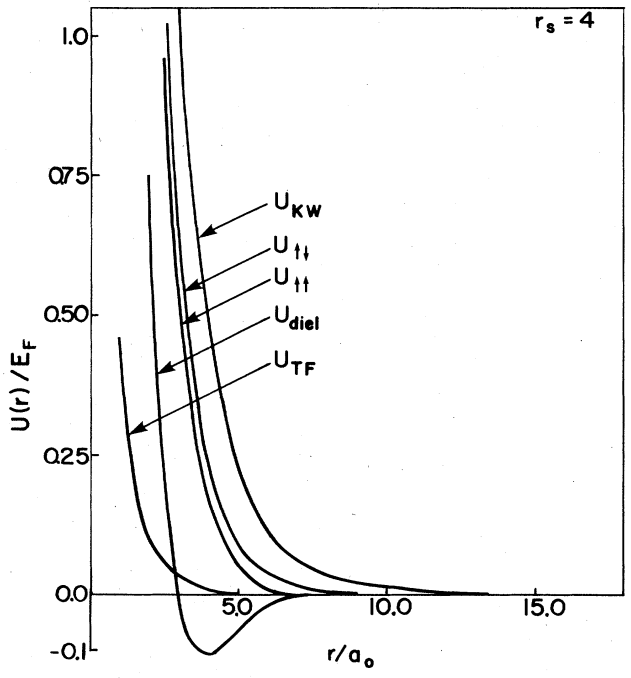

FIG. 7. Static effective interactions in real space. r is in units of Bohr radius and $U(r)$ in units of Fermi energy. Labels have the same meaning as in Fig. 6.

In Fig. 7 we have plotted the Fourier transforms of the various static interactions as functions of r. At small r, they all reduce, as they should, to the bare Coulomb interaction $1/r$; at large r they decay with Friedel oscillations (with the exception of TF). These oscillations are too small to be seen on the scale of our graph. Notice, however, the strong oscillation in $U_{\mathrm{diel}}(r)$ related to the qualitatively different behavior of $U_{\mathrm{diel}}(q)$ at small q. The relative strengths of the various interactions in real space are the the same as in q space.

VII. GENERALIZATION TO MULTICOMPONENT SYSTEMS

The results of the previous sections can be easily extended to multicomponent systems. Let ϕ_{ij} denote the bare interaction between a particle of species "i" and a particle of species "j". Let us define the pseudopotentials:

$$\psi^s_{ij}(q)=\phi_{ij}(q)[1-G^s_{ij}(q)]\ , \tag{26a}$$

$$\psi^a_{ij}(q)=-\phi_{ij}(q)G^a_{ij}(q)\ , \tag{26b}$$

where $G^{s(a)}_{ij}(q)$ are the local-field factors. The density-density response matrix is given in the generalized random-phase approximation by

$$[\underline{\chi}^{-1}(q)]_{ij}=\chi^{-1}_{0i}(q)\delta_{ij}-\psi^s_{ij}(q) \tag{27a}$$

and the longitudinal and transverse spin susceptibilities by

$$[\underline{\chi}^{\mathrm{L}}_a(q)]_{ij}=2[\underline{\chi}^{\mathrm{T}}_a(q)]_{ij}=\frac{\chi_{0i}(q)}{1-\psi^a_{ii}(q)\chi_{0i}(q)}\delta_{ij}\ , \tag{27b}$$

where χ_{0i} is the Lindhard function of the ith component. The two-body effective interaction between two particles of the ith species is then given in terms of the three functions $U^{ii}_{\uparrow\uparrow}$, $U^{ii}_{\uparrow\downarrow}$, and $U^{ii,\mathrm{T}}_{\uparrow\downarrow}$ to be combined as in Eqs. (5):

$$U^{ii}_{\uparrow\uparrow}(q)=\phi_{ii}(q)+\sum_{k,l}\psi^s_{ik}(q)\chi^s_{kl}(q)\psi^s_{li}(q)+\sum_{k,l}\psi^a_{ik}(q)\chi^{a,\mathrm{L}}_{kl}(q)\psi^a_{li}(q)\ , \tag{28a}$$

$$U^{ii}_{\uparrow\downarrow}(q)=\phi_{ii}(q)+\sum_{k,l}\psi^s_{ik}(q)\chi^s_{kl}(q)\psi^s_{li}(q)-\sum_{k,l}\psi^a_{ik}(q)\chi^{a,\mathrm{L}}_{kl}(q)\psi^a_{li}(q)\ , \tag{28a}$$

$$U^{ii,\mathrm{T}}_{\uparrow\downarrow}(q)=[2\psi^a_{ii}(q)]^2\chi^{a,\mathrm{T}}_{ii}(q)\ . \tag{28c}$$

The detailed derivation of these results is given in Appendix B.

Although Eqs. (26)—(28) represent a complete expression of the effective interaction in terms of local-field factors, they can be put in a different form which may be

useful in some applications. Suppose we are interested in the behavior of one component and ask the following question: What is the effect of the other components on the effective interaction between two particles of this species? It is then natural to try and separate from $U_{ii}^{\uparrow\uparrow}$ and $U_{ii}^{\uparrow\downarrow}$ of Eqs. (28) the terms that would describe the effective interaction if the other species were not present. As an example of this procedure let us consider a two-component system with one species much lighter than the other ($m_1 \ll m_2$). Such a situation can occur with s and d electrons in transition metals, electrons and holes in semiconductors and semimetals, and liquid-metallic hydrogen. From Eqs. (28), after a certain amount of algebra, we find that the effective interaction between two light particles is given by

$$U_{11}^{\uparrow\uparrow\,(\uparrow\downarrow)}(q,\omega)=U_c^{\uparrow\uparrow\,(\uparrow\downarrow)}(q,\omega)+U_h(q,\omega)\ , \tag{29a}$$

where $U_c^{\uparrow\uparrow\,(\uparrow\downarrow)}(q,\omega)$ are given by Eqs. (17) with $\psi^{s(a)}$ replaced by $\psi_{11}^{s(a)}$ and $\chi^{s(a)}$ by $\chi_{11}^{s(a)}$ and

$$U_h(q,\omega)=\left[\frac{\psi_{12}(q)}{1-\psi_{11}^s(q)\chi_{01}(q,\omega)}\right]^2\frac{\chi_{02}(q,\omega)}{1-f_s(q,\omega)\chi_{02}(q,\omega)}\ , \tag{29b}$$

where

$$f_s(q,\omega)=\psi_{22}^s(q)+[\psi_{12}(q)]^2\frac{\chi_{01}(q,\omega)}{1-\psi_{11}^s(q)\chi_{01}(q,\omega)}\ . \tag{29c}$$

Due to the large mass difference we can take the static limit for the Lindhard function of the light component, χ_{01}. In that case Eqs. (29) reduce to the well-known result of the theory of the electron-ion system. The first term is the purely electronic part of the interaction and the second term corresponds to the phonon-mediated interaction, namely, the product of a squared electron-phonon matrix element (screened and vertex renormalized) times the full phonon propagator. Indeed, the poles of the last term determine the dispersion of the acoustic branch of the density-fluctuation spectrum. In addition to this, incoherent particle-hole excitations, characteristic of a degenerate Fermi system, also contribute to the interaction. They do not have an exact analog in the electron-ion system.

A very interesting feature of the effective interaction in a two-component system is the appearance of a large difference between the 1-1 and the 2-2 interaction. Thus, for example, in a system with two kinds of electrons the effective interaction between two heavy electrons will be very different from that between two light electrons. This difference would exist even if spin fluctuations were not included, but then the interaction between heavy electrons would be more repulsive than the one between light electrons. The most interesting effect, however, arises from the presence of spin fluctuations. The latter are very sensitive to the mass ratio and can lead to a strong attraction between heavy electrons in the triplet states. The relevance of this fact to the problem of superconductivity in heavy-fermion materials has not yet been investigated.

In an electron-hole liquid, with moderate value of the mass ratio, the second term of Eq. (29) gives an attractive interaction. Thus, superconductivity can occur. This question will be examined in detail elsewhere.

We conclude this section by showing how the result of Eqs. (29) can be used to include the contribution of lattice polarization to the effective interaction in a simple metal. In the limit of large-ion-to-electron-mass ratio we can replace the response function $\chi_{02}(q,\omega)$ by its large-ω limit:

$$\chi_{02}(q,\omega)\to n_i q^2/m_i\omega^2\ , \tag{30}$$

where n_i is the ionic density. This is just the classical response function of a continuous deformable background of charge. Next we can eliminate the interaction function $f_s(q)$ in favor of the phonon dispersion $\omega_s(q)$, which is given by

$$\omega_s(q)=n_i q^2 f_s(q)/m_i\ . \tag{31}$$

And finally we replace the function $\psi_{12}(q)$ by the electron-ion pseudopotential $v_{e\text{-}i}(q)$. This correspondence is strongly suggested by the form of the phonon-mediated interaction which we obtain as follows:

$$U_{\text{lattice}}(q,\omega)=\left[\frac{v_{e\text{-}i}(q)}{1-\psi_{11}^s(q)\chi_{01}(q,0)}\right]^2\frac{n_i q^2}{2m_i\omega_s(q)}\times\frac{2\omega_s(q)}{\omega^2-\omega_s^2(q)}\ . \tag{32}$$

Now recall that

$$\frac{v_{e\text{-}i}(q)}{1-\psi_{11}^s(q)\chi_{01}(q,0)}=\frac{v_{e\text{-}i}(q)\tilde{\Lambda}(q,0)}{\epsilon(q,0)}\ . \tag{33}$$

Thus, Eq. (32) coincides with the standard form[8] of the phonon-mediated interaction with a "dressed" pseudopotential which includes screening and vertex corrections.[9] It is expressed in terms of the local-field factor $G_{11}^s(q)$ of the electron gas, of the phonon dispersion $\omega_s(q)$, and of the electron-ion pseudopotential $v_{e\text{-}i}(q)$. All these quantities can be calculated in principle. The polarizability of the ion core can be included by rescaling r_s to r_s/ϵ_B where ϵ_B is the dielectric constant of the ion core ($\epsilon_B\sim 1$). Notice that the static limit of Eq. (32) diverges when the sound velocity tends to zero.

VIII. CONCLUSION

In this paper we have developed a simple theory of the two-body effective interaction in a degenerate Fermi liquid of one or more components interacting *via* a Coulomb force.

The main weakness of this theory is that it assumes the irreducible particle-hole interaction to depend only on the momentum transfer in the particle-hole channel. This assumption is needed to reduce the evaluation of the Feynman diagrams to the simple summation of geometric series. We also have assumed the validity of a quasiparticle picture by associating a free-particle propagator with effective mass m^* to each internal line and by neglecting finite lifetime effects. Thus, the interaction that we have derived is expected to be valid in the vicinity of the Fermi

surface. Furthermore, we have excluded from our analysis certain diagrams corresponding to nonlinear polarization processes, whose importance is, at present, difficult to estimate. On the other hand, the main virtue of this theory is that it follows in a logical way from first principles and, in the end, makes contact with some of the most successful theories of interacting Fermi liquids: the kinetic equation approach[5] and the (phenomenological) polarization-field approach,[4] both of which are built upon the concept of "local-field factor." The final form of the effective interaction is simple, calculable, and easy to understand on physical grounds. In the small-q limit its singular part reduces to the singular part of the $l=0$ Landau scattering amplitude. A new, nonlocal term, due to the excitation of transverse spin waves appears quite naturally in the $\uparrow\downarrow$ interaction. This term was missing in previous theories of the effective interaction and its effect should be investigated.

We wish to comment on the possible applications of the effective interaction. One possibility is to calculate the effect of electron-electron interactions on the transport properties of simple metals. The contribution of electron-electron scattering to the thermal resistivity has been measured[10] in various metals and can thus provide a test of the theory. Kukkonen and Wilkins[7] and MacDonald and Geldart[11] have done such a calculation, the first using the form of Eq. (18) and the second using the s-p approximation of Dy and Pethick. Their results are in reasonable agreement with the experimental data which still, have, however, rather large error bars. A more stringent test is definitely needed.

Recently, Penn[12] has used several forms of the effective interaction to calculate the mean free path of low-energy electrons in a simple metal. The various forms give very similar results with the exception of the KO form, which gives a much shorter mean free path. Remarkably enough, the KO form seems to be in reasonable agreement with the still uncertain experimental data.

The effective electron-electron interaction that we have derived here enables one to calculate the Coulomb parameter μ occurring in the expression for the superconducting transition temperature and particularly the effect of spin fluctuations on the latter. This may be helpful in understanding the superconducting properties of heavy-fermion materials. An exciting possibility concerns the question of superconductivity in an electron-hole liquid in which the mass of the holes is greater than the mass of the electrons. We have investigated this problem with interesting results.

APPENDIX A: CONNECTION WITH THE LANDAU THEORY OF FERMI LIQUIDS

In this appendix we discuss the relation between the small-q limit of the effective interaction and the scattering amplitude of the Landau theory of Fermi liquids, generalized to include Coulomb forces. The Landau scattering amplitude is defined as

$$A^r_{\mathbf{p}\mathbf{p}'}=2\pi i z_{\mathbf{p}} z_{\mathbf{p}'}\Gamma^r_{\mathbf{p}\mathbf{p}'}\ , \tag{A1}$$

where the superscript r indicates that the small-q and -ω limit of the full scattering function $\Gamma(p,p',q)$, with arguments p and p' on the Fermi surface, is taken for $\mathbf{q}/\omega=\mathbf{r}=\text{const}$. The Landau interaction function in a Coulomb system is defined as

$$f_{\mathbf{p}\mathbf{p}'}=2\pi i z_{\mathbf{p}} z_{\mathbf{p}'}(\Gamma^0_{sc})_{\mathbf{p}\mathbf{p}'}\ , \tag{A2}$$

where Γ_{sc} is the *proper* part of Γ. The above formulas apply both to the spin-symmetric and spin-antisymmetric components of the f's and A's. The connection between the Landau scattering amplitude and the Landau interaction function is given by

$$A^r_{\mathbf{p}\mathbf{p}'}=f^c_{\mathbf{p}\mathbf{p}'}+\sum_{\mathbf{p}''} f^c_{\mathbf{p}\mathbf{p}''}\chi^r_{0\mathbf{p}''}A^r_{\mathbf{p}''\mathbf{p}'}\ , \tag{A3}$$

where the symmetric part of f^c is

$$f^{c,s}_{\mathbf{p}\mathbf{p}'}=f^s_{\mathbf{p}\mathbf{p}'}+v(q)\ ,$$

and the antisymmetric one is

$$f^{c,a}_{\mathbf{p}\mathbf{p}'}=f^a_{\mathbf{p}\mathbf{p}'} \tag{A4}$$

and

$$\chi^r_{0\mathbf{p}}=\frac{\mathbf{r}\cdot\mathbf{v}_p}{1-\mathbf{r}\cdot\mathbf{v}_p+i\eta}\ , \tag{A5}$$

where $\mathbf{v}_p$ is the quasiparticle velocity. The density-density response function can be expressed in terms of the spin-symmetric scattering amplitude, either by using the formal definition of χ of the Ward identities[2] or by solving the transport equation for quasiparticles, using Eqs. (A3) to connect the A's to the f's. The result is

$$\chi^r=\sum_{\mathbf{p}}\chi^r_{0p}\left[1+\sum_{\mathbf{p}'}A^{r,s}_{\mathbf{p}\mathbf{p}'}\chi^r_{0p'}\right]\ . \tag{A6}$$

Expanding $A^{r,s}_{pp'}$ in Legendre polynomials of the angle between $\mathbf{p}$ and $\mathbf{p}'$ and retaining only the component with $l=0$, we find

$$\chi^r=\chi^r_0(1+A^{r,s}_0\chi^r_0)\ . \tag{A7}$$

Comparing this with the small-q and -ω limit of the density-density response function in Eq. (8), we find

$$A^{r,s}_0=\lim_{\substack{q\to 0\\ q/\omega=r}}\left[\psi^s(q)+[\psi^s(q)]^2\frac{\chi_0(q,\omega)}{1-\psi^s(q)\chi_0(q,\omega)}\right]\ . \tag{A8}$$

This gives the spin-symmetric component of the $l=0$ scattering amplitude. Similarly, expressing the spin susceptibility in terms of the spin-antisymmetric amplitudes and retaining only the $l=0$ component, we find

$$A^{r,a}_0=\lim_{\substack{q\to 0\\ q/\omega=r}}\left[\psi^a(q)+[\psi^a(q)]^2\frac{\chi_0(q,\omega)}{1-\psi^a(q)\chi_0(q,\omega)}\right]\ . \tag{A9}$$

The singular part of the Landau scattering amplitude [i.e., the part proportional to $\chi_0(q,\omega)$] has precisely the same form and value as the singular part of the effective interaction [see Eq. (17)]. This result is rigorous in the small-q limit as long as the local fields satisfy the sum rules and the $l>0$ components are negligible. The local

approximation introduced in this paper is equivalent to extrapolating the validity of this relationship to finite wave vectors and frequencies.

The nonsingular parts of the scattering amplitudes, namely, ψ^s and ψ^a do not coincide with the nonsingular parts of the effective interaction. This is expected, since the full Landau scattering amplitude must also include diagrams reducible in the particle-particle channel, whereas our interaction must not. This difference between the nonsingular parts accounts precisely for the particle-particle ladders.

APPENDIX B: EFFECTIVE INTERACTION IN A TWO-COMPONENT COULOMB LIQUID

In this appendix we derive the two-component generalization of the effective interaction, i.e., Eqs. (27) of the main text. Rather than separating the proper and improper parts of the interaction we shall treat them on equal footing using, in the ladder diagrams of Fig. 2(a), the full particle-hole irreducible interaction I instead of its proper part $\tilde{I}$. In this way the diagrams of Fig. 1(a) become formally part of the ladder-diagram contribution. The relation between $\tilde{I}$ and I is

$$I_{i,\sigma_i;j,\sigma_j}(p_1,p_2,q)=\phi_{ij}(q)+\tilde{I}_{i,\sigma_i;j,\sigma_j}(p_1,p_2,q)\ , \tag{B1}$$

where i,σ_i and j,σ_j denote the species and spins of the incoming particles. Using the local approximation and the expression of $\tilde{I}$ in terms of the local fields this can be written as

$$\begin{aligned} I_{i,\sigma_i;j,\sigma_j}(q)&=\phi_{ij}(q)-\phi_{ij}(q)G_{i,\sigma_i;j,\sigma_j}(q)\\ &\equiv\psi_{i,\sigma_i;j,\sigma_j}(q)\ . \end{aligned} \tag{B2}$$

Now we evaluate the sum of the diagrams reducible in the direct particle-hole channel and denote it by $J^d_{i,\sigma_i;j,\sigma_j}(q)$. The quantities I, J^d, and ψ are 4×4 matrices in a two-component system and we denote them by $\underline{I}$, $\underline{J}^d$, and $\underline{\psi}$. We also introduce $\underline{\chi}_0$ such that

$$(\chi_0)_{i,\sigma_i;j,\sigma_j}=(\chi_{0i}/2)\delta_{i,j}\delta_{\sigma_i,\sigma_j}\ .$$

The summation of the series shown in Fig. 2(a) is then equivalent to the matrix equation

$$\underline{J}^d(q)=\underline{\psi}(q)\underline{\chi}_0(q)\underline{\psi}(q)+\underline{\psi}(q)\underline{\chi}_0(q)\underline{J}^d(q)\ , \tag{B3}$$

whose solution is

$$\underline{J}^d(q)=\underline{\psi}(q)\underline{\chi}(q)\underline{\psi}(q)\ , \tag{B4}$$

where $\underline{\chi}(q)$ is defined *via* the relation

$$\underline{\chi}^{-1}(q)=\underline{\chi}_0^{-1}(q)-\underline{\psi}(q)\ . \tag{B5}$$

Any matrix entering Eq. (B4) can be regarded as a set of four 2×2 matrices in spin space: $(\underline{\psi}_{i,j})_{\sigma_i,\sigma_j}=\underline{\psi}_{i,\sigma_i;j,\sigma_j}$. From isotropy in spin space it follows that any of the $\underline{\psi}_{ij}$'s has the form

$$\underline{\psi}_{ij}=\begin{bmatrix}\psi_{ij}^{\uparrow\uparrow} & \psi_{ij}^{\uparrow\downarrow}\\ \psi_{ij}^{\uparrow\downarrow} & \psi_{ij}^{\uparrow\uparrow}\end{bmatrix} \tag{B6}$$

with only two independent components $\psi_{ij}^{\uparrow\uparrow}$ and $\psi_{ij}^{\uparrow\downarrow}$. Such a matrix can be diagonalized with eigenvalues

$$\begin{aligned} &\psi_{ij}^{+}=\psi_{ij}^{\uparrow\uparrow}+\psi_{ij}^{\uparrow\downarrow}=2\psi_{ij}^{s}\\ &\text{and}\\ &\psi_{ij}^{-}=\psi_{ij}^{\uparrow\uparrow}-\psi_{ij}^{\uparrow\downarrow}=2\psi_{ij}^{a}\ . \end{aligned} \tag{B7}$$

In a similar way we can diagonalize χ_{ij} and, comparing Eqs. (B5) with Eqs. (27a) and (27b), we find the eigenvalues

$$\begin{aligned} &\chi_{ij}^{+}=\chi_{ij}^{\uparrow\uparrow}+\chi_{ij}^{\uparrow\downarrow}=\tfrac{1}{2}\chi_{ij}^{s}\ ,\\ &\chi_{ij}^{-}=\chi_{ij}^{\uparrow\uparrow}-\chi_{ij}^{\uparrow\downarrow}=\tfrac{1}{2}\chi_{ij}^{a}\ . \end{aligned} \tag{B8}$$

Returning to Eq. (B3) we see that it can be rewritten as

$$\underline{J}^d_{ij}(q)=\sum_{k,l=1}^{2}\underline{\psi}_{ik}(q)\underline{\chi}_{kl}(q)\underline{\psi}_{lj}(q)\ . \tag{B9}$$

Diagonalizing both sides of this equation, we find

$$J_{ij}^{d,\pm}(q)=\sum_{k,l=1}^{2}\psi_{ik}^{\pm}(q)\chi_{kl}^{\pm}(q)\psi_{lj}^{\pm}(q)\ , \tag{B10}$$

from which, using Eqs. (B7) and (B8), we find at once

$$\begin{aligned} J_{ij}^{d,\uparrow\uparrow(\uparrow\downarrow)}(q)&=\tfrac{1}{2}[J_{ij}^{d,+}(q)\pm J_{ij}^{d,-}(q)]\\ &=\sum_{k,l=1}^{2}[\psi_{ik}^{s}(q)\chi_{kl}(q)\psi_{lj}^{s}(q)\\ &\qquad\pm\psi_{ik}^{a}(q)\chi_{kl}^{a,\mathrm{L}}(q)\psi_{lj}^{a}(q)]\ , \end{aligned} \tag{B11}$$

where the plus sign is for the $\uparrow\uparrow$ case and the minus sign is for the $\uparrow\downarrow$ case. Adding the direct term $\phi_{ij}(q)$ and specializing to the case i=j we immediately find $U^{\uparrow\uparrow}$ and $U^{\uparrow\downarrow}$ in the form of Eqs. (28a) and (28b). Finally, we must evaluate the term of interaction *via* transverse spin fluctuations. The corresponding diagrams are shown in Fig. 2(c). The intermediate particle-hole pairs must necessarily be of the same kind as the incoming and outgoing ones (say, of species 1). The identification of the transverse I block with $2\psi_{ii}^{a}(q)$ still follows from isotropy. Thus, the summation performed in the one-component case is still valid in the two-component case and $U_{ii}^{\mathrm{T}}(q)$ is given by Eq. (13) with G^a replaced by G_{ii}^{a}. This is Eq. (28c).

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