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Phase diagrams of the random-field Potts model in three dimensions

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We develop and perform $1/q$ expansions for the q-state Potts model in a random field. The expansion parameter is $z=1/q^{1/3}$ in three dimensions, and we calculate to $O(z^{12})$ using the Padé method for resummation. A phase diagram is constructed and a transition into an ordered state is found. For $q \geqslant 3$ the transition is of first order, and for $q=2$ (Ising model) we find the location of the tricritical point.

Much attention has been given recently to the properties of the random-field Ising model (RFIM). Interest has been high both from the theoretical and experimental points of view.[1] From the theoretical side one would like to be able to calculate the critical behavior and the lower critical dimension of this model. Field-theoretic arguments based on perturbation theory and ϵ expansion about six dimensions,[2] supported by arguments based on supersymmetry[3] predicted a reduction in the effective dimensionality of the RFIM by two, as a result of the quenched random field. Hence, pushing the arguments to their extreme the lower critical dimension d_l of the RFIM ought to be three (since $d_l=1$ for the pure system). On the other hand, a beautiful intuitive domain argument by Imry and Ma[4] predicted $d_l=2$. Support for the validity of the latter argument has been accumulated in recent years,[5] and possible flaws of the field-theoretical arguments were pointed out.[6] Recently, a proof has been given to the existence of ferromagnetic long-range order in the model at $d=3$ and zero temperature $(T=0)$.[7] This proof, if extended to finite T will determine d_l conclusively. Various Monte Carlo calculations also support the assertion of $d_l=2$.[8] Several high-temperature series analyses of the susceptibility have also been performed recently.[9,10] From an experimental point of view, realizations of the RFIM occur as diluted antiferromagnets in a constant field,[11] or possibly as gel solutions in binary mixtures.[12] In the experiments d_l seems to be above three, but a very plausible explanation for these results has been proposed recently.[13] The claim is that as a result of strong pinning of the interfaces by the random field, as one quenches the temperature from above, it takes the system an extremely long time to display long-range order. Thus, Villain[13] distinguishes between the "dynamical" lower critical dimension (LCD), which is four, and the static LCD, which is two.

Recently, there has also been interest in the random-field Potts model (RFPM).[14,15] In this model each spin variable can take q values. For $q=2$ the RFPM coincides with the RFIM. The phase diagram of the RFPM has been calculated in mean-field theory and the transition is found to be of first order for all $q \geqslant 3$. Fluctuations in the real model are capable of modifying the order of the transition, and it has been conjectured[14] that in low dimensions q_c (the value below which the transition is of second order) is larger for the RFPM compared to a pure Potts model. Thus it was speculated that in three dimensions for some range of the random field, the transition for $q=3$ is of second order. As will be discussed below, our results do not support this conjecture.

In this work we extend the technique of the $1/q$ expansion in the Lagrangian fomulation to the RFPM. The Lagrangian $1/q$ expansion has been developed by Ginsparg, Goldschmidt, and Zuber[16] and extended by one of us to the Potts model in an applied field.[17] The Hamiltonian $1/q$ expansion developed by Kogut[18] is not directly applicable to the RFPM since the field is random along all dimensions, including the "time." The $1/q$ method has been shown to yield accurate phase diagrams and is very precise in determining the latent heat across the transition and in evaluating q_c.[16–19] The expansion parameter in the Lagrangian version is $z=1/q^{1/d}$ in d dimensions, and in particular $z=q^{-1/3}$ in three dimensions. One has to scale properly the temperature and the random field with q in order to achieve a well-behaved $1/q$ expansion, i.e., that a finite number of diagrams contribute at each order in z.

The effective Hamiltonian describing the RFPM is given by

$$-\beta\mathscr{H} \equiv \bar{\mathscr{H}} = \beta J \sum_{\langle ij\rangle} \delta_{\lambda_i,\lambda_j} + h_R \sum_i \delta_{\lambda_i,\tau_i} \quad , \tag{1}$$

where $\beta=1/kT$ and $h_R=\beta H_R$, where H_R is the magnitude of the random field. In (1) λ_i and τ_i are (complex) qth order roots of unity at site i. The λ's represent the spin variables and τ_i are the random-field variables. δ is a Kronecker delta function and the sum $\langle ij\rangle$ runs over nearest neighbors. J is the nearest-neighbor coupling which we will set equal to one. Equation (1) corresponds to a discrete RF distribution. The case of a flat distribution will be discussed briefly below. The free energy of the RFPM is given by

$$-\beta F(\beta,h_R) = \frac{1}{S} \prod_{\text{sites}} \frac{1}{q} \sum_{\tau_i} \ln \prod_{\text{sites}} \frac{1}{q} \sum_{\lambda_i} \exp(-\beta\mathscr{H}) \quad , \tag{2}$$

where S is the number of sites. The average over the random field (RF) is taken at the end since it is a quenched disorder. As mentioned above, it is necessary for the large q expansion to scale the coupling constants with q. Thus we introduce rescaled temperature and RF variables v and f. In Refs. 16 and 17 it was shown that the correct scaling of the temperature for the pure model is

$$e^{\beta} = 1 + q^{1/d} v \quad , \tag{3}$$

where v is a parameter of order unity and d is the number of spatial dimensions. As we argue below, the correct scaling of the RF is

$$e^{h_R} = 1 + q^b f, \quad \text{with } b=0 \quad ; \tag{4}$$

i.e., there is no q dependence here and the parameter f is defined only for convenience of the calculation. To see this we describe the method of constructing the $1/q$ expansion. The $1/q$ expansion consists of a rearrangement of the high- and low-temperature series, respectively, to include terms to a given order in $z(\equiv q^{1/d})$. Each order in z contains terms to all orders in β and h_R. In the high-temperature region one writes Eq. (2) in the form

$$-\beta F_{\rm I}=\frac{1}{S}\prod_{\rm sites}\frac{1}{q}\sum_{\tau_i}\ln\left[(1+q^{-(d-1)/d}v)^{Sd}(1+q^{b-1}f)^S\prod_{\rm sites}\frac{1}{q}\sum_{\lambda_i}\prod_{\langle ij\rangle}\left\{1+A\sum_r\chi_r(\lambda_i)\chi_r^*(\lambda_j)\right\}\prod_i\left\{1+B\sum_r\chi_r(\lambda_i)\chi_r^*(\tau_i)\right\}\right] , \quad (5)$$

with

$$A=\frac{vz^{d-1}}{1+vz^{d-1}};\quad B=\frac{fq^{b-1}}{1+fq^{b-1}} . \quad (6)$$

In Eq. (5) S is the number of sites and χ_r, $r=1,\ldots,q-1$ are the nontrivial characters of Zq: $\chi_r(\lambda)=\lambda^r$. These characters satisfy

$$\frac{1}{q}\sum_\lambda \chi_{r_1}(\lambda)\cdots\chi_{r_n}(\lambda)=\delta_{r_1+\cdots+r_n,0}\ (\mathrm{mod}\ q) \quad (7)$$

and contributions to (5) can be described as closed graphs on the lattice. Nonvanishing graphs containing RF contributions must contain two or more vertices at the same site, which occur after the logarithm in (5) is performed. An example for such a graph is depicted in Fig. 1. This graph has a weight proportional to $(q-1)(AB^2)^2\sim q^s$ for large q, with $s=4b-\frac{13}{3}$. More generally, if considering a graph with L_m links, S_m sites, and S_h field vertices, and this graph occurs k times at the same location, its contribution is of order q^s with

$$s=k(L_m/d+bS_h-S_m)-S_h+1 . \quad (8)$$

In the case $S_h=S_m$ and for large diagrams satisfying $L_m\sim dS_m$ we find $s=(kb-1)S_m+1$ so one must choose $b<1/k$ in order to tame the q dependence of large graphs. Since k is arbitrary we must choose $b=0$.

To construct the $1/q$ expansion for the free energy at low temperatures we expand about a ferromagnetic ground state for which all $\lambda_i=1$. The existence of a ferromagnetic ground state is justified *a posteriori* by the fact that we find (in $d=3$) that for certain range of temperature and random field the free energy of this state is lower than the free energy of the paramagnetic phase as computed from Eq. (5). The ferromagnetic region is stable as we include higher and higher orders in the $1/q$ expansion and perform a Padé analysis. This finding is in agreement with the current belief, discussed at the outset, that $d=3$ is above the lower critical dimension of the RFPM.

Corrections to the free energy come from perturbing the state with all $\lambda_i=1$ with more and more spin flips. The diagrams consist of flipped spins at certain sites together with frustrated links attached to these sites. Special care must be given to "excluded volume" diagrams because of the quenching procedure. The free energy in the ferromagnetic region is given by

$$-\beta F_{\rm II}=\frac{1}{S}\prod_{\rm sites}\frac{1}{q}\sum_{\tau_i}\ln\left\{\frac{1}{q^S}\exp\left(\beta Sd+h_R\sum_{i=1}^{S}\delta_{\tau_i,1}\right)\left[1+\sum_{\{G\}}\prod_{j\in\{G\}}\frac{1}{q}\sum_{\lambda_j\neq 1}\exp\left(-\beta L_m+h_R\sum_{j\in\{G\}}(\delta_{\lambda_j,\tau_j}-\delta_{\tau_j,1})\right)\right]\right\} , \quad (9)$$

where $\{G\}$ is the set of points belonging to the graph G.

When calculating $F_{\rm I}$ and $F_{\rm II}$ from Eqs. (5) and (9) we substitute expressions (3) and (4) for β and then collect terms in powers of z. We have calculated $F_{\rm I}$ and $F_{\rm II}$ in three dimensions for a simple cubic lattice up to order $z^{12}(=1/q^4)$. Up to this order random-field contributions to $F_{\rm I}$ enter only through the term $\ln(1+z^3f)=z^3f+\cdots$, since the lowest-order RF diagram contributing to $F_{\rm I}$ (Fig. 1) occurs only at $O(z^{13})$. The RF contribution to $F_{\rm II}$ starts at $O(z^3)$, and in this case all diagrams contain RF dependence. The phase boundary is obtained by equating the values of $F_{\rm I}$ and $F_{\rm II}$, and the latent heat is obtained by taking the derivative with respect to temperature of the difference $F_{\rm I}-F_{\rm II}$ at the phase boundary. For $q=\infty$ the phase boundary is a straight line of constant $T=T_c(q=\infty)$ in the H_R-T plane. However, when $1/q$ corrections are incorporated and a Padé analysis is performed the following things happen.

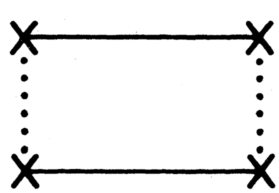

FIG. 1. A free-energy diagram contributing to $F_{\rm I}$, consisting of one link and two RF sites (denoted by $\times$), each repeating itself at the same site.

(1) The transition temperature T_c is shifted from its $q=\infty$ value and it becomes also dependent on the RF strength. More specifically, the phase boundary curves towards low temperatures as H_R increases.

(2) The latent heat across the transition is reduced compared with $q=\infty$ and for $q=2$ and weak RF the transition becomes second order.

(3) The latent heat across the transition is a slightly *increasing* function of H_R starting from $H_R=0$ with a tendency to increase more rapidly for higher values of H_R. For $q=3$ and higher, the transition is always first order in the range of H_R for which our analysis is valid (see below), and there is no apparent tendency for the latent heat to decrease beyond that range. For $q=2$ there is a tricritical point in the H_R-T plane. A tricritical point has been predicted in mean-field theory[20] for the case of a discrete RF distribution, and our analysis indicates that such a point exists even in the presence of fluctuations.

(4) We cannot obtain the whole H_R-T phase boundary

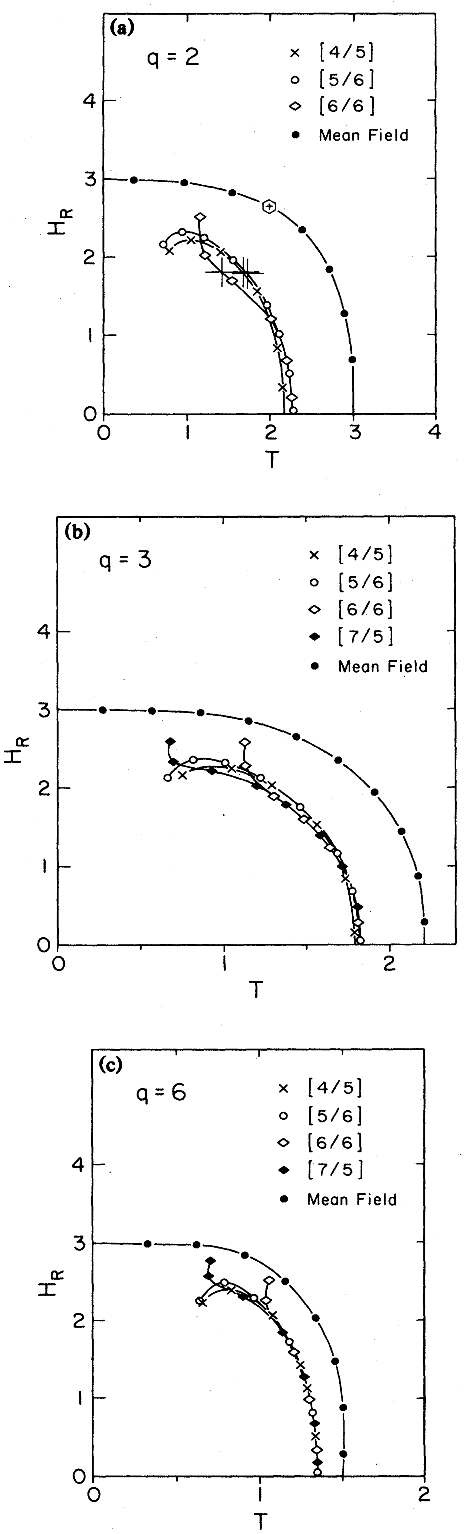

FIG. 2. (a) The phase boundary for the $q=2$ RFPM as obtained from different Padé approximants and from mean-field theory. The location of the tricritical point is indicated with crossbars (Ref. 21) for different approximants and also indicated on the mean-field curve. At this point the transition changes from second order for weak RF to first order for stronger RF. (b) The phase boundary for $q=3$. (c) The phase boundary for $q=6$.

since $f=\exp(\beta H_R)-1$ cannot become too big, otherwise the coefficients of the z expansion become too large and the Padé analysis breaks down. It is thus expected, and verified by comparing the agreement among different Padé approximants, that the expansion is valid up to a value $f\sim q$, which implies $\beta H_R \sim \ln q$. This means that the phase boundary can be obtained only for $T \gtrsim cH_R/\ln q$, c being some constant. This is the area to the right of a diagonal line in the H_R-T plane. We practically determine the region of validity by following the phase boundary till different approximants start to disperse.

We provide now some details of the analysis. The phase boundary is given by an expression of the form $v=\sum A_i(f)z^i$. The coefficients A_i are determined by plugging this series into $F_{\rm I}-F_{\rm II}$ and demanding that the result vanishes order by order in z. We used an algebraic manipulation program (REDUCE) to manipulate the series. The coefficients $A_i(f)$ include terms to all orders in the RF H_R. In fact they include all orders in f since terms like $1/(1+f)$ and $\ln(1+f)$ appear in the coefficients. Details of the coefficients will be presented elsewhere.

Here we display the results of different Padé approximants in Figs. 2(a), 2(b), and 2(c) for $q=2,3,6$, respectively.[21] We also display on the same figures the mean-field phase boundary.[14,15] We consider the approximation valid up to the point where the different Padé approximants start to disperse. We see that fluctuations have the effect of reducing the size of the ferromagnetic region, and this tendency decreases with increasing q. In order to estimate the latent heat Δ across the transition we did not use a straightforward Padé analysis of the series but we first made the assumption that as $q\rightarrow q_c$, $\Delta\sim a(z_c-z)^\sigma$ with $\sigma>0$. This assumption gives the best fit to the data for zero RF.[19] For $H_R\neq 0$, σ may depend on the RF but our analysis shows that it is almost independent of H_R for a large range of RF and its value is approximately 1.6, varying only slightly among various Padé approximants. We then calculated the series for $\Delta^{1/1.6}$ and only then carried out the Padé analysis. For $q=3$ and above the transition is always first order. [We find that $q_c(d=3)$ remains ~ 2.6 for weak RF and decreases for stronger H_R.] For $q=2$ we find a tricritical point for $H_R/J=1.8\pm 0.2$ and $T/J=1.6\pm 0.2$. This is in excellent agreement with the estimated value $H_R/J=1.8$ by Rasmussen *et al.*,[8] using a Monte Carlo renormalization-group technique for the RFIM.

We also carried out a $1/q$ analysis for the case of a uniformly distributed random fields whose magnitude varies between 0 and H (flat distribution). For $q=3$ and above the transition is still first order for the range of validity of the analysis. For $q=2$ the latent heat appears to vanish for all values of the RF in the range of validity; thus, a tricritical point, if existing in this case, might occur only for $H>2.4$, which is the limit of validity of the calculation; but a more accurate investigation is necessary.

Investigation of the two-dimensional RFPM is now in progress and results will be reported elsewhere.

To conclude, we have used the $1/q$ expansion method to obtain the phase boundary and latent heat of the RFPM. The method is powerful enough to extrapolate to low values of q and to obtain answers to many interesting questions about the RFPM.

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[21]Notice that with our Hamiltonian of Eq. (1), our T/J is smaller by a factor of 2 than that of Refs. 20 or 8, so one has to multiply our value of T by 2 in order to compare it with these references.