RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 32, NUMBER 3 1 AUGUST 1985

Spinor exponents for the two-dimensional Potts model

Bernard Nienhuis
Medische en Fysiologische Fysica, Laboratorium voor Experimentele Fysica, Postbus 80 000, 3508 TA Utrecht, The Netherlands

Hubert J. F. Knops
Instituut voor Theoretische Fysica, Katholieke Universiteit Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands

(Received 9 May 1985)

The spinor operator in the two-dimensional Ising model can be readily generalized to other self-dual models as the product of the order parameter and its dual image, the disorder operator. Recently, the exponent of this and other operators in the q-state Potts model received renewed attention, as the theory of conformal invariance produces complete lists of critical exponents for some of these models. In this paper we calculate the critical and tricritical indices of the spinor operator in the two-dimensional q-state Potts model, via a well-known mapping into a solid-on-solid model. Our results give a physical identification for the exponents predicted by the conformal theory.

The two-dimensional q-state Potts model has played the role of a testing ground for many theories and calculations in the field of phase transitions. This is due in part to the fact that this model is one of the simplest generalizations[1] of the Ising model, and also to some intriguing relations with other models,[2,3] resulting in some exactly known properties of the phase transition.[4] Approximate calculations of the critical behavior became sufficiently accurate to inspire some conjectures to the true values of the thermal[5] and magnetic[6] exponents, both of the critical and tricritical[7] points. It did not take much time before these conjectures were understood and demonstrated to be indeed exact.[8-11] The theory behind this understanding was based on an equivalence between the Potts model and a special kind of solid-on-solid (SOS) model.[2,3,12] According to renormalization group (RG) theories, SOS models viewed on asymptotically large scales, are equivalent to Gaussian models in the presence of what has become known as spin-wave fields.[13] Since the critical behavior of the Gaussian model is understood in great detail, these equivalences led not only to confirmations of the conjectures of the thermal and magnetic exponents, but also to further predictions of other critical exponents.[14]

All exponents thus obtained correspond precisely to values recently produced by a completely different approach, the so-called conformal theory. In this theory one extends the global invariance for spatial scaling and rotation in a critical system to a local invariance for arbitrary conformal transformations. It has been stressed by Belavin, Polyakov, and Zamolodchikov[15] that in particular in two dimensions the possible critical exponents of a model are greatly reduced by the adoption of conformal invariance. In fact, it has been shown (with the extra assumption of reflection positivity of the correlation functions) that all exponents of a given model can be expressed in terms of a single (rational valued) parameter C called the central charge, which is specified by the model under consideration. The list of exponents produced[16] in this way does indeed contain all exponents known so far for the critical and tricritical points of the two- and three-state Potts models. However, many other exponents are predicted for which there exists as yet no physical identification.[16,17] Physical operators considered up till now were restricted to scalar operators i.e., operators for which the correlation function does not depend on orientation. In this Rapid Communication it is shown that the list of identifications can be completed by incorporating operators that behave like spinors under rotation.

Spinor operators (also called parafermions) have an interest in their own right both in statistical physics and field theory.[18] A spinor operator is defined in general as a product of the order parameter and its dual image the disorder operator. The most well-known example is the spinor operator in the Ising model, which is directly related to the Fermion operators that diagonalize the transfer matrix for this model.[19] Correlation functions built from spinors are characterized by two indices; one is the usual anomalous dimension x describing the transformation properties under rescaling, the other index l represents the behavior under rotations like the angular quantum number. In the case of the Ising model the spinor indices are given[19] by $x=l=\frac{1}{2}$, also for p-state clock models the spinor indices are known[18] for $p>4$. In this paper we compute the spinor exponents for the critical and tricritical q-state Potts model using the asymptotic equivalence to the Gaussian model. We find that a conjecture of Dotsenko[17] for the case $q=3$ proves to be wrong.

The transformation of the Potts model to the random cluster expansion or Whitney polynomial,[2,3,20] though well known, is reviewed here because some of the steps in the derivation are useful in discussing the spinor operator. The partition sum of the q-state Potts model is defined as usual,

$$Z=\sum_{t}\exp\sum_{\langle j,k\rangle}J\delta(t_j,t_k)=\sum_{t}\prod_{\langle j,k\rangle}[1+u\delta(t_j,t_k)] \ . \qquad (1)$$

The first sum is over all variables t, each assuming q values $0,1,\ldots,q-1$, and the sum in the exponent is over all nearest-neighbor edges of the lattice. The expression can be simply rewritten as given, with $u=\exp(J)-1$. The product can be expanded as a sum over graphs by placing bonds arbitrarily on some edges of the lattice, representing the $u\delta(t_j,t_k)$ term. Each term in the graphical sum can be readily summed over t, by noting that spins connected by bonds must assume the same value. The result is the Whitney polynomial

$$Z=\sum_G u^b q^c \quad , \tag{2}$$

where the sum is over all possible graphs consisting of bonds placed arbitrarily on the lattice edges, b is the number of such bonds, and c is the number of clusters of connected sites, into which these bonds partition the lattice. Figure 1 shows a typical example of a graph in this sum. Duality can be conveniently demonstrated in this graphical language. Each graph of the sum in Eq. (2) can be turned into a graph of the dual model by placing a bond on all and only those edges of the dual lattice that are not cut by a bond on the original lattice. It can be shown that this mapping between graphs is isomorphic, and that the dual model is again a Potts model. The spin-spin correlation function

$$\langle s_l s_m\rangle \equiv \left\langle \exp\left[\frac{2\pi i}{q}(t_l-t_m)\right]\right\rangle \tag{3}$$

is up to a factor Z equal to a sum like Eq. (2) with the restriction that only those terms occur for which the sites l and m are contained in the same cluster. In all other terms the variables t_l and t_m are freely summed over, so that those terms do not contribute to Eq. (3).

In order to introduce the spinor operators we first consider the correlation function of disorder operators located at two faces of the lattice (sites of the dual lattice) λ and μ. The disorder correlation function can be defined as $G[\sigma(\lambda)\sigma(\mu)]\equiv Z'/Z$, where Z' denotes the partition sum of a Potts model with a Hamiltonian modified in the following way. Choose a path between λ and μ, along the edges of the dual lattice visiting no edge more than once (see Fig. 1). Select all the edges $\langle j,k\rangle$ (of the original lattice) that are crossed by this path, such that the site j is to the left of the traveler from λ to μ. On these edges the δ function of Eq. (1) is altered according to

$$\delta(t_j,t_k)\to\delta(t_j+1,t_k) \quad , \tag{4}$$

where the addition is modulo q. Thus, the Hamiltonian

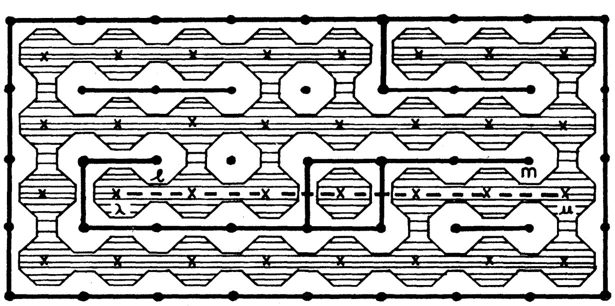

FIG. 1. Graphical representation of a Potts configuration. Dots are sites of the original lattice, crosses are the sites of the dual lattice. Heavy straight lines indicate the spins that are connected by the graph. The broken line is the domain wall between λ and μ. The polygon decomposition of the graph is drawn light.

favors unequal spins on either side of this path, which we accordingly refer to as domain wall. It is a direct consequence of the symmetry of the Potts Hamiltonian that the correlation function defined in this way only depends on the points λ and μ but not on the shape of the wall joining them. It is well known[19,21] that the disorder operator defined in this way is the dual image of the spin operator of Eq. (3). Therefore, graphs that contribute to the disorder correlation are precisely those graphs for which the dual has λ and μ on the same cluster.

Of special interest here are those correlation functions, which contain both spin and disorder operators, in particular their dependence on the relative position of these operators. Consider for instance the correlation function

$$G(r,\rho,\ \cdots)\equiv\langle s(r)\sigma(\rho)\cdots\rangle \quad , \tag{5}$$

where σ stands for a disorder operator, and r and ρ are positions in the plane. In the scaling limit these positions can be varied continuously. Now consider the dependence of G on r. Where r crosses the domain wall, which is issued from ρ, G must have a cut, and abruptly change phase by $\exp(2\pi i/q)$. If r travels to the other side by circling ρ, G changes phase continuously. Thus, if it were not for the cut, G would pick up a phase factor $\exp(2\pi i/q)$ for each time that r goes around ρ. Hence, the rotational symmetry of the plane and the internal symmetry of the spin, are jointly represented in this type of correlation function.

The spinor operator is defined as the product of a spin operator on a site and a disorder operator on an adjacent face. When one spinor circles another, the corresponding correlation function gains a phase factor of $\exp(4\pi i/q)$, since in the process both spins must cross the domain wall once, each contributing a factor $\exp(2\pi i/q)$. It is straightforward to verify that the diagrams that contribute to the two-point spinor correlation function are obtained by the following rules: (i) both spins at l and m are on the same cluster, (ii) both disorder operators at λ and μ are on the same cluster of the dual diagram, and (iii) each diagram carries a phase $\exp(2\pi i n/q)$ where n is the number of (oriented) intersections with the domain wall of a path from l to m along the cluster. [The path exists by rule (i), n is unique by rule (ii).] The phase of the diagram in Fig. 1 for example, is $\exp(2\pi i/q)$. It is clear that these phase factors are responsible for the rotational behavior of the correlation function. Before we continue to calculate the spinor exponents we notice that there is, in fact, a whole family of spinor operators. The disorder operator may be changed by replacing $t+1$ in Eq. (4) by $t+n$ or by any permutation P of $t=0,\ldots,q-1$. At the same time the order parameter is replaced by $\exp(2\pi ikt/q)$ or by a function $f(t)$ that induces a unitary representation of P, i.e., $f[P(t)]=\exp(2\pi ip)f(t)$, where p is some rational number of which the denominator is the order of one of the cyclic permutations, into which P can be decomposed. The phase factor picked up as the corresponding spinors circle one another is then $\exp(4\pi ip)$. We will henceforth label the spinor operators by this index p.

Many critical exponents of the q-state Potts model have been calculated using an equivalence between this model at its critical point and a special kind of SOS model. To compute the spinor exponent here we use the same technique. Rather than rederive the entire process we will simply state the few facts we need here, and refer the reader to the literature[3,9–14,22] for details. The first step is to use an alternative representation of the graphs of Eq. (2): the polygon decomposition of the surrounding lattice.[3] The sites as well as the faces of the original lattice correspond to faces of the surrounding lattice, and the edges of the original lattice are the sites of the surrounding lattice. In order to keep track of the difference between the original and the dual lattice, we have shaded the faces corresponding to the dual sites. For brevity it is convenient to refer to the shaded region as water (lake), and to the unshaded regions as land (island). Figure 1 shows the polygon decomposition corresponding to the original graph. The rule is simply that each vertex of the surrounding lattice is cut open to allow the bond, either on the original or on the dual lattice, to go through unintersected. An orientation is assigned to these polygons after which they can be interpreted as steps (of $\pm\pi/2$) in the configuration of a SOS model. The only contribution of a SOS configuratiuon to the Boltzmann weight comes from the corners in the domain walls. If one follows a domain wall keeping the higher level to the left, a turn to the left contributes a factor $\exp(y\pi i/8)$ and a turn to the right $\exp(-y\pi i/8)$, where $\sqrt{q}=2\cos(\pi y/2)$. Since each domain wall closes upon itself without crossing, the total winding number is ± 1, contributing a net phase factor of $\exp(\pm y\pi i/2)$. At times we refer to the polygons as shorelines of lakes inside islands, and islands inside lakes.

Here we need to inspect the diagrams that contribute to the spinor correlation function. Since the spinor operator is a product of adjacent spin and disorder operators, based on land and water, respectively, it is convenient to consider the spinor operator as sitting right on the shoreline, as in Fig. 2. The only diagrams that contribute are those in which the two spins are located in the same island, and the two disorder operators in the same lake. This is only possible if the two spinors are located on the very same shoreline (see Fig. 2). This restriction can be formulated in the SOS language, by imposing that the spinors correspond to screw dislocations or vortices with opposite charges. The two sections into which the spinors partition the shoreline together constitute the domain wall of this screw dislocation. The magnetic charge m or vorticity measured in multiples of 2π, is thus $m_{1,2}=\pm\frac{1}{2}$. The arbitrariness in the definition of height in the SOS model in the presence of the screw dislocation can be lifted if one assumes that the Potts domain

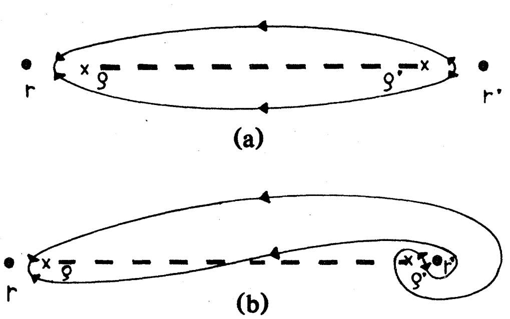

FIG. 2. Examples of diagrams (in a continuum description) that contribute to the spinor correlation function. Note that in (b) any path from r to r' has to cut either the domain wall (broken line) or the SOS walls yielding a height difference π between r and r'.

wall between ρ and ρ' serves as a domain wall in the SOS model also (with sign $\mp\frac{1}{2}$).

This prescription selects all graphs that we need according to rules (i) and (ii). Next we have to take care of the proper phase, rule (iii). Consider the diagram that results from Fig. 2(a) by rotating one of the spinors about its own center. The weight of the diagram in the SOS model changes by a phase factor, due to the fact that the two sections of the domain wall contribute oppositely to the winding number. This is illustrated in Fig. 2(b). Each full turn contributes a factor $\exp(y\pi i)$ to the weight of the domain wall that connects the spinors. But since the rotation automatically lifts or depresses the SOS height at the site of the spin by π (see Fig. 2), the phase factor can be compensated for by a spin wave with charge $-y$. For this purpose the same spin-wave charge is necessary on both of the spinors. However, we know that according to rule (iii) a spinor does indeed gain a phase factor as it is turned around, namely, the factor $\exp(2\pi pi)$. This can be accomplished by an additional spin wave of charge $\pm 2p$, resulting in a total electric charge of $e_{1,2}=-y\pm 2p$. For a discussion of the apparent violation of charge neutrality induced by this procedure the reader is referred to Ref. 11, where a similar situation is met in the context of the magnetic exponent. Note that electric charges in this SOS model are defined[11] only modulo 2, implying that p should be defined modulo 1, the lowest value yields the leading exponent, higher values should be considered as corrections to scaling. The critical exponents governing the scaling and rotation behavior of the spinor correlation function x_p and l_p, respectively, then follow from the general form[22]

$$x_p=-\frac{1}{4x}e_1e_2-xm_1m_2\ ,$$
$$l_p=-\tfrac{1}{2}(e_1m_2+e_2m_1)\ , \tag{6}$$

where x is the coupling constant of the Coulomb gas, or Gaussian model, which for the Potts model is given by $x=2-y$. It follows that

$$x_p=1+\frac{p^2-1}{x},\quad l_p=p\ . \tag{7}$$

The value of l_p implies indeed that the two-point correlation function picks up a phase factor of $\exp(4p\pi i)$ when one spinor circles the other.

Table I lists the possible values of the spinor exponents with $p<1$, for the two-, three-, and four-state Potts critical and tricritical points, as given by Eq. (7). In order to compare these values with the results of the conformal theory one should notice that the exponents (denoted by Δ and $\bar{\Delta}$) presented in this theory refer to a complex notation and reflect the z and $\bar{z}$ dependence of the correlation functions, respectively. The spinor exponents in our notation are found as $x=\Delta+\bar{\Delta}$ and $l=\Delta-\bar{\Delta}$. Both conformal exponents are for a given model (or central charge) determined by a coordination number (k,m) in the so-called conformal grid. For the central charge $C=1-6/n(n-1)$ these exponents are given by

$$4n(n-1)\Delta_{k,m}=[nk-(n-1)m]^2-1\ , \tag{8}$$

with $1\leqslant k<n-1$ and $1\leqslant m<n$. From the results of Friedan, Qiu, and Shenker[17] it seems a reasonable guess

TABLE I. The values of the spinor exponents for the Potts critical and tricritical transition.

q	x		$x_{1/2}$	Δ	$\bar{\Delta}$	$x_{1/3}$	Δ	$\bar{\Delta}$	$x_{2/3}$	Δ	$\bar{\Delta}$	$x_{1/4}$	Δ	$\bar{\Delta}$	$x_{3/4}$	Δ	$\bar{\Delta}$
2	$\frac{3}{2}$	crit	$\frac{1}{2}$	$\frac{1}{2}$	0												
3	$\frac{5}{3}$	crit	$\frac{11}{20}$	$\frac{21}{40}$	$\frac{1}{40}$	$\frac{7}{15}$	$\frac{2}{5}$	$\frac{1}{15}$	$\frac{2}{3}$	$\frac{2}{3}$	0						
4	2		$\frac{5}{8}$	$\frac{9}{16}$	$\frac{1}{16}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{13}{18}$	$\frac{25}{36}$	$\frac{1}{36}$	$\frac{17}{32}$	$\frac{25}{64}$	$\frac{9}{64}$	$\frac{25}{32}$	$\frac{49}{64}$	$\frac{1}{64}$
3	$\frac{7}{3}$	tricrit	$\frac{19}{28}$	$\frac{5}{56}$	$\frac{33}{56}$	$\frac{13}{21}$	$\frac{10}{21}$	$\frac{1}{7}$	$\frac{16}{21}$	$\frac{5}{7}$	$\frac{1}{21}$						
2	$\frac{5}{2}$	tricrit	$\frac{7}{10}$	$\frac{3}{5}$	$\frac{1}{10}$												

that the exponent of the Potts energy is given by $2\Delta_{2,1}(n)$ for the critical point and by $2\Delta_{1,2}(n)$ for the tricritical point. It follows that the relation between the central charge and the number of states of the Potts model can be conveniently expressed as $n=2/(2-x)$ for the critical point (yielding $n=4,6,\infty$ for $q=2,3,4$), and $n=x/(x-2)$ for the tricritical point (yielding $n=5,7$ for $q=2,3$), respectively. With this correspondence we can compare our spinor exponents against the results of conformal invariance. We find that all exponents of Table I occur among the exponents allowed by Eq. (8) for the appropriate value of n. Some care must be taken with the four-state model. As in this case $n\rightarrow\infty$ it is not hard to show that all squares of rational numbers are allowed by Eq. (8) (take both k and m proportional with n), in agreement with Table I.

The precise way in which the various spinor operators are represented in the conformal grid is rather intriguing. Consider the spinor operator $\psi_{1/2}$ with $p=\frac{1}{2}$ for the Ising model. It can be represented as

$$\psi_{1/2}=O_{2,1}(z)O_{2,3}(\bar{z}) \quad , \tag{9}$$

where the indices refer to lattice points of the conformal grid. Due to the essential degeneracy of the exponents in the conformal grid an alternative representation is

$$\psi_{1/2}=O_{1,3}(z)O_{1,1}(\bar{z}) \quad . \tag{10}$$

Turning to the three-state model we find that both representations still correspond to spinors but now to different ones namely, $\psi_{1/3}$, respectively, $\psi_{2/3}$. The same pattern is found for the tricritical models. Dotsenko[17] assumed in his conjecture for the spinor exponents for the three-state model that $\psi_{1/3}$ could be represented by an operator product depending only on z. We see that this only happens to occur for the spinor $\psi_{2/3}$ ($O_{1,1}$ is the identity) in accordance with the fact that $\Delta=\frac{2}{3}$ does, but $\Delta=\frac{1}{3}$ does not belong to the exponents allowed by the conformal grid.

Note added in proof. We observe that for $n=4,6$ all pairs $\Delta_{k,m}$ with $\Delta_{k,m+2}$ form spinor exponents, and likewise for $n=5,7$ the combinations $\Delta_{k,m}$ with $\Delta_{k+2,m}$. Some of the corresponding spinor operators have $p>1$. Therefore, all exponents predicted by the conformal theory except the magnetic Ising exponents are members of one or two spinor pairs.

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