Critical dynamics of the kinetic Ising model on the fractal Koch curves

Yaakov Achiam*
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
(Received 25 March 1985)

The critical slowing down of the kinetic Glauber-Ising model on different fractal geometries with quasilinear lattices is studied. The classes of fractals which are examined are the nonbranching Koch curves and the branching Koch curves. The relaxation of two different perturbations from equilibrium is examined. The dynamic critical exponent is calculated for these lattices using an exact renormalization-group transformation. The value $z=1/\nu+D_f$ is found for both fractals.

I. INTRODUCTION

Recently, a systematic study of critical phenomena in systems which have a fractal geometry[1] has been carried out.[2–7] There are a number of incentives for this growing interest in the critical behavior of spin system on fractals. One of them is the realization that there are physical systems in which the extensive properties are scaled according to a noninteger power law of the geometric linear length scale.[1,8,9] This fractal dimensionality was found to be essential in the determination of the behavior of the system. Among these systems one can find the diluted magnetic systems, which exhibit such fractal nature at the vicinity of the percolation threshold.[10,11]

Another motivation is that systems with noninteger dimensionality have been used for a long time in the study of critical phenomena. The dimensionality of a system near its critical point is one of the few parameters which are utilized to classify the critical behavior in the appropriate universality class.[12] For more than a decade, a formal analytic continuation of the dimensionality was used to study the effect of the dimensionality on critical behavior.[13–16] Underlying this analytic continuation there is a noninteger, unphysical Euclidean space which is translationally invariant and thus differs from the standard fractals.[1] Therefore, it was quite natural to study the idea of universality in the framework of fractals, which have noninteger dimensionality and are real geometric objects. Moreover, different fractals which have the same dimensionality can belong to different universality classes. This is a result of the fact that unlike the usual Euclidean space, the fractals are not translationally invariant, and the dimensionality alone is not sufficient to determine the influence of the geometry on the universality class.[2–5]

There is one more, almost trivial incentive for the study of critical phenomena on fractals. There are some families of fractals which are quite simple to analyze using standard methods.[3] For some of them, one can even obtain exact results which describe their critical behavior.[3,4] Thus, even if their physical origin is a little obscured, fractals are still intriguing.

Most of the literature that deals with the critical (magnetic) behavior on fractals is restricted to the study of the behavior of the systems at equilibrium. In a series of papers,[2–7] the statics of different families of fractals have been studied, and their relation to physical systems has been discussed.[1,6,10,11] However, there are only a few studies of the kinetic spin model on the fractals. Harris and Stinchcombe[17] generalized the study of the one-dimensional (1D) kinetic Ising model[18,19] to fractals, as well as to other lattices. However, this approach, which had previously been used to study 2D systems by Suzuki *et al.*[20] and by Droz and co-workers,[21] was shown in Ref. 22 to be at most a bad approximation to the conventional theory.[23,24] A different approach was carried out by Luscombe.[25,26] He calculated a lower bound for the dynamic exponent of the Sierpinski gasket, and obtained the exact dynamic exponent of the nonbranching Koch curves. Another approach to the study of the dynamics of random fractals had been presented by Hanley.[27] We shall discuss it in a separate paper, together with our study of the Sierpinski gasket.

In a previous paper[28] we reported a preliminary study of the critical dynamics of magnetic Ising spins' system on some fractals. We used the real-space time-dependent renormalization-group (TDRG) approach,[29,30] which has been found to be very convenient for the study of the critical slowing down of the kinetic Ising model on lattices with low dimensionality. In the following we describe the application of this technique to the study of Koch curves,[1] both the nonbranching and the branching. These fractals have a common feature resulting from their quasi-1D nature. In a separate paper the dynamics of the Sierpinski gasket and the Sierpinski carpet will be discussed.

The fractals that we are studying are nonrandom and have defined geometrical shapes. They are constructed by an iterative procedure in which each part of the object (a segment in our case) is replaced by a more complex shape (the *generator* of the fractal). Examples of some fractals are given in Fig. 1. Contrary to other hierarchical models,[31] each stage of the iteration is described by a rescaling of the length by a factor of b. In the iteration the number of the segments in the lattice, N', increases to N by a factor

$$N/N'=b^{D_f}\ , \tag{1.1}$$

which defines the fractal dimensionality D_f.

The fractals which are studied in the present paper have

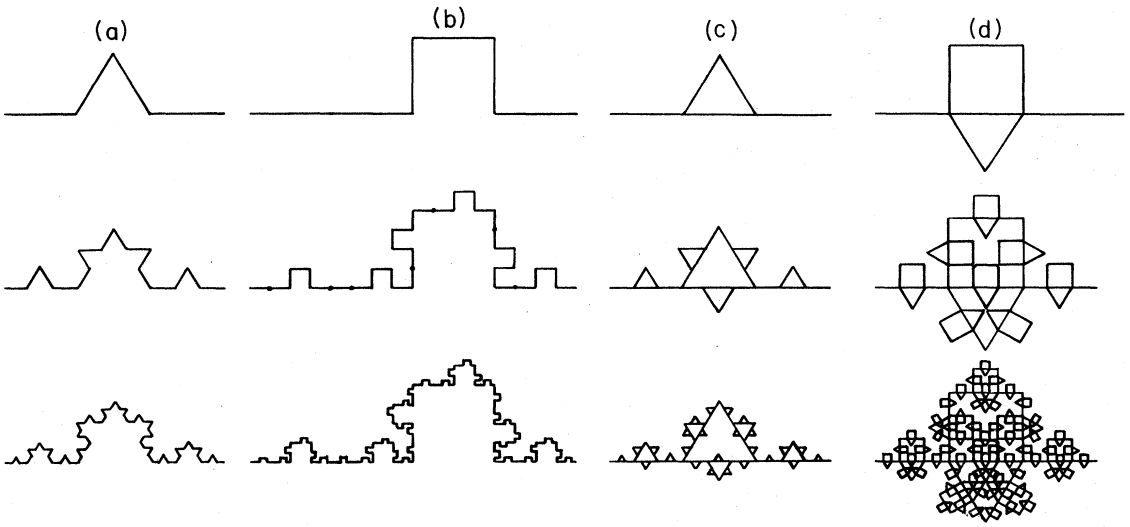

FIG. 1. Three stages in the construction of fractals. The generator of the fractal appears in the first line. The second and the third lines correspond to the second and third stages of iteration. Different fractals are placed in different columns: (a) A NBKC with $D_f=\ln 4/\ln 3$. (b) A NBKC with $D_f=\ln 6/\ln 4$. (c) A BKC with $D_f=\ln 5/\ln 3$. (d) A BKC with $D_f=\ln 8/\ln 3$.

different D_f. However, they all have the topological dimensionality[1] $D_T=1$. The value of D_T is determined by the recursion relation $D_T=D_T'+1$, where D_T' is the topological dimensionality of the cutting set which cuts the original lattice into two separate pieces. The value $D_T=1$ corresponds to 1D curves in Euclidean geometry; thus, this paper is restricted to the study of quasilinear fractals.

Another parameter which is used to characterize the topological properties of the fractal is R, the order of ramification.[1] R at a point X measures the smallest number of interactions which have to be cut to separate a part of the lattice which includes X. The two extreme values of R obey the inequality, $R_{\max}\geq 2R_{\min}-2$. We are studying two families of the $D_T=1$ fractals having finite R. One is the nonbranching Koch curves (NBKC), which have $R_{\max}=R_{\min}=2$, and thus are wiggling chains [Figs. 1(a)–1(b)]. The second is the branching Koch curves (BKC) which are inhomogeneous ($R_{\max}\neq R_{\min}$) but still have finite R, and are quasilinear in a sense that points which have $R_{\min}=2$ are spread all over the fractal [Figs. 1(c) and 1(d)]. For more details on the topological properties of these fractals, and their influence on the statics of magnetic spin systems located on these fractals, the reader is referred to Refs. 3–6.

The paper is organized as follows: In Sec. II the kinetic model is represented, and the TDRG is briefly reviewed. The NBKC are discussed in Sec. III. A modified model, which is based on a NBKC to which noniterative bonds have been added, is discussed in Sec. IV. In Sec. V we present the dynamics of the BKC. We conclude in Sec. VI.

II. KINETIC MODEL AND THE TIME-DEPENDENT RENORMALIZATION GROUP

We study the nearest neighbors (NN) Ising spin-half system, which is described by the Hamiltonian

$$\overline{H}=K\sum_{\langle i,j\rangle}\sigma_i\sigma_j\ , \tag{2.1}$$

where K is the (minus) NN interaction (over the temperature and in units of the Boltzmann constant). The spins $\{\sigma_i=\pm 1\}$ are located on the junctions of the Koch curve lattice, and the sum is taken over all $\langle i,j\rangle$ NN.

The normalized spins' equilibrium probability distribution is $P_e(\{\sigma\})=(1/Z)\exp[\overline{H}(\{\sigma\})]$, where Z normalizes the probability. P_e is the infinite time limit of the spins' time-dependent probability distribution $P(\{\sigma\};t)$:

$$P_e(\{\sigma\})=\lim_{t\to\infty}P(\{\sigma\};t)\ . \tag{2.2}$$

The time evolution of P is given by a master equation,

$$\tau_0\left[\frac{d}{dt}\right]P(\{\sigma\};t)=-\tilde{L}P(\{\sigma\};t)\ , \tag{2.3}$$

where τ_0 is a bare time scale characterizing the coupling to a heat bath. For the Liouville operator $\tilde{L}$ we use an empirical expression which guarantees the correct equilibrium limit (2.2). We have chosen a kinetic model which is a generalization of the Glauber model[32] to $D\neq 1$: The system is first brought into a state of constrained equilibrium. Then at time $t=0$ the constraint is removed, and the system relaxes towards the final equilibrium (2.2) via an interaction with a heat bath. Only one spin is allowed to flip at a time, with a transition probability rate $W_i(\{\sigma\})$. The master equation (2.3) which describes this procedure is

$$\begin{aligned}\tau_0\left[\frac{d}{dt}\right]P(\{\sigma\};t)&=-\sum_i^N\left[(1-p_i)W_i(\{\sigma\})P(\{\sigma\};t)\right]\\&\equiv-\sum_i^N\tilde{L}_iP(\{\sigma\};t)\\&\equiv-\sum_i^N L_i\Phi(\mathbf{h},\{\sigma\};t)\ ,\end{aligned} \tag{2.4}$$

where p_i is a spin flip operator, $p_i f(\{\sigma_{j\neq i}\},\sigma_i) = f(\{\sigma_{j\neq i}\},-\sigma_i)$, and Φ is the perturbation from equilibrium $\Phi \equiv P(\{\sigma\};t)/P_e$. The transition probability rate satisfies a detailed balance which ensures the ergodicity of the system:

$$\tilde{L}_i P_e(\{\sigma\}) = 0 . \tag{2.5}$$

This relation does not determine W_i uniquely. We use[29,30]

$$W_i = [P_e(\{\sigma_{j\neq i}\},-\sigma_i)/P_e(\{\sigma_{j\neq i}\},\sigma_i)]^{1/2} . \tag{2.6}$$

To study the critical slowing down, we can limit ourselves to the relaxation of an infinitely small perturbation from equilibrium. The form of Φ is determined by a trial and error procedure. The slowest mode, near the critical point, is a homogeneous one. We include in Φ a set of operators which constitute an *invariant subspace*[29,30] in the parameter space of the TDRG transformation. The *invariant subspace* is the smallest subspace in the parameter space, which has the right symmetry, and where the master equation can be described before and after the RG transformation. We study two forms of Φ: The odd, magneticlike perturbation,

$$\Phi^M(\mathbf{h},\{\sigma\}) = 1 + \sum_q h_q \sum_i \sigma_i^q , \tag{2.7}$$

where q distinguishes between points which have different R, and the even, energylike perturbation

$$\Phi^E(\mathbf{h},\{\sigma\}) = 1 + h^E \sum_{\langle i,j\rangle} \sigma_i \sigma_j . \tag{2.8}$$

The behavior of the system, which is described near the critical point by (2.1), (2.4), and (2.6), can be studied using the TDRG.[29,30,33,34] The TDRG is composed of two stages. First, a renormalization of the space by a factor b^{D_f} is carried out. This stage of the RG transformation is performed using the decimation transformation.[34,35] The equation of motion is multiplied by $T(\mu,\sigma)$,

$$T(\mu,\sigma) \equiv \prod_{j=1}^{N'} \delta(\mu_j - \sigma_j) , \tag{2.9}$$

where σ_j are the spins at the edges of the generator. Then a trace over the $\{\sigma\}$ is performed. Equation (2.4) becomes

$$\tau_0 \left[\frac{d}{dt}\right] \mathrm{Tr}_{\{\sigma\}} \left[T(\mu;\sigma) P(\{\sigma\};t)\right] = -\mathrm{Tr}_{\{\sigma\}} \left[T(\mu;\sigma) \sum_i^N L_i \Phi(\{\sigma\};t)\right] . \tag{2.10}$$

The left-hand side is nothing other than the standard static RG transformation[33–35] which transforms P into $P' = P(K';\mathbf{h}')$. In the parameter space $(K,\mathbf{h})$ the RG transformation is described by the recursion relations

$$K' = RK , \quad \mathbf{h}' = \mathbf{\Lambda h} . \tag{2.11}$$

The transformation of P and L, which appear in the right-hand side of (2.10), is determined via (2.6) by the static recursion relations (2.11). In the *invariant subspace* of the parameter space, the RG transformation of the right-hand side of (2.10) results in $L \to L'$, $P_e \to P_e'$, and a transformation in the invariant subspace,

$$\mathbf{h}_d = \mathbf{\Omega h} . \tag{2.12}$$

In the second stage of the TDRG the invariant form of the master equation (2.10) is restored by presenting $\mathbf{h}_d$ (2.12) in terms of $\mathbf{h}'$ (2.11), and the time rescaling $\tau_0' = b^z \tau_0$ is performed.

If $\mathbf{\Lambda}$ and $\mathbf{\Omega}$ commute, and this is the situation with the NBKC, the eigenvalues of $\mathbf{\Omega\Lambda}^{-1}$ are the time rescaling factor. The dynamic exponent for each eigenvalue can be obtained from the standard RG argument[30,33]

$$\frac{\omega}{\lambda} = b^{-z} , \tag{2.13}$$

where ω/λ is the eigenvalue of $\mathbf{\Omega\Lambda}^{-1}$.

In some cases (e.g., in the BKC case), $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ do not commute. The system is close to equilibrium, and we look for the invariant form at the limit of the order $n \to \infty$ of the RG transformation. The largest eigenvalues ω and λ of the matrices $\mathbf{\Omega}$ and $\mathbf{\Lambda}$, respectively, control the scaling properties as $t \to \infty$.[30] Hence, (2.13) again determines the dynamic exponent.

III. NONBRANCHING KOCH CURVES

The NBKC are homogeneous curves with a finite $R = 2$. The implication of this ramification number is that the spins are arranged in a chain with nearest-neighbor interactions between them. Thus, as far as the magnetic properties are concerned, this is a linear 1D chain, and the Glauber solution[32] is applied to it. The effect of the wiggliness is that between two spins which are x apart (x is the geometrical distance), there are x^{D_f} bonds. In the Glauber solution, the distance is measured in terms of the number of bonds between the spins. Hence, the correlation (bonds) length $\tilde{\xi}$ should be replaced by the real correlation length ξ, $\xi = \tilde{\xi}^{1/D_f}$. The time scale, which behaves according to the Glauber solution as $\tau \sim \tilde{\xi}^2$, is $\tau \sim \xi^{2D_f}$. Expressing this result in terms of the dynamic exponent z leads to $z = 2D_f$.

Although the dynamic exponent is obtainable from the Glauber solution, we shall use the TDRG approach to rederive it. The TDRG gives directly the value $z = 2D_f$. It is interesting to see what the sources are for the different contributions to this expression, and to express them in a scaling way which also fits the BKC.

The iterative procedure which defines the NBKC is as follows: The segments of the n order are replaced by the *generator* of the fractal, each of which is composed from l segments of the $n+1$ order and whose length is reduced by a factor of b. An example for such a procedure is given in Fig. 1(b), where $b = 4$ and $l = 6$.

Under the RG we transform the $n+1$ stage to the n stage by tracing over the $l-1$ internal σ_k^{n+1} of the *generator* and thus reduce the number of spins from N to N' [see Eq. (1.1)]. By using the notation

$$y = \tanh K \; ; \; y' = \tanh K' , \tag{3.1}$$

the left-hand side of the master equation (2.10) becomes

$$\left[\frac{d}{dt}\right]\sum_{\{\sigma\}} T(\mu,\sigma)P(\{\sigma\};y,h)=\left[\frac{d}{dt}\right]P'(\{\mu\};y',h'), \quad (3.2)$$

where

$$y'=y^l\ , \quad h'=b^{D_f}h \quad (3.3)$$

and the contribution to the normalization of P, $Z'=Z/A^{N'}$, is

$$A=2^{l-1}(\cosh K)^l/\cosh K' \ .$$

The right-hand side of the master equation is the sum of the terms $hP^{(i)}\sigma_i$, where $P^{(i)}\equiv P_e W_i$. Due to the detailed balance (2.5), $P^{(i)}$ is independent of σ_i. Thus, $\mathrm{Tr}_\sigma P_e^{(i)}\sigma_i=0$. The only terms which survive are the $P^{(j)}\mu_j$. The σ part of $P^{(j)}$ is the same as in P_e, except for the interactions around μ_j which are missing.[30] The two generators in $P^{(j)}$ around μ_j are composed from $l-1$ segments (instead of l). The trace over the spins in this "unfull" generator gives $2^{l-1}(\cosh K)^{l-1}$ instead of the factor A which is needed in order to retain the right normalization. Thus, the right-hand side becomes

$$\sum_{\{\sigma\}} T(\mu,\sigma)\sum_{i=1}^{N} h_i P_e^{(i)}\sigma_i=\left[\frac{1}{A}(2\cosh K)^{l-1}\right]^2\sum_{j=1}^{N'} h_j P'^{(j)}\mu_j$$

$$\equiv\frac{\omega_M}{\lambda_M}L'P' \ , \quad (3.4)$$

where

$$\omega_M=\cosh^2 K'/\cosh^2 K$$

$$=(1+y^2+y^4+\cdots+y^{2(l-1)})^{-1}. \quad (3.5)$$

At the zero-temperature fixed point $y^*=1$ and $\omega=l^{-1}=b^{-D_f}$. The rescaling factor of the bare time scale (2.13) is $b^{-z}=\omega/\lambda=b^{-2D_f}$. Thus,

$$z_M=2D_f \ . \quad (3.6)$$

By reexamining the expression (3.5), one can see that as $T\to 0$ it reduces to $\omega=(dy'/dy)^{-1}\equiv b^{-1/\nu}$, where ν is the static exponent of the correlation length.[12] Of course, $1/\nu=D_f$ for the NBKC. Hence, (3.6) can be rewritten as

$$z_M=\frac{1}{\nu}+D_f \ . \quad (3.7)$$

We have studied the relaxation of a perturbation which near equilibrium does not contain spin operators which are even under a spin reversal. Thus, the result (3.6) describes a perturbation caused by a magnetic field, and not by a changing of the temperature. To study the effect of a temperaturelike perturbation we introduce the perturbation (2.8). The recursion relation for the interaction K is as before. The recursion relations for the other parameters are

$$A=2^{l-1}\cosh^l(K+h^E)/\cosh K' \ ,$$

$$h'^E\equiv\lambda_E h^E=l h^E=b^{D_f}h^E \ .$$

The RG transformation of a typical term in the right-hand side of the master equation is

$$\sum_{\{\sigma\}} T(\mu,\sigma)P^{(j)}\mu_j\sigma_1$$

$$=P'^{(j)}(\omega_M)^{1/2}\,2^{l-1}\sinh^{l-1}(K+h^E)A^{-1}\mu_j\mu_{j+1}$$

$$\sim P'^{(j)}\omega_M y^{*l-1}\mu_j\mu_{j+1} \ ,$$

where the $(\omega_M)^{1/2}$ results from the generator which started at μ_i and does not include the σ_1. At the fixed point we reveal the previous time rescale factor,

$$z_E=-\left[\ln\frac{\omega}{\lambda_E}\right]\Big/\ln b=2D_f \ . \quad (3.8)$$

By comparing (3.8) and (3.6) one can see that in the NBKC the magnetization and the energy have a critical slowing down which is characterized by the same dynamic exponent. This result is consistent with the Glauber solution of the 1D chain.

There are some differences between the Glauber solution and the information that is obtained using the TDRG that we would like to mention. The transition rate which is used here is not the same as the one Glauber used. With the TDRG which is applied here, one can extend the result to other transition rates, including the Glauber transition rate. The Glauber solution is limited to the one-spin and two-spin time average. Our results can be extended using the standard RG argument to describe the slowest time scale in more complicated time-dependent quantities. The discussion of these points is the same as in the simple 1D case.[30,36] The reader is referred to these references for a comprehensive discussion of the link with other kinetic models.

IV. MODIFIED NONBRANCHING KOCH CURVES

A modified NBKC was introduced by Gefen *et al.*[3] Their idea was to decorate a NBKC with noniterative bonds, and thus to have a fractal with a less trivial relationship to the 1D problem. However, the decoration with the noniterated bonds was done only at the last iteration that generates the fractal. These noniterative bonds disappear in the first RG transformation, leaving a standard (unmodified) NBKC. Then, new noniterative bonds were reimplanted artificially under the criterion that after two repeating RG stages with these noniterative bonds the free energy will be the same as with the noniterative bonds which disappeared after the first stage.

Such a procedure is *a priori* forced to reproduce the NBKC results. Hence, we decided not to study this construction, but rather another in which the noniterative bonds appear in the generator, and thus will exist at any order of the RG. A similar procedure can be found in some hierarchical lattices.[31] The noniterative interactions are not affected by the RG transformation, and only the iterative bonds are transformed. Examples for such curves can be found in Fig. 2. Although the noniterative bonds are fixed during the RG transformation, only at one initial value will the recursion relations have a nontrivial fixed point. At the fixed point the noniterative interactions have the same value as the iterated ones. Near the fixed point, both interactions have similar order of magnitude. Hence, the system is more likely to have the

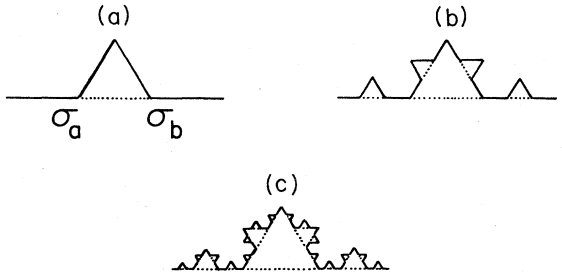

FIG. 2. The modified NBKC. (a) The generator. (b),(c) The fractal after two and three steps of iterations, respectively. The solid lines represent iterative bonds K. The dotted lines represent noniterative bonds K_0. The bold dots represent the location of the spins. In this example $l=2$ [the two bonds which compose the edges of the triangle in (a), and $m=1$].

properties of the BKC rather than the NBKC.

The two branching points, σ_a and σ_b, on the two sides of the noniterative bond K_0 are linked by a chain of l iterative bonds. A tracing over the intermediate $l-1$ spins in the "link" leads to an effective interaction $K_b=\tanh^{-1}y^l$ between σ_a and σ_b. Taking into account the $2m$ spins in the two external sides of σ_a and σ_b we find the recursion relation for K and Z:

$$y'=y^{2m}\tanh(K_0+K_b)\ ,$$
$$A=2^{2m+l-1}\cosh^{2m+l}(K)\cosh(K_0)\cosh^{-1}(K')(1+y^l y_0)\ , \tag{4.1}$$

where $y_0=\tanh K_0$. The recursion relations have a nontrivial fixed point $y^*=1$ only if $y_0=1$. The fixed point has a critical exponent $1/\nu=2m$.

The transformation of the magnetic field is derived in the Appendix. At the fixed point we find

$$(h^M)'=(2m+l)h^M=b^{D_f}h^M\ . \tag{4.2}$$

The above recursion relations define the RG transformation of the left-hand side of the master equation with either Φ^M or Φ^E.

The RG transformation of the right-hand side of (2.10) will be performed with Φ^M, the magneticlike perturbation. A typical term $P^{(i)}\sigma_i$ is linear in σ_i. Thus, as in the NBKC, the only terms which do not vanish under the RG are $P^{(j)}\mu_j$. In these terms $P^{(j)}$ is the P of a generator with the first bond starting from the site of μ_j removed. It is even in the spins, and has only one μ. Thus, the tracing over the spins leaves only the spin-independent term

$$2^{2m+l-1}\cosh^{2m+l-1}(K)\cosh(K_0)(1+y^l y_0)\ .$$

This term should written as $\omega^{1/2}A$ in order to restore the form of the right-hand side of (2.10). The power $\frac{1}{2}$ enters because the total contribution to ω comes from two neighboring generators. By comparing it to (4.1) we find

$$\omega=\left[\frac{\cosh K'}{\cosh K}\right]^2=b^{-1/\nu}\ \text{as}\ T\to T_c. \tag{4.3}$$

Like the NBKC the invariant subspace is composed from one field, h^M. The matrices Λ and Ω are reduced to the two scalars, λ and ω. The dynamic exponent is obtained via (2.13), (4.2), and (4.3),

$$z=\frac{1}{\nu}+D_f\ . \tag{4.4}$$

The expression (4.4) for z is the same as (3.7) for the NBKC. However, in the NBKC $1/\nu=D_f$, while here $1/\nu=(\ln 2m)/\ln b<D_f$.

V. BRANCHING KOCH CURVES

The BKC are characterized by an iterative procedure in which the n-order segment is replaced by a generator having segments coming to, and going from, a blob of polygons (Fig. 3). The nonequilibrium Hamiltonian is parametrized using the interaction K [or y, (3.1)], and a set of fields h_q where q is the coordination number. As examples, the BKC's which appeared in Fig. 1 have the following construction: In Fig. 1(c), $n=1$, $m=1$, $r=2$, $l=1$ and the fields are $h_{q=2}$ and $h_{q=3}$. In Fig. 1(d), $n=1$, $m=1$, $l_1=2$, $l_2=1$, $l_3=3$ and the fields are $h_{q=2}$ and $h_{q=4}$. The n,m,r,l are defined in Fig. 3(a).

We first study generators having only one polygon [Fig. 3(a)]. The RG transformation of the equilibrium probability distribution is given by the recursion relations,

$$y'=y^{n+m}(y^r+y^l)(1+y^{r+l})^{-1} \tag{5.1}$$

and

$$A=2^{(n+m+r+l-2)}(\cosh K)^{(n+m+r+l)}\cosh^{-1}K'(1+y^{r+l})\ . \tag{5.2}$$

The RG transformation of Φ is a little more complex. There are sites with different coordination number ($q=2$, $q=3$), and the RG transformation creates different renormalized fields at the different sites.

The renormalized field at the site μ_i is obtained in the Appendix. At the fixed point we get

$$h'_{q=2}=h_{q=2}(n+m+r+l-3)+2h_{q=3}\ ,$$
$$h'_{q=3}=h_{q=2}[\tfrac{3}{2}(n+m+r+l-4)]+4h_{q=3}\ . \tag{5.3}$$

In the recursion relation for $h_{q=3}$ we assumed a symmetrical generator $n=m$. If $n\neq m$ one can either distinguish between the different $h_{q=3}$ fields according to the

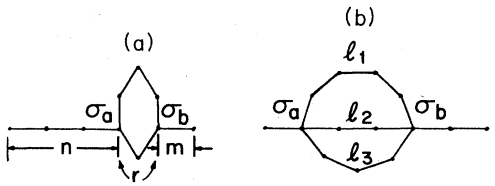

FIG. 3. The BKC fractals. (a) A generator with $h_{q=3}$. The number of the bonds in different parts of the generator is denoted by the letters on the figure. (b) The generator with $h_{q>3}$. The number of the bonds in each "link" is denoted by the l_i. The arms are to the left of σ_a and to the right of σ_b, and include $k=1+2=3$ bonds.

symmetry of the vertex, or one can average the contributions.

At the fixed point, the recursion relation for $\mathbf{h}$ reduces to the one given in Ref. 3, where it was obtained by a general argument which uses the quasi-first-order nature of the transition at $T_c=0$.

The eigenvalues and the eigenvectors of the recursion relation (5.3) are

$$\lambda_M^1=n+m+r+l=b^{D_f}\ ,\quad \lambda_M^2=1\ ,$$
$$\boldsymbol{\lambda}^1=\{q_2,q_3\}\ ,\quad \boldsymbol{\lambda}^2=\{1,2-b^{D_f}/2\}. \tag{5.4}$$

The RG transformation of the right-hand side of (2.10) is straightforward. The typical terms in the expression are $P^{(i)}\sigma_i$ and $P^{(j)}\mu_j$. As in all other cases where the RG is a decimation, the first terms are linear in σ and vanish under the operation of the trace. The RG contribution of $P^{(j)}\mu_j$ from each generator that includes the μ_j arises as the constant term in the renormalization of P_e, except one factor of $\cosh K$ which is missing in this case. By keeping the right normalization of the probability distribution we get, for a site with a coordination number q,

$$\omega_M=\left[\frac{\cosh K'}{\cosh K}\right]^q=\left[\frac{dy'}{dy}\right]^{-q/2}\quad \text{for } T_c=0.$$

The limit as $T\to 0$ of ω, the largest $q=2$ eigenvalue is

$$\omega_M=(n+m)^{-1}\equiv b^{-1/\nu}\ . \tag{5.5}$$

The dynamic exponent is now obtained from (2.13), (5.4), and (5.5):

$$z=\frac{1}{\nu}+D_f\ . \tag{5.6}$$

The BKC which we have studied has a generator which is composed from a "blob" and two "arms," where the "blob" creates two parallel links between the "arms." Different BKC could have a larger number of parallel links in the "blob" [Fig. 3(b)]. We will now show that this modification does not change the expression for z, given by (5.6) in terms ν and the fractal dimensionality.

The generator of the fractal has in its two "arms" a total of k bonds, and has links with length l_i. In the coming calculations it is convenient to express the contribution of the blob between σ_a to σ_b, after performing the trace over the internal spins in the blob, as $A_b\exp[(\sum K_{l_i})\sigma_a\sigma_b]$, where $K_{l_i}=\tanh^{-1}(\tanh^{l_i}K)$. Using this notation, the calculation reduces to that of a linear chain. In particular, the recursion relation for the bond strength is

$$y'=y^k\tanh\left[\sum_i \tanh^{-1}(\tanh^l K)\right]\ . \tag{5.7}$$

The transformation of the magnetization was discussed at length by Gefen *et al.*[3] To save ourselves calculations which are similar to those in the preceding two sections, we shall cite their result, namely, that the largest eigenvalue is b^{D_f}, and that the corresponding eigenvector is composed from all the fields h_q.

To calculate ω we should recall that the transformation of $P^{(j)}$ is the same as the constant term from the RG of a generator with $k-1$ bonds in its "arms," $2^{k-1}\cosh^{k-1}(K)A_b\cosh(K_b)$, where $K_b=\sum_i K_{l_i}$. Thus, the difference between this constant and the contribution from the free energy (of the full generator),

$$A=2^{k-1}A_b\cosh^k(K)\cosh(K_b)/\cosh(K')\ , \tag{5.8}$$

is the factor $\cosh K'/\cosh K$. This is the contribution to ω from each generator. If μ_j has a coordination number q, the total contribution to ω is $\omega_M=(\cosh K'/\cosh K)^q\approx b^{-q/(2\nu)}$. Using (2.13) we find that the dynamic exponent is

$$b^{-z}=b^{-1/\nu}/b^{D_f}\ ,\quad z=\frac{1}{\nu}+D_f\ , \tag{5.9}$$

which reveals the NBKC result.

We proceed with the study of the relaxation of the even perturbation Φ^E. This is a perturbation of the interaction K. Thus, the RG of the left-hand side of the master equation is given by the recursion relation (5.7) and (5.8) which leads to $\lambda_E=b^{1/\nu}$. To calculate ω_E we have to renormalize the right-hand side of the master equation. $\Phi^E(h_E)$ is an even perturbation. Hence, there is a linear contribution to A. However, (2.10) is already linear in h_E, thus A can be taken in the zero order of h_E, which is (5.8). Around the renormalized lattice site μ_j there are q generators. The multiplication of these generators by W_j creates a structure similar to generators with only $k-1$ bonds in their "arms." The spin σ_a, the NN of μ_1, which appears in the perturbation $\mu_1\sigma_a$ is in only one generator. The renormalization of the other $q-1$ generators is just as in the magneticlike perturbation, i.e., $(\omega_M)^{(q-1)/2}$. The last generator is multiplied by σ_a. The trace over its intermediate spins is proportional to the spin-dependent term in the renormalized probability distribution of the $k-1$ generator,

$$2^{k-1}\cosh^{k-1}(K)A_b\cosh(K_b)y^{m-1}\tanh(K_b)\mu_1\mu_2$$
$$=2^{k-1}\cosh^{k-1}(K)A_b\cosh(K_b)\frac{y'}{y}\mu_1\mu_2\ . \tag{5.10}$$

This term should be written as $\tilde{\omega}A(h^E)\mu_1\mu_2$. By comparing (5.8) and (5.10) we get

$$\tilde{\omega}=(\cosh K'/\cosh K)(y'/y)\sim b^{-1/(2\nu)}=(\omega_M)^{1/2}\ .$$

Now the contributions from all the q generators should be collected together. The scaling factor depends on the coordination number of the two sides of the specific generator. The perturbation Φ^E should be written using a set of fields $\mathbf{h}^E$ with $q\geq 2$, which are defined according to the specific geometry of the fractal. However, the lowest q is always 2, thus the largest eigenvalue in all geometries, ω_E, is $\omega_E=\tilde{\omega}(\omega_M)^{1/2}=b^{-1/\nu}$ and the dynamic exponent is $z_E=2/\nu$. Since for the BKC $1/\nu<D_f$, $z_E<z_M$.

VI. CONCLUSIONS

We have studied three families of fractals. All of these fractals have topological dimensionality of 1 and a finite ramification, and are, thus, quasilinear. This property implies a static behavior exhibiting a quasicritical phase

transition at zero temperature.[3] If the fractal is not homogeneous it has a larger ν compared to the homogeneous fractal with the same fractal dimensionality.

We calculated the dynamic exponent for these fractals, and found that it can be written in terms of the static exponents,

$$z=\frac{1}{\nu}+D_f \ . \tag{6.1}$$

The factor D_f results from the scaling of the most slowly relaxing perturbation. The factor $1/\nu$ results from the scaling of the Liouville operator. The common behavior of the three families of fractals is the above scaling law which connects z with the static exponents. $1/\nu$ is directly linked to the topology of the fractals. Hence, even if the different fractals have the same fractal dimensionality, they can have different $1/\nu$, different z, and different relaxation of energylike perturbation.

In the first family, the NBKC, $1/\nu=D_f$. The expression for z reduces to $2D_f$. This result can be obtained directly from the Glauber solution of the kinetic Ising chain. Noticing that in 1D, $1/\nu=D_f=1$, we find that the expression for z reduces to the known one, $z=2$ at 1D. Two kinds of perturbations [(2.7) and (2.8)] were studied. In the NBKC both relax with the same time scale.

In the BKC family $1/\nu$ is no longer D_f. Moreover, $1/\nu$ obeys the relation $1/\nu<1<D_f$. Thus, the BKC will relax faster than a NBKC with the same fractal dimensionality. In the NBKC the two invariant subspaces, the magneticlike and the energylike, were constructed from one parameter only. This was not the case in the BKC. Each of the invariant subspaces of the BKC includes a few parameters which depend on the coordination number. The matrices $\mathbf{\Lambda}$ and $\mathbf{\Omega}$ do not commute. The scalar product of their eigenvectors corresponding to their largest eigenvalues does not vanish. This implies that the procedure to determine z, which was described in Sec. II, applies to this situation, and that the relaxation has transients. For a detailed discussion of these transients the reader is referred to Ref. 30. We also found that the energylike perturbation relaxes faster than the magneticlike perturbation.

We introduced another family of fractals, the modified NBKC. These fractals behave more like the BKC than the NBKC in the sense that $1/\nu<D_f$ and the energy perturbation relaxes as in the BKC. On the other hand, $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ reduce to scalars as for the NBKC. If we compare a NBKC with a modified NBKC which had been prepared from the former by introducing noniterative bonds we find the same D_f. If the noniterative bonds are replaced by iterative bonds we obtain a BKC with larger D_f. Thus, of the two fractals the modified NBKC will relax the fastest, while the NBKC will relax the slowest. The BKC has relaxation time between these two.

We would like to comment on the method that we used for the calculations. The TDRG transformation of these fractals is an exact one. On the other hand, there are some assumptions in the model. One is the assumption of the existence of the linear response regime. This assumption fits the critical relaxation close to equilibrium. We also assumed that there are no other invariant subspaces which have slower relaxation time. From our experience with 1D problems, this assumption is very likely valid.

The appearance of $1/\nu$ in (6.1) causes the difference between families of quasi-1D fractals. To understand its origin, we recall the arms and blob picture which was mentioned in Sec. V. The blob is composed of a few parallel chains of bonds. The blob can be substituted by an effective bond, much stronger than the bonds in the arms. As the system approaches $T_c=0$ the effective bond becomes so strong that the two spins at its edges can be taken as totally correlated. Thus, all the relaxation procedure takes place along the arms, whose length is given by $1/\nu$.

It appears not unreasonable to us to extend the result obtained here to randomly diluted systems, i.e., to assume that the randomness will affect the value of ν, but not the scaling law (6.1) between the dynamic and the static exponents.

The diluted magnetic system in the vicinity of the percolation threshold[10,11] is a random fractal system with $T_c=0$. If the arms and blob picture is applied to it, as was suggested by Refs. 37 and 38, such a system can be used to test the scaling law.

We know of only one such system, for which experimental information regarding the static and the dynamic exponents exists. This is the percolating Ising antiferromagnet, $Rb_2Co_cMg_{1-c}F_4$ with $c=0.58$. The static exponent $\nu=1.32\pm0.05$ of this system was measured by Cowley and co-workers.[39] The theoretical value for D_f is[10,11] $D_f=1.58$, and the dynamic exponent was measured by Aeppli *et al.*,[40] $z=2.4^{+0.2}_{-0.1}$. Unfortunately, the brackets of uncertainty are too large to test (6.1) in random systems. However, this experimental z is very close to $z=2.34$ which is obtained by plugging the experimental values into (6.1).

The agreement between the experimental and the calculated values of z is much better than results derived from our study of the Sierpinski gasket.[28] However, we must stress again that the applicability of (6.1) to random systems needs further support than is given here.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Grant No. DMR-82-16718. I would like to thank Professor T.C. Lubensky and Professor A. B. Harris for stimulating discussions and M. Filipkowski for reading the manuscript.

APPENDIX

1. Modified NBKC: The recursion relation for Φ^M

The magnetic field near the fixed point is treated as a small perturbation. The contribution of the field at the $2m+l-1$ spins in the generator is

$$2^{2m+l-1}\cosh^{2m+l}(K)\cosh(K_0)\left\{y^{m+1}\frac{1-y^{l-1}}{1-y}(1+y_0)+y\frac{1-y^{m-1}}{1-y}[(1+y^l y_0)+y^m(y^l+y_0)]\right.$$
$$\left.+y^m(1+y^l)(1+y_0)\right\}h^M(\mu_1+\mu_2)\ . \tag{A1}$$

This should be written as $P_e^{1-2}\tilde{\lambda}h^M(\mu_1+\mu_2)$ where P^{1-2} is the contribution to the probability distribution from the $\mu_1-\mu_2$ normalized interaction. Using (4.1) we get

$$\tilde{\lambda}=\left\{y^{m+1}\frac{1-y^{l-1}}{1-y}(1+y_0)+y\frac{1-y^{m-1}}{1-y}[(1+y^l y_0)+y^m(y^l+y_0)]+y^m(1+y^l)(1+y_0)\right\}(1+y^l y_0)^{-1}(1+y')^{-1}\ . \tag{A2}$$

The new field at μ_1 is

$$h'=(2\tilde{\lambda}+1)h\ , \tag{A3}$$

where the first term stands for the contribution from the two generators around μ_1, and the second term comes from the original field at that site. At T_c (A.3) reduces to (4.2).

2. The BKC: The recursion relation for Φ^M

The recursion relations for the magnetization can be found using the contributions to the renormalization of the field in the internal $j=(n+m+r+l-2)$ spins of the generator between μ_1 and μ_2,

$$TP_e^{1-2}(h_{q_2}\sigma_2+\cdots+h_{q_j}\sigma_j)=\sum_q[(a_q^n-y'a_q^m)\mu_1+(a_q^m-y'a_q^n)\mu_2](1+y^{r+l})^{-1}(1-y'^2)^{-1}P_e'^{1-2}\ , \tag{A4}$$

where,

$$a_2^n=y[(1-y^{n-1})(1+y^{r+l})+y^n(1-y^{m-1})(y^r+y^l)]/(1-y)\ ,\quad a_3^n=y^n(1+y^{r+l}+y^r+y^l)\ ,$$

and a_q^m are obtained from the above expressions by interchanging n and m. The P_e^{1-2} ($P_e'^{1-2}$) are the equilibrium probability distributions for the generator (segment) between μ_1 and μ_2. The renormalized field at the site μ_i is obtained from the above internal contribution, together with the old field at μ_i. At the fixed point this expression is reduced to (5.3).

---

*On leave from Nuclear Research Centre—Negev, P.O. Box 9001, 84190 Beer-Sheva, Israel.

[1]B. B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (Freeman, San Francisco, 1977); *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[2]Y. Gefen, B. B. Mandelbrot, and A. Aharony, Phys. Rev. Lett. **45**, 855 (1980).
[3]Y. Gefen, A. Aharony, and B. B. Mandelbrot, J. Phys. A **16**, 1267 (1983).
[4]Y. Gefen, A. Aharony, Y. Shapir, and B. B. Mandelbrot, J. Phys. A **17**, 435 (1984).
[5]Y. Gefen, A. Aharony, and B. B. Mandelbrot, J. Phys. A **17**, 1277 (1984).
[6]A. Aharony, Y. Gefen, and Y. Kantor, J. Stat. Phys. **36**, 795 (1984); and references therein.
[7]S. Havlin, D. Ben-Avraham, and D. Movshovitz, J. Stat. Phys. **36**, 831 (1984).
[8]I. Procaccia, J. Stat. Phys. **36**, 649 (1984); and references therein.
[9]A. Brandstater, J. Swift, H. L. Swinney, A. Wolf, J. Doyne Farmer, E. Jen, and P. J. Crutchfield, Phys. Rev. Lett. **51**, 1442 (1983).
[10]S. Kirkpatrick, in *Ill-Condensed Matter,* 1978 Les Houches Lectures, edited by R. Balian, R Maynard, and G. Toulouse (North-Holland, New York, 1979).
[11]Y. Gefen, A. Aharony, B. B. Mandelbrot, and S. Kirkpatrick, Phys. Rev. Lett. **47**, 1771 (1981).
[12]H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, New York and Oxford, 1971).
[13]M. E. Fisher, Rev. Mod. Phys. **46**, 597 (1974).
[14]K.G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).
[15]E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. **36**, 691 (1976).
[16]D. J. Wallace and R. K. P. Zia, Phys. Rev. Lett. **43**, 808 (1979).

[17]C. K. Harris and R. B. Stinchcombe, University of Oxford Report No. 14/83, 1983 (unpublished).
[18]Y. Achiam, J. Phys. A **11**, 975 (1978).
[19]R. B. Stinchcombe, Phys. Rev. Lett. **50**, 200 (1983).
[20]M. Suzuki, K. Sogo, I. Matsuba, H. Ikeda, T. Chikama, and H. Takano, Prog. Theor. Phys. **61**, 864 (1979).
[21]M. Droz, Phys. Lett. **73A**, 407 (1979); M. Droz and A. Malaspinas, J. Phys. C **13**, 4365 (1980); Helv. Phys. Acta (Switz.) **53**, 214 (1980).
[22]J. O. Indekeu and A. L. Stella, Phys. Lett. **78A**, 160 (1980).
[23]L. Van Hove, Phys. Rev. **93**, 1374 (1954).
[24]P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977); and references therein.
[25]J. H. Luscombe (unpublished).
[26]J. H. Luscombe and R. C. Desai (unpublished).
[27]C. L. Hanley (private communication).
[28]Y. Achiam, Phys. Rev. B **31**, 4732 (1985).
[29]Y. Achiam and J. M. Kosterlitz, Phys. Rev. Lett. **41**, 128 (1978).
[30]Y. Achiam, Physica **120A**, 279 (1983).
[31]M. Kaufman and R. B. Griffiths, Phys. Rev. B **24**, 496 (1981).
[32]R. J. Glauber, J. Math. Phys. (N.Y.) **4**, 294 (1963).
[33]K. G. Wilson and J. Kogut, Phys. Rep. **12C**, 75 (1974).
[34]T. Niemeijer and J. M. J. van Leeuwen, in *Phase Transition and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6.
[35]D. R. Nelson and M. E. Fisher, Ann. Phys. (N.Y.) **91**, 226 (1975).
[36]Y. Achiam, J. Phys. A **13**, L93 (1980).
[37]A. Coniglio, Phys. Rev. Lett. **46**, 250 (1981).
[38]R. Pike and H. E. Stanley, J. Phys. A **14**, L169 (1981).
[39]R. A. Cowley, R. J. Birgeneau, G. Shirane, H. G. Guggenheim, and H. Ikeda, Phys. Rev. B **21**, 4038 (1980).
[40]G. Aeppli, H Guggenheim, and Y. J. Uemura, Phys. Rev. Lett. **52**, 942 (1984).