Phase transitions in fully frustrated spin systems

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We study the phase structure and critical behavior of fully frustrated systems. The Hamiltonians considered have global O(n) symmetry, as well as the discrete symmetries associated with the space group. The fully frustrated XY ($n=2$) model on the square and triangular lattices are two of the more popular models that belong to this class of problems. We derive a Landau-Ginzburg Hamiltonian for the general case, assuming that both the (continuous) O(n) symmetry, and some discrete symmetry are broken in the low-temperature phase. This Hamiltonian is studied in $4-\epsilon$ and $2+\epsilon$ dimensions by standard renormalization-group procedures. For $n=d=2$ we establish connection to a microscopic double-layer model, which is mapped onto a Coulomb-gas problem. Renormalization-group recursion relations are derived, and the resulting flows are used to restrict the kinds of transitions that can be observed in various cases. In particular, for the fully frustrated XY model on the square and triangular lattices, we expect a *single* transition from the disordered phase to one with Ising-like long-range order and algebraically decaying XY-type correlations.

I. INTRODUCTION

Interest in fully frustrated spin systems has been triggered originally by the close connection of these systems with various models of spin glasses. Peculiar phase structure of spin glasses is generally believed to appear as a result of competition between randomly distributed ferromagnetic and antiferromagnetic interactions.[1,2] Owing to this competition, the low-temperature phase of spin glasses is highly degenerate. Edwards and Anderson,[3] considering a model with randomly interacting XY-like spins, have estimated that there may be $O(2^N)$ ground states for a model with N spins. Each one of these states is continuously degenerate, but only discrete transformations can connect these 2^N states.

Fully frustrated continuous-spin systems are somewhat intermediate between usual phase-transition models and spin glasses. In these models a high degree of bond competition results in double degeneracy of the ground state, even though their Hamiltonian is translationally invariant.

More recent interest in uniformly frustrated models has been motivated by their connection to experimental systems such as arrays of coupled-system Josephson junctions in a transverse magnetic field.

A simple system exhibiting this discrete degeneracy of the low-temperature phase consists of XY spins on a two-dimensional square lattice with the Hamiltonian (the negative inverse temperature $-\beta$ will be absorbed in coupling constants)

$$H=J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathscr{R}(f_{ij})\mathbf{S}_j=J\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j-f_{ij})\ , \tag{1.1}$$

where $\langle i,j\rangle$ denotes a pair of nearest neighbors and $\mathscr{R}(f_{ij})$ is the operator of rotation by an angle f_{ij}. The f_{ij} are chosen to satisfy

$$f_{ij}=-f_{ji}=\begin{cases}0 & \text{if } \langle i,j\rangle \text{ is in } x \text{ direction,}\\ r_i\cdot\hat{x}F & \text{if } \langle i,j\rangle \text{ is in } y \text{ direction,}\end{cases} \tag{1.2}$$

where lattice spacing is taken to be unity and F is the frustration. Periodicity of interactions in (1.1) and reflection invariance provide a natural cutoff for frustration: $0\leq F\leq\pi$. The case $F=\pi$ corresponds to fully frustrated model for which (1.1) and (1.2) reduce to ferromagnetic coupling in the x direction and alternating rows of ferromagnetic and antiferromagnetic bonds in the y direction (Fig. 1).

In addition to usual continuous degeneracy, the low-temperature phase of (1.1) may have a high degree of discrete degeneracy. For $F=2\pi(m/n)$, where m/n is an irreducible fraction, multiplicity of the ground state is of order n.[4] In particular, the ground state of the fully frustrated model has double (discrete) degeneracy, in addition to the obvious invariance under rotations.

Models exhibiting double degeneracy of the low-temperature phase have been studied by Villain,[5–7] who pointed out that they possess some new type of long-range order which he called "chiral" order. Consider, for exam-

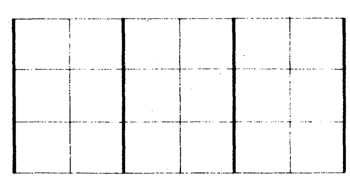

FIG. 1. Fully frustrated square lattices. Thick lines correspond to antiferromagnetic coupling, thin lines to ferromagnetic couplings.

ple, the fully frustrated model on a two-dimensional square lattice. In the ground state the square lattice splits into four sublattices, A, B, C, and D. Each sublattice is ordered ferromagnetically, but the direction of magnetization differs from sublattice to sublattice. Two different relative orientations (helicities) are possible [Figs. 2(a) and 2(b)]. Notice that the ground state of Fig. 2(a) cannot be transformed into the state of Fig. 2(b) by a continuous transformation. The order parameter associated with the spontaneously broken discrete symmetry can be identified as follows.

On each plaquette we define two vectors ϕ_1 and ϕ_2 by

$$\phi_1=S^A+\mathscr{R}\left[\frac{\pi}{4}\right]S^B+\mathscr{R}\left[\frac{\pi}{2}\right]S^C+\mathscr{R}\left[\frac{3\pi}{4}\right]S^D\,,$$

$$\phi_2=S^A+\mathscr{R}\left[-\frac{\pi}{4}\right]S^B+\mathscr{R}\left[-\frac{\pi}{2}\right]S^C+\mathscr{R}\left[-\frac{3\pi}{4}\right]S^D\,. \tag{1.3}$$

In the ground state of Fig. 2(a), $|\phi_1|=1$ and $\phi_2=0$, while in the ground state of Fig. 2(b), $|\phi_2|=1$ and $\phi_1=0$. Villain argued that at sufficiently low temperature any ground state is stable with respect to introduction of a domain of the other ground state. His argument is as follows. Consider a narrow boundary separating two domains of different helicity. The energy of such a boundary per unit length (unit area for three-dimensional systems) is of order $T\,|J\,|$ and its entropy is of order unity. Therefore, in the low-temperature phase ($|J\,|>1$) most plaquettes will have the same helicity, and the "chiral" correlation function

$$\Gamma_{\text{ch}}(R)=\langle[\phi_1^2(0)-\phi_2^2(0)][\phi_1^2(R)-\phi_2^2(R)]\rangle \tag{1.4}$$

will have a nonvanishing limit as $R\to\infty$. This long-range order will vanish at some transition temperature T_c, which does not have to be the same as the critical temperature of a Kosterlitz-Thouless transition, also expected in this system.

Monte Carlo simulation of this model[8] produced evidence for a transition similar to a Kosterlitz-Thouless (KT) transition, except for one important feature. Whereas in the KT transition the specific heat per site C has a peak with a maximum value $C_{\max}$ that is independent of lattice size N, in the frustrated model $C_{\max}$ diverges logarithmically with N. The logarithmic scaling of the specific heat is characteristic of the Ising transition. However, the Monte Carlo data were not precise enough to distinguish whether the peak value of the specific heat occurs at the same temperature as the jump in vorticity or slightly above.

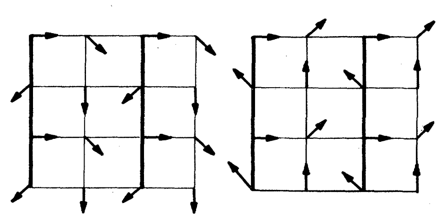

FIG. 2. Ground states of the fully frustrated XY model on the square lattice.

The fully frustrated XY model on the triangular lattice has been studied recently by mean-field techniques[9] and Monte Carlo simulation.[10] These calculations favor a single transition of a novel type rather than two consecutive transitions.

While the precise form of the definition of ϕ_1,ϕ_2 depends on the details of each model, the possibility of such a definition and Villain's argument depend only on the existence of two different types of ground state which cannot be connected by rotation. In this work we argue that all models exhibiting this kind of low-temperature phase are described by the same Landau-Ginzburg-Wilson (LGW) Hamiltonian. Assuming that the strong universality hypothesis[11] is satisfied, that is, all models described by the same LGW Hamiltonian belong the same universality class, we study this Hamiltonian by renormalization-group methods.

The most general LGW Hamiltonian, appropriate for transitions in fully frustrated models, is derived in Sec. II. It should be noted that, up to the leading anisotropy term (fourth order), the XY antiferromagnet on the triangular lattice, the fully frustrated XY model on the square lattice, and the XY helimagnet are all characterized by the same LGW Hamiltonian. In Sec. III this Hamiltonian is studied by means of an exact renormalization group in dimension $d=4-\epsilon$ and $d=2+\epsilon$. In both cases no stable fixed points are present in the vicinity of the Gaussian fixed point. In dimension $4-\epsilon$ this is interpreted as a first-order transition, while for two-dimensional systems, renormalization-group recursion relations allow us to establish a close connection between our models and a model defined by the Hamiltonian

$$H=J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j(1+\tau_i\tau_j)\,, \tag{1.5}$$

where the $\mathbf{S}_i$ are n-component vectors ($n\geq 2$), τ_i are Ising-like variables, and $\langle i,j\rangle$ runs over pairs of nearest neighbors on the two-dimensional lattice. An Ising-like transition is predicted for the model (1.5) with $n\geq 3$. In Sec. IV the Hamiltonian (1.5) with $n=2$ is rewritten in the Coulomb-gas representation and analyzed using a Kosterlitz-type position-space-renormalization group.

Such renormalization-group procedures are based on expansion in various small fugacities. The initial Hamiltonian does not lie in the region where these fugacities are small. Therefore, we cannot draw unambiguous conclusions concerning the nature of the transition from the disordered phase to the low-temperature one. However, the possible scenarios we find are either a single first-order transition or a single continuous transition of novel critical behavior (Ising-like and Kosterlitz-Thouless-like simultaneously). Which of these two scenarios is chosen by any particular system may depend on various nonuniversal details of the underlying model.

II. SYMMETRY ANALYSIS

A. General considerations

Consider a system of classical n-component ($n \geq 2$) unit vectors $\mathbf{S}_i$ located at sites i of a d-dimensional lattice. The Hamiltonian of such a system will be assumed to satisfy usual requirements of translational invariance, global $O(n)$ invariance, and sufficiently short-range interactions. As a result of the rotational invariance of the Hamiltonian the ground state is continuously degenerate: under arbitrary global rotation it transforms into another ground state. However, it may happen that not all ground states can be connected by rotation. In this case, in addition to rotation, some discrete transformation is required to connect all ground states.

In this paper we will concentrate on the "minimal" case: we assume that apart from the usual rotational degeneracy, the ground state has additional double degeneracy. Then these ground states form two "pockets" in phase space, separated by a potential barrier, and the symmetry group of the Hamiltonian contains some discrete transformation η, satisfying $\eta^2 = E$, which connects these two "pockets."

Let $\mathcal{N}$ be the number of spins per unit cell *in the low-temperature phase.* Consider the expectation values

$$\langle \mathbf{S}^j_{L\alpha} \rangle, \quad \alpha = 1,2,\ldots,n, \quad L = 1,\ldots,\mathcal{N} \tag{2.1}$$

where j runs over unit cells, L labels spins within a cell, and α labels spin components. Let us assume that these quantities do not depend on j for all temperatures. Then we have only $n\mathcal{N}$ independent quantities $\langle S_{L\alpha} \rangle$. Standard theorems of linear algebra tell that we always can choose $n\mathcal{N}$ independent linear combinations,

$$\phi_{M\beta} = \sum_{L,\alpha} a^{L\alpha}_{M\beta} \langle \mathbf{S}_{L\alpha} \rangle, \quad \beta = 1,2,\ldots,n, \quad M = 1,2,\ldots,\mathcal{N} \tag{2.2}$$

satisfying the following requirements:

(a) For each M, n quantities $\phi_{M\beta}$ transform under rotation as components of an n-dimensional vector ϕ_M.

(b) In one ground state,

$$|\phi_1| > 0, \quad \phi_2 = 0, \quad \phi_3 = \cdots = \phi_{\mathcal{N}} = 0 , \tag{2.3}$$

while in the other ground state,

$$\phi_1 = 0, \quad |\phi_2| > 0, \quad \phi_3 = \cdots = \phi_{\mathcal{N}} = 0 . \tag{2.4}$$

(c) η transforms ϕ_1 into ϕ_2 and vice versa.

Assuming that $\phi_3, \phi_4, \ldots, \phi_{\mathcal{N}}$ vanish at all temperatures, we see that the order parameter can have at most $2n$ independent components

$$\phi_{1\beta}, \quad \phi_{2\beta}, \quad \beta = 1,2,\ldots,n . \tag{2.5}$$

The appropriate Landau-Ginzburg-Wilson effective Hamiltonian[12] is constructed from all possible invariants that can be built from the order parameter and its derivatives.[13]

Consider first those terms in the expansion which do not contain derivatives. Any such term which is invariant under both rotation and the operation η can be constructed from three basic invariants:

$$\phi_1^2 + \phi_2^2, \quad \phi_1 \cdot \phi_2, \quad \phi_1^2 \phi_2^2 . \tag{2.6}$$

It will be convenient to define the following set of invariants,

$$O_k = |\phi_1|^k |\phi_2|^k \cos(k\alpha), \quad k = 0, 1, 2, \ldots \tag{2.7}$$

where α is defined by

$$\cos\alpha = \frac{\phi_1 \cdot \phi_2}{|\phi_1| \, |\phi_2|} .$$

O_k can also be written as

$$\begin{aligned} O_0 &= 1 , \\ O_1 &= \phi_1 \cdot \phi_2 , \\ O_2 &= 2(\phi_1 \cdot \phi_2)^2 - \phi_1^2 \phi_2^2 , \\ O_{k+2} &= O_k O_2 + (O_{k+1} - O_{k-1} \phi_1^2 \phi_2^2) \phi_1 \cdot \phi_2 . \end{aligned} \tag{2.8}$$

Then all terms which do not contain derivatives are of the form

$$(\phi_1^2 + \phi_2^2)^i (\phi_1^2 \phi_2^2)^j O_k, \quad i,j,k = 0,1,2,\ldots . \tag{2.9}$$

Inserting the derivative terms, one finds that the most general Hamiltonian is

$$\begin{aligned} \mathcal{H} = {} & r(\phi_1^2 + \phi_2^2) + w_1 \phi_1 \cdot \phi_2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + a(\nabla\phi_1)(\nabla\phi_2) \\ & + u(\phi_1^2 + \phi_2^2)^2 + v\phi_1^2\phi_2^2 + w_2(2(\phi_1 \cdot \phi_2)^2 - \phi_1^2 \phi_2^2) \\ & + w_1'(\phi_1^2 + \phi_2^2)(\phi_1 \cdot \phi_2) + \cdots , \end{aligned} \tag{2.10}$$

where the ellipsis denotes terms of sixth and higher order. Notice that for our models $u > 0$ and $v > 0$.

The partition function is obtained by performing the functional integral over ϕ_1 and ϕ_2,

$$\mathbb{Z} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp\left[-\int d^d x \, \mathcal{H}(\phi_1, \phi_2, \nabla\phi_1, \nabla\phi_2, \ldots) \right]. \tag{2.11}$$

The appearance of two second-order invariants indicates that the order parameter transforms as a reducible representation of the $SO(n) \times \mathbb{Z}_2$ group generated by rotation and η. However, in most cases the group of transformations which do not leave the ground state invariant is larger than $SO(n) \times \mathbb{Z}_2$, i.e., the full symmetry group of the Hamiltonian contains a transformation ζ which acts on the order parameter as follows:

$$\zeta \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \mathcal{R}(\theta)\phi_2 \end{bmatrix} . \tag{2.12}$$

Here, $\mathcal{R}(\theta)$ denotes the operator of rotation by a certain angle $\theta \neq 2\pi$. Since O_k is invariant under ζ if $\zeta^k = E$ ($k\theta$ is a multiple of 2π), several invariants including $\phi_1 \cdot \phi_2$ disappear and the Hamiltonian (2.10) becomes

$$\begin{aligned} \mathcal{H} = {} & r(\phi_1^2 + \phi_2^2) + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + u(\phi_1^2 + \phi_2^2)^2 \\ & + v\phi_1^2\phi_2^2 + w(2(\phi_1 \cdot \phi_2)^2 - \phi_1^2\phi_2^2) + \cdots . \end{aligned} \tag{2.13}$$

The existence of ζ also assures that the order parameter transforms as an irreducible representation of the group generated by rotation, η, and ζ.

B. Examples of frustrated systems

In this subsection several frustrated models are treated explicitly. We determine the order parameters of these models and construct corresponding LGW Hamiltonians. Only two-dimensional lattices are considered.

1. Triangular lattice

The simplest example of a frustrated system is provided by the antiferromagnetic XY model on the triangular lattice (see Fig. 3). This model is defined in terms of two-component unit vectors $\mathbf{S}_j$ located at the sites j of a triangular lattice. The Hamiltonian of this model is

$$H=J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j\ , \tag{2.14}$$

where $\langle i,j\rangle$ denotes a pair of nearest neighbors and the coupling constant $J<0$ favors antiferromagnetic alignment. This Hamiltonian is invariant under the group O(2) of rotations and reflections of the spins, and also under the translations, rotations, and reflections which form the space group $P6mm$ of a triangular lattice.[14] Since the operations in O(2) and $P6mm$ commute, the symmetry group G_0 of H is simply their direct product.

The ground state can be obtained[15] by a Fourier transformation of $\mathbf{S}_j$:

$$\mathbf{S}_j=\sum_k \mathbf{S}(k)e^{ik\cdot r_j},\quad \mathbf{S}(-k)=\mathbf{S}^*(k) \tag{2.15}$$

where k runs over the Brillouin zone of the triangular lattice. Then we have

$$H=JN\sum_k \mathbf{S}(k)\cdot\mathbf{S}(-k)\{\cos(k\cdot t_1)+\cos(k\cdot t_2)+\cos[k\cdot(t_1+t_2)]\}\ , \tag{2.16}$$

where N is the number of lattice sites and t_1,t_2 are the lattice vectors. Since $\mathbf{S}_j^2=1$, the $\mathbf{S}(k)$ satisfy N conditions,

$$\sum_{k,k'}\mathbf{S}(k)\cdot\mathbf{S}(k')e^{i(k+k')\cdot r_j}=1,\quad j=1,2,\ldots,N\ . \tag{2.17}$$

The ground state can be determined by minimizing (2.16) under the "weak" condition

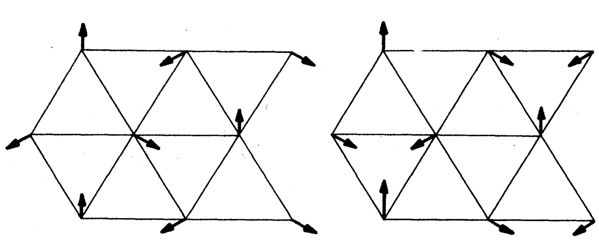

FIG. 3. Ground states of the fully frustrated XY model on the triangular lattice.

$$\frac{1}{N}\sum_j\sum_{k,k'}\mathbf{S}(k)\cdot\mathbf{S}(k')e^{i(k+k')\cdot r_j}=1\ , \tag{2.18}$$

and then looking for solutions satisfying (2.17). Under the weak condition the minimum satisfies

$$\mathbf{S}(k)\{\cos(k\cdot t_1)+\cos(k\cdot t_2)+\cos[k\cdot(t_1+t_2)]\}=\lambda\mathbf{S}(k)\ , \tag{2.19}$$

where λ is the Lagrange multiplier. In order to obtain a nonvanishing solution, we must have

$$\lambda=\cos(k\cdot t_1)+\cos(k\cdot t_2)+\cos[k\cdot(t_1+t_2)]\ . \tag{2.20}$$

Multiplying both sides of (2.19) by $\mathbf{S}(-k)$ and summing over k we see that $-JN\lambda$ is just the total energy of the system. Therefore, $\mathbf{S}(k)$ must vanish unless k is such that λ reaches its maximum. The maximum is reached for $k=\pm Q$, $Q=-\frac{1}{3}g_1+\frac{2}{3}g_2$, where g_1,g_2 are reciprocal-lattice vectors defined by $g_i\cdot t_j=2\pi\delta_{ij}$. Note that $\pm Q$ lies at the corners of the first Brillouin zone. The ground state corresponding to $k=Q$ is given by

$$\mathbf{S}_j=\mathbf{u}\cos(Q\cdot r_j)+\mathbf{v}\sin(Q\cdot r_j)\ , \tag{2.21}$$

where $\mathbf{u}$ and $\mathbf{v}$ are two orthogonal unit vectors. The symmetry group $P6mm$ contains a global rotation of the lattice by 180°. Action of this transformation on the ground state (2.21) gives us another ground state,

$$\mathbf{S}_j=\mathbf{u}\cos(Q\cdot r_j)-\mathbf{v}\sin(Q\cdot r_j)\ , \tag{2.22}$$

which corresponds to $k=-Q$. Since $\mathbf{u}$ and $\mathbf{v}$ are two-component vectors, the pair $(\mathbf{u},\mathbf{v})$ cannot be continuously transformed into $(\mathbf{u},-\mathbf{v})$.

The order parameter of this model can have at most six independent components, that is, expectation values of two spin components for each one of the sublattices. The six quantities

$$S_{L\alpha}=\langle \mathbf{S}_{L\alpha}\rangle,\ \ L=A,B,C\ \text{and}\ \alpha=x,y \tag{2.23}$$

transform as a reducible representation of $P6mm\times$O(2) which can be decomposed into a four- and a two-dimensional irreducible representation spanned, respectively, by the following linear combinations:

$$\begin{aligned}
\phi_{1x}&=S_{Ax}-\frac{1}{2}S_{Bx}+\frac{\sqrt{3}}{2}S_{By}-\frac{1}{2}S_{Cx}-\frac{\sqrt{3}}{2}S_{Cy}\ ,\\
\phi_{1y}&=S_{Ay}-\frac{\sqrt{3}}{2}S_{Bx}-\frac{1}{2}S_{By}+\frac{\sqrt{3}}{2}S_{Cx}-\frac{1}{2}S_{Cy}\ ,\\
\phi_{2x}&=S_{Ax}-\frac{1}{2}S_{Bx}-\frac{\sqrt{3}}{2}S_{By}-\frac{1}{2}S_{Cx}+\frac{\sqrt{3}}{2}S_{Cy}\ ,\\
\phi_{2y}&=S_{Ay}+\frac{\sqrt{3}}{2}S_{Bx}-\frac{1}{2}S_{By}-\frac{\sqrt{3}}{2}S_{Cx}-\frac{1}{2}S_{Cy}\ ,
\end{aligned} \tag{2.24}$$

and

$$\phi_{3x}=S_{Ax}+S_{Bx}+S_{Cx},\quad \phi_{3y}=S_{Ay}+S_{By}+S_{Cy}\ . \tag{2.25}$$

Since $\phi_{3\alpha}$ is just the net magnetization which vanishes both in the ordered and disordered phases, we will be concerned only with the representation spanned by ϕ_1,ϕ_2. This four-dimensional representation is a Kronecker product of a two-dimensional representation of $P6mm$ and a two-dimensional representation of O(2). In a compact vector notation,

$$\phi_1=S_A+\mathscr{R}\left[\frac{2\pi}{3}\right]S_B+\mathscr{R}\left[-\frac{2\pi}{3}\right]S_C ,$$
$$\phi_2=S_A+\mathscr{R}\left[-\frac{2\pi}{3}\right]S_B+\mathscr{R}\left[\frac{2\pi}{3}\right]S_C , \quad (2.26)$$

where ϕ_1,ϕ_2 and S_L are understood as two-component vectors, $\phi_i=(\phi_{ix},\phi_{iy})$ and $S_L=(S_{Lx},S_{Ly})$.

Under global spin rotation, ϕ_1,ϕ_2 transform as two-component vectors while elements of $P6mm$ act as follows. Since the elements of $P31m$ leave ϕ_1,ϕ_2 unchanged, we need to consider only the factor group $P6mm/P31m$, which is just the group of all permutations of sublattice indices A,B,C. This group is generated by the permutations ζ: $[ABC]\rightarrow[CAB]$ and η: $[ABC]\rightarrow[ACB]$, which act on ϕ_1,ϕ_2 as

$$\zeta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}\mathscr{R}(2\pi/3)\phi_1\\ \mathscr{R}(-2\pi/3)\phi_2\end{bmatrix},\ \eta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}\phi_2\\ \phi_1\end{bmatrix} . \quad (2.27)$$

Therefore O_k invariants disappear, unless k is a multiple of 3, and the Hamiltonian is

$$H=a(\phi_1^2+\phi_2^2)+u(\phi_1^2+\phi_2^2)^2+v\phi_1^2\phi_2^2$$
$$+w_3(4(\phi_1\cdot\phi_2)^2-3\phi_1^2\phi_2^2(\phi_1\cdot\phi_2))+\cdots . \quad (2.28)$$

2. *Square lattice*

The fully frustrated XY model on the square lattice[5,6,8] is defined by the Hamiltonian

$$H=\sum_{\langle i,j\rangle}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j , \quad (2.29)$$

where the absolute value of the coupling constant is fixed, $|J_{ij}|=J$. On horizontal bonds, $J_{ij}>0$, while vertical bonds form alternating rows of ferromagnetic and antiferromagnetic couplings (Fig. 1). In the ground state the original lattice splits into four sublattices, A, B, C, and D (Fig. 2).

In addition to the usual $P2mm\times$O(2) symmetries, the symmetry group of the Hamiltonian (2.29) contains a "gauge" symmetry[16,17] ζ. This transformation can be carried out in the following two steps:

(a) Perform the gauge transformation

$$S_i\rightarrow -S_i,\ \ J_{ij}\rightarrow -J_{ij} \quad (2.30)$$

on all sites of sublattice D (Fig. 2).

(b) Rotate the lattice by 90° to obtain the initial configuration of antiferromagnetic bonds.

Under the action of $P2mm$, O(2), and ζ, the eight quantities

$$S_{L\alpha}=\langle \mathbf{S}_{L\alpha}\rangle,\ \ L=A,B,C,D \text{ and } \alpha=x,y \quad (2.31)$$

transform as two four-dimensional irreducible representations. The relevant representation is spanned by

$$\phi_{1x}=S_{Ax}+\frac{\sqrt{2}}{2}S_{Bx}-\frac{\sqrt{2}}{2}S_{By}-S_{Cx}-\frac{\sqrt{2}}{2}S_{Dx}-\frac{\sqrt{2}}{2}S_{Dy} ,$$
$$\phi_{1y}=S_{Ay}+\frac{\sqrt{2}}{2}S_{Bx}+\frac{\sqrt{2}}{2}S_{By}+S_{Cx}+\frac{\sqrt{2}}{2}S_{Dx}-\frac{\sqrt{2}}{2}S_{Dy} ,$$
$$(2.32)$$
$$\phi_{2x}=S_{Ax}+\frac{\sqrt{2}}{2}S_{Bx}+\frac{\sqrt{2}}{2}S_{By}+S_{Cx}-\frac{\sqrt{2}}{2}S_{Dx}+\frac{\sqrt{2}}{2}S_{Dy} ,$$
$$\phi_{2y}=S_{Ay}-\frac{\sqrt{2}}{2}S_{Bx}+\frac{\sqrt{2}}{2}S_{By}-S_{Cx}-\frac{\sqrt{2}}{2}S_{Dx}-\frac{\sqrt{2}}{2}S_{Dy} .$$

The action of the symmetry group on ϕ_1,ϕ_2 is generated by rotation, ζ, and the permutation of sublattice indices η: $[ABCD]\rightarrow[DCBA]$, which act as follows:

$$\zeta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}\mathscr{R}(\pi/2)\phi_1\\ \mathscr{R}(-\pi/2)\phi_2\end{bmatrix},\ \eta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}\phi_2\\ \phi_1\end{bmatrix} . \quad (2.33)$$

Therefore, the O_k invariant of lowest order appearing in the Hamiltonian is O_4,

$$O_4=8(\phi_1\cdot\phi_2)^4-8\phi_1^2\phi_2^2(\phi_1\cdot\phi_2)^2+\phi_1^4\phi_2^4 . \quad (2.34)$$

Now consider effects of various lattice anisotropies. Introduction of different coupling on alternating horizontal rows will break ζ, but ζ^2 will remain a good symmetry. In this case the

$$O_2=2(\phi_1\cdot\phi_2)^2-\phi_1^2\phi_2^2$$

invariant will appear in the LGW Hamiltonian. Another way to break ζ is to introduce antiferromagnetic bonds different in absolute value from the coupling on ferromagnetic bonds. Then ζ^2 will also be broken and the LGW Hamiltonian will be of the most general form, (2.10).

It is of special interest to note that the four-dimensional irreducible representation (2.32) contains two functions that belong to $k=(0,0)$ and two that belong to $k=(0,\pi)$. While usually symmetry operations of a space group do not mix functions with $|k_1|\neq|k_2|$, the operation ζ does. A similar observation was made by Blankschtein *et al.*,[18] who studied fully frustrated *three*-dimensional Ising models.

3. *XY helimagnets (Refs. 19 and 20)*

In some models all O_k invariants disappear from the LGW Hamiltonian. Consider, for example, a system of XY spins at the sites of a square lattice interacting via the following Hamiltonian:

$$H=J_1\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j+J_2\sum_{\langle i,j\rangle'}\mathbf{S}_i\cdot\mathbf{S}_j , \quad (2.35)$$

where $\langle i,j\rangle$ runs over pairs of nearest neighbors and $\langle i,j\rangle'$ runs over pairs of next-nearest neighbors in the x direction. If J_1,J_2 satisfy

$$J_1>0,\ \ J_2<0,\ \ |J_2|>\tfrac{1}{4}J_1 , \quad (2.36)$$

the ground state will be given by

$$\mathbf{S}_j=\mathbf{u}\cos(Q\cdot r_j)+\mathbf{v}\sin(Q\cdot r_j) , \quad (2.37)$$

or by

$$\mathbf{S}_j = \mathbf{u}\cos(Q\cdot r_j) - \mathbf{v}\sin(Q\cdot r_j)\ , \tag{2.38}$$

where the wave vector Q points along the x direction and $\cos|Q| = -J_1/4J_2$ (lattice spacing is taken to be unity). Equations (2.37) and (2.38) describe spiral magnetic structures with opposite helicities.

For $|Q| = 2\pi(m/\mathcal{N})$, where $m/\mathcal{N}$ is an irreducible fraction, the unit cell in the low-temperature phase contains $\mathcal{N}$ spins, and the components of the order parameter can be defined as

$$\phi_1 = \frac{1}{\mathcal{N}} \sum_{L=0}^{\mathcal{N}-1} \mathcal{R}\left[2\pi\frac{m}{\mathcal{N}}L\right] S(L\hat{x})\ ,$$
$$\phi_2 = \frac{1}{\mathcal{N}} \sum_{L=0}^{\mathcal{N}-1} \mathcal{R}\left[-2\pi\frac{m}{\mathcal{N}}L\right] S(L\hat{x})\ . \tag{2.39}$$

Here, $S(L\hat{x})$ denotes average magnetization on sites with the x coordinate equal to L. For incommensurate wave vectors we must take the limit $\mathcal{N}\to\infty$. Translation by a lattice spacing in the x direction acts on ϕ_1,ϕ_2 as follows:

$$\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \to \begin{bmatrix}\mathcal{R}(2\pi(m/\mathcal{N}))\phi_1\\ \mathcal{R}(-2\pi(m/\mathcal{N}))\phi_2\end{bmatrix}\ . \tag{2.40}$$

Therefore, the O_k invariant of lowest order appearing in the Hamiltonian is $O_{\mathcal{N}/2}$ for $\mathcal{N}$ even and $O_{\mathcal{N}}$ for $\mathcal{N}$ odd. For incommensurate wave vectors, all O_k disappear and the LGW Hamiltonian is constructed from powers of $\phi_1^2+\phi_2^2$ and $\phi_1^2\phi_2^2$:

$$H = a(\phi_1^2+\phi_2^2) + u(\phi_1^2+\phi_2^2)^2 + v\phi_1^2\phi_2^2 + \cdots\ . \tag{2.41}$$

III. ϵ EXPANSION NEAR FOUR AND TWO DIMENSIONS

A. $4-\epsilon$ dimensions

The mean-field approximation predicts that systems described by the Hamiltonian (2.13) exhibit a second-order phase transition. However, since the mean-field theory is not very reliable, we will study the critical behavior of (2.13) by the powerful methods of the renormalization group.

Recursion relations for three-dimensional systems can be obtained by Wilson's ϵ-expansion version of the renormalization-group (RG) transformation.[21] In this approach the Hamiltonian is Fourier-transformed and a spherical Brillouin zone is introduced. Performing the functional integral over the $\phi(q)$ which have wave vectors q in the range $\Lambda/l < |q| < \Lambda$, with $l>1$, and rescaling q and $\phi(q)$, a partition function of the same form but with new parameters r', u', v', and w' is obtained. Terms of sixth and higher orders are known[21] to be irrelevant near $d=4$, so that (2.13) reduces to

$$\mathcal{H} = r(\phi_1^2+\phi_2^2) + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + u(\phi_1^2+\phi_2^2)^2 + v\phi_1^2\phi_2^2 + w(2(\phi_1\cdot\phi_2)^2 - \phi_1^2\phi_2^2)\ . \tag{3.1}$$

The Hamiltonian (3.1) was studied previously in a different context by Aharony.[22] Recursion relations for u, v, and w were found to be

$$\frac{du}{dl} = \epsilon u - \frac{1}{8\pi^2}[\,8(n+4)u^2 + nv^2 - 2(n+14)w^2 + 4nuv - 8(n-1)uw - 4(n-2)vw]\ ,$$
$$\frac{dv}{dl} = \epsilon v + \frac{1}{4\pi^2}[(n-4)v^2 - 24uv - 4(n-1)vw]\ , \tag{3.2}$$
$$\frac{dw}{dl} = \epsilon w + \frac{1}{2\pi^2}[(6-n)w^2 - 12uw + 2vw]\ .$$

From the recursion relation for v, it follows that if we start from a non-negative initial value of v, the renormalized value of v will also be non-negative. However, all stable fixed points of these recursion relations reside in the region $v<0$. Under the action of a RG transformation, systems with a positive initial value of v flow to the instability region bounded by the planes $u=0$ and $4u+v=|w|$. Renormalization-group trajectories for models with $n=2$ and $w=0$ are shown in Fig. 4. This runaway to the instability region is interpreted as a first-order transition.[23,24] Therefore we conclude that a first-order transition is expected in three-dimensional fully frustrated systems.

B. $2+\epsilon$ dimensions

The phase transition in two-dimensional systems can be studied by means of Polyakov's $2+\epsilon$ expansion.[25] As opposed to the soft spin field of the $4-\epsilon$ expansion, the $2+\epsilon$ theory deals with a fixed-length spin field. Therefore, the partition sum now reads

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2\, \delta(1-\phi_1^2-\phi_2^2) \times \exp\left[-\int d^dx (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + v\phi_1^2\phi_2^2 + \sum_{k\geq 2} w_k O_k + \cdots\right]\ . \tag{3.3}$$

Assuming that the magnetization is along the ϕ_1 direction, we parametrize the spin field by

$$\phi_1(x) = e^{-a(x)}\varphi_1(x),\quad \phi_2(x) = (1-e^{-2a(x)})^{1/2}\varphi_2(x)\ , \tag{3.4}$$

where $\varphi_1^2=\varphi_2^2=1$. Inserting (3.4) into (3.3), one finds that the a field acquires mass v. By power counting the mass is found to be strongly relevant (to zeroth order in $\epsilon, \lambda_v=2$). Thus we can set $v=\infty$ and neglect the a field. Since all O_k invariants are proportional to some power of

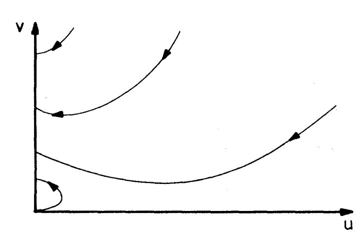

FIG. 4. Renormalization-group trajectories in $d=4-\epsilon$ for models with $n=2$ and $w=0$.

a, these invariants are irrelevant.

The partition function (3.3) is a continuous limit of a microscopic model,

$$H=J\sum_{\langle i,j\rangle}\phi_i\cdot\phi_j-v\sum_i\phi_{i1}^2\phi_{i2}^2\ , \tag{3.5}$$

where we neglected O_k invariants. Defining

$$\mathbf{S}=\frac{1}{\sqrt{2}}(\phi_1+\phi_2),\ \ \mathbf{S}'=\frac{1}{\sqrt{2}}(\phi_1-\phi_2)\ , \tag{3.6}$$

and taking into account that $\phi_1^2+\phi_2^2=1$, (3.5) transforms into

$$H=J\sum_{i,j}\mathbf{S}_i\cdot\mathbf{S}_j+J\sum_{\langle i,j\rangle}\mathbf{S}_i'\cdot\mathbf{S}_j'+v\sum_i[(\mathbf{S}_i\cdot\mathbf{S}_i')^2-\tfrac{1}{4}]\ . \tag{3.7}$$

From the constraint $\phi_1^2+\phi_2^2=1$, it follows that $(\mathbf{S}_i\cdot\mathbf{S}_i')^2\le\frac{1}{4}$ with equality if and only if

$$\mathbf{S}_i'=\tau_i\mathbf{S}_i,\ \ \tau_i=\pm1\ . \tag{3.8}$$

In the limit $v=+\infty$ only configurations satisfying (3.8) contribute to the partition sum. Substitution of (3.8) into (3.7) yields

$$H=J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j(1+\tau_i\tau_j)\ . \tag{3.9}$$

This Hamiltonian describes an ($n\ge2$)-component vector model coupled to an Ising model. For $n\ge3$ this Hamiltonian is expected to exhibit a single transition of an Ising universality class. The $n=2$ case will be studied in detail in Sec. IV.

C. Effects of symmetry breaking

Consider the LGW Hamiltonian (2.10). The quadratic part can be diagonalized by introducing

$$\psi_1=\frac{1}{\sqrt{2}}(\phi_1+\phi_2),\ \ \psi_2=\frac{1}{\sqrt{2}}(\phi_1-\phi_2)\ . \tag{3.10}$$

In terms of these two representations, the Hamiltonian takes the form

$$\mathscr{H}=\left[r+\frac{w_1}{2}\right]\psi_1^1+\left[r-\frac{w_1}{2}\right]\psi_2^2+\left[1+\frac{a}{2}\right](\nabla\psi_1)^2+\left[1-\frac{a}{2}\right](\nabla\psi_2)^2+\left[u+\frac{v}{4}+\frac{w_2}{4}+\frac{w_2'}{2}\right]\psi_1^4$$
$$+\left[u+\frac{v}{4}+\frac{w_2}{4}-\frac{w_2'}{2}\right]\psi_2^4+\left[\frac{v}{2}-\frac{3w_2}{4}\right]\psi_1^2\psi_2^2-(v-w_2)(\psi_1\cdot\psi_2)^2+\cdots\ . \tag{3.11}$$

In $d=4-\epsilon$ one finds that this Hamiltonian has, for large symmetry breaking, two transitions. The first (high temperature) is associated with ordering of the O(n) type. This is followed by an Ising-like transition. Such a situation arises in systems with a tetracritical point. In our case, however, on the $w_1=0$ line the transition is first order. Therefore, the phase diagram has a first-order line at low symmetry, and two lines of continuous transitions at large symmetry breaking; these lines join in some manner.

For $d=2$ and $n=2$, one expects, for large symmetry breaking, a high-temperature Kosterlitz-Thouless transition, followed by an Ising transition. So far we can deduce only that one of the three possibilities of Fig. 5 [(a)—(c)] can occur.

The fact that one has to choose among only these three possibilities, combined with the results of Sec. IV, will allow us to exclude possibility (a).

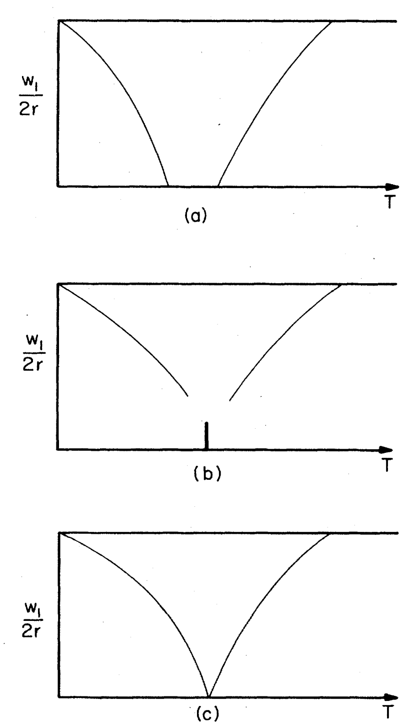

FIG. 5. Possible phase diagrams for models with $w=0$. Thick line denotes first-order transition. The manner in which transition lines meet in (b) has not been specified.

IV. FRUSTRATED XY MODEL IN TWO DIMENSIONS

In this section we study the model (3.9) with two-component spins S_i. First, a variational approximation is presented, followed by scaling arguments applied to this model. Subsequently, the partition function is transformed into the Coulomb-gas representation. Standard position-space RG techniques are employed to derive generalized Kosterlitz recursion relations. These recursion relations are used to establish the phase diagram.

A. Decoupled variational approximation

The model (3.9) consists of an XY model and an Ising model with some peculiar coupling between them. One of the possibilities that arises here is that this coupling is irrelevant and the model (3.9) has two independent phase transitions, one of XY type and one Ising-like. If indeed this is the case, the natural question to ask is which transition occurs first.

In order to estimate the critical couplings of XY and Ising transitions, we will minimize the trial free energy

$$\beta F = \sum_{\{\mathbf{S},\tau\}} (-\rho_t H + \rho_t \ln \rho_t) \tag{4.1}$$

under the constraint

$$\rho_t = \rho_{XY}(K)\rho_I(L) = \frac{\exp\left[K \sum_{\langle i,j\rangle} \mathbf{S}_i \mathbf{S}_j\right] \exp\left[L \sum_{\langle i,j\rangle} \tau_i \tau_j\right]}{Z_{XY}(K) Z_I(L)} . \tag{4.2}$$

Here, K and L are the parameters to be determined. Substitution of (4.2) into (4.1) yields

$$\beta F = \frac{1}{Z_{XY}(K)Z_I(L)} \sum_{\{\mathbf{S},\tau\}} \left[\sum_{\langle i,j\rangle} (-J\mathbf{S}_i\cdot\mathbf{S}_j - J\tau_i\tau_j\mathbf{S}_i\cdot\mathbf{S}_j + K\mathbf{S}_i\cdot\mathbf{S}_j + L\tau_i\tau_j) \times \exp\left[K \sum_{\langle i,j\rangle} \mathbf{S}_i\cdot\mathbf{S}_j\right] \exp\left[L \sum_{\langle i,j\rangle} \tau_i\tau_j\right]\right] - \ln Z_{XY}(K) - \ln Z_I(L) . \tag{4.3}$$

Using the relation

$$\frac{1}{Z_{XY}(K)} \sum_{\{\mathbf{S},\tau\}} \left[\sum_{\langle i,j\rangle} K\mathbf{S}_i\cdot\mathbf{S}_j\right] \exp\left[K \sum_{\langle i,j\rangle} \mathbf{S}_i\cdot\mathbf{S}_j\right] - \ln Z_{XY}(K) = \langle H_{XY}(K)\rangle + \beta F = -S_{XY}(K) , \tag{4.4}$$

where $S_{XY}(K)$ is the entropy of the XY model at coupling K, and a similar relation for the Ising model, we obtain

$$\beta F = -S_{XY}(K) - S_I(L) - \frac{J}{K}\langle H_{XY}(K)\rangle - \frac{J}{NKL}\langle H_{XY}(K)\rangle\langle H_I(L)\rangle . \tag{4.5}$$

Here, N is the number of bonds. In order to minimize the free energy, K and L must satisfy

$$K = J\left[1 - \frac{1}{N}\langle H_I(L)\rangle\right], \quad L = -J\frac{1}{N}\langle H_{XY}(K)\rangle . \tag{4.6}$$

The functional dependence of $(1/N)\langle H_I(L)\rangle$ on L is known exactly,[26] while $(1/N)\langle H_{XY}(K)\rangle$ can be taken from Monte Carlo simulation of the XY model.[27] After substitution of these functions into (4.6), this system can be solved numerically, giving K and L as some monotonic functions of J. When $L(J)$ reaches the critical value $L_c = \frac{1}{2}\ln(1+\sqrt{2})$, $K(J)$ is equal to 1.07, which is slightly below the critical value[27] $K_c = 1.12$. Therefore we conclude that within the decoupled approximation the model (3.8) possesses an intermediate phase with ordered Ising spins and disordered XY spins.

B. Double-layer model

The model (3.9) is a limit of a more general model described by the partition function

$$Z = \left[\prod_r \int_{-\pi}^{\pi} d\theta(r) d\phi(r)\right] \exp\left[J \sum_{\langle r,r'\rangle} \cos[\theta(r)-\theta(r')] + J \sum_{\langle r,r'\rangle} \cos[\phi(r)-\phi(r')] + L \sum_r \cos p[\theta(r)-\phi(r)]\right] , \tag{4.7}$$

where p is some positive integer. The model (3.9) is recovered if $p=2$ and $L\to\infty$. For finite L, (4.7) describes a system consisting of two layers of XY spins with an interaction $L\cos p[\theta(r)-\phi(r)]$ on links between the layers.

To check the relevance of the coupling between the layers, we calculate the correlation function

$$\langle \cos p[\theta(0)-\phi(0)] \cos p[\theta(\rho)-\phi(\rho)]\rangle = \langle e^{ip[\theta(0)-\theta(\rho)]}\rangle_\theta \langle e^{-ip[\phi(0)-\phi(\rho)]}\rangle_\phi . \tag{4.8}$$

In the spin-wave approximation the correlation function $\langle e^{ip[\theta(0)-\theta(\rho)]}\rangle$ was calculated by José *et al.*,[28]

$$\langle e^{ip[\theta(0)-\theta(\rho)]}\rangle \sim \rho^{-p^2/2\pi J} . \tag{4.9}$$

Consequently, the correlation function (4.8) decays as $\rho^{-p^2/\pi J}$. From this result and from the scaling theory of phase transitions, it follows that the operator $L\cos p(\theta-\phi)$ has a scaling index $2-p^2/2\pi J$, i.e., it is irrelevant for $2\pi J < p^2/2$. Since vortices are known[29] to be irrelevant for $4 < 2\pi J$, the Gaussian fixed line is stable against both vortex perturbations and coupling between the layers for $4 < 2\pi J < p^2/2$. Therefore we expect that the model (4.7) with $p \geq 3$ will exhibit an intermediate massless phase similar to that of the p-state clock model with $p \geq 5$.

In order to proceed further with the investigation in the double-layer model, we rewrite it in an equivalent lattice Coulomb-gas representation. In this representation typical problems of two-dimensional statistical mechanics are formulated in terms of two sets of integer variables $m(R)$

and $q(r)$ residing on sites r of the original lattice and sites R of the dual lattice. Interaction between charges of the same kind at large separations is proportional to the logarithm of the distance. On the other hand, the $m(R)-q(r)$ interaction is proportional to the angle between the vector $R-r$ and some fixed direction in the plane. This form resembles the interaction between electric charges and magnetic monopoles. The duality transformation for a model written in a Coulomb-gas representation has a particularly simple form. It reduces to a interchange of "charges" m and "monopoles" q.

The purpose of rewriting our model in Coulomb-gas language is twofold. First of all, recursion relations for a problem with a single Coulomb gas were derived by Kosterlitz.[29] Derivation of recursion relations for a model with any number of Coulomb gases can be done by straightforward generalization of Kosterlitz's method.[30] Another advantage of the Coulombic formulation lies in the unification of various models: the difference between problems is reflected only in changes in the parameters of the interactions. This unification suggests that under renormalization-group transformation our model may flow toward some simpler model with known behavior.

The general prescription for the transformation of two-dimensional models into the Coulomb-gas representation was given by Kadanoff.[31] Following his methods we find that the double-layer model can be reformulated in terms of two sets of "charges" $m(R)$ and $n(R)$, and a set of "monopoles" $q(r)$.

The first step in the transformation into the Coulomb-gas representation is the replacement of all cosine interactions by a Villain form,

$$e^{k\cos(x)} \to \sum_{m=-\infty}^{\infty} e^{-k(x-2\pi m)^2/2} , \tag{4.10}$$

which can also be written (up to a normalization factor) as

$$\sum_{q=-\infty}^{\infty} e^{-q^2/2k+iqx} . \tag{4.11}$$

Substituting (4.10) for the coupling within each layer and (4.11) for coupling between layers, we obtain

$$\mathbb{Z} = \left[\prod_r \int_{-\pi}^{\pi} d\theta(r)d\phi(r)\right] \sum_{\substack{m(r,r')\\ n(r,r')\\ q(r)}} \exp\left[-\frac{J}{2}\sum_{\langle r,r'\rangle}[\theta(r)-\theta(r')-2\pi m(r,r')]^2 -\frac{J}{2}\sum_{\langle r,r'\rangle}[\phi(r)-\phi(r')-2\pi n(r,r')]^2-\frac{1}{2L}\sum_r q^2(r)+ip\sum_r q(r)[\theta(r)-\phi(r)]\right] . \tag{4.12}$$

Note that $m(r,r')=-m(r',r)$.

Since the Villain interaction has the same symmetries as the cosine interaction, and closely approximates it numerically, one can argue that models (4.7) and (4.12) belong to the same universality class. Indeed, the differences between Villain and cosine interactions were shown to be irrelevant to the critical behavior of the XY model.[32] On the other hand, the Villain interaction allows exact decomposition of the configurations to spin waves and the Coulomb gases of vortices.

For future convenience we will carry out the transformation to the Coulomb-gas representation for a more general model. We introduce in the Hamiltonian a term

$$K\sum_{\langle r,r'\rangle}[\theta(r)-\theta(r')-2\pi m(r,r')][\phi(r)-\phi(r')-2\pi n(r,r')] , \tag{4.13}$$

with $0\le K\le J$. The initial model corresponds to the value $K=0$, but since a nonzero value of this parameter will be generated by the renormalization procedure, it is convenient to include it explicitly from the start. This term is missing in the recursion relations of Parga and van Himbergen,[33] who studied the $p=1$ case.

On each site R of the dual lattice we define integer variables $m(R)$ and $n(R)$ which are essentially the circulation of $m(r,r')$ and $n(r,r')$ around the plaquette containing the dual lattice site R:

$$\begin{aligned} m(R)=\sum_R m(r,r') &= m(R-\tfrac{1}{2}\hat{x}-\tfrac{1}{2}\hat{y},R+\tfrac{1}{2}\hat{x}-\tfrac{1}{2}\hat{y})+m(R+\tfrac{1}{2}\hat{x}-\tfrac{1}{2}\hat{y},R+\tfrac{1}{2}\hat{x}+\tfrac{1}{2}\hat{y})\\ &\quad +m(R+\tfrac{1}{2}\hat{x}+\tfrac{1}{2}\hat{y},R-\tfrac{1}{2}\hat{x}+\tfrac{1}{2}\hat{y})+m(R-\tfrac{1}{2}\hat{x}+\tfrac{1}{2}\hat{y},R-\tfrac{1}{2}\hat{x}-\tfrac{1}{2}\hat{y}) ,\\ n(R)=\sum_R m(r,r') &= n(R-\tfrac{1}{2}\hat{x}-\tfrac{1}{2}\hat{y},R+\tfrac{1}{2}\hat{x}-\tfrac{1}{2}\hat{y})+n(R+\tfrac{1}{2}\hat{x}-\tfrac{1}{2}\hat{y},R+\tfrac{1}{2}\hat{x}+\tfrac{1}{2}\hat{y})\\ &\quad +n(R+\tfrac{1}{2}\hat{x}+\tfrac{1}{2}\hat{y},R-\tfrac{1}{2}\hat{x}+\tfrac{1}{2}\hat{y})+n(R-\tfrac{1}{2}\hat{x}+\tfrac{1}{2}\hat{y},R-\tfrac{1}{2}\hat{x}-\tfrac{1}{2}\hat{y}) . \end{aligned} \tag{4.14}$$

Now the partition function (4.12) reads

$$\mathbb{Z} = \sum_{\substack{m(R),n(R),\\ q(r)}} \mathbb{Z}(m,n,q)\exp\left[-\frac{1}{2L}\sum_r q^2(r)\right], \tag{4.15}$$

with

$$\mathbb{Z}(m,n,q)= \sum_{\substack{m(r,r')\\ n(r,r')}} \left[\prod_r \int_{-\pi}^{\pi} d\theta(r)d\phi(r)\right]\left[\prod_r \delta_{m(R),\sum_R m(r,r')}\delta_{n(R),\sum_R n(r,r')}\right]$$
$$\times\exp\left[-\frac{J}{2}\sum_{\langle r,r'\rangle}[\theta(r)-\theta(r')-2\pi m(r,r')]^2-\frac{J}{2}\sum_{\langle r,r'\rangle}[\phi(r)-\phi(r')-2\pi n(r,r')]^2\right.$$
$$\left.+K\sum_{\langle r,r'\rangle}[\theta(r)-\theta(r')-2\pi m(r,r')][\phi(r)-\phi(r')-2\pi n(r,r')]+ip\sum_r q(r)[\theta(r)-\phi(r)]\right]. \tag{4.16}$$

In order to obtain the Coulomb-gas representation, we need to perform summations on $m(r,r'),n(r,r')$ and integrations on $\theta(r),\phi(r)$ in $\mathbb{Z}(m,n,q)$.

This is done in detail in the Appendix. We obtain, after some algebra, the final form,

$$\mathbb{Z} = \sum_{m,n,q}{}' \exp\left[-\frac{\pi^2}{2}J\sum_R m^2(R)-\frac{\pi^2}{2}J\sum_R n^2(R)+\pi^2K\sum_R m(R)n(R)\right.$$
$$-\left[\frac{1}{2L}+\frac{\pi p^2}{2\pi J+2\pi K}\right]\sum_r q^2(r)+2\pi J\sum_{(R,R')} m(R)\ln\left[\frac{|R-R'|}{a}\right]m(R')$$
$$+2\pi J\sum_{(R,R')} n(R)\ln\left[\frac{|R-R'|}{a}\right]n(R')-2\pi K\sum_{R\neq R'} m(R)\ln\left[\frac{|R-R'|}{a}\right]n(R')$$
$$\left.+ip\sum[m(R)-n(R)]\Theta(R-r)q(r)+\frac{2p^2}{2\pi J+2\pi K}\sum_{(r,r')} q(r)\ln\left[\frac{|r-r'|}{a}\right]q(r')\right], \tag{4.17}$$

where a is the lattice spacing. This expression is derived assuming $K\le J$. The prime on the summation implies, for $K<J$, "charge neutrality" for each of $m(r)$, $n(r)$, and $q(r)$. For $K=J$, only the constraints $\sum q(r)=0$ and $\sum_R[m(R)-n(R)]=0$ survive (see the Appendix). The function $\Theta(R)$ is given by

$$\Theta(R_x,R_y)=\tan^{-1}(R_y/R_x)\,. \tag{4.18}$$

The integer-valued variables $m(R)$ and $n(R)$ are identified[28,34] with vortices on different layers of the double-layer model (4.7), while the $q(r)$ mediate the interaction between the layers.

Correlation functions can be transformed into the Coulomb-gas representation by the same techniques. Two different correlation functions can be defined for the model (3.9)—the correlation function for the Ising variables,

$$\langle\tau(0)\tau(\rho)\rangle\,, \tag{4.19}$$

and the XY correlation function,

$$\langle e^{i\theta(0)}e^{i\theta(\rho)}\rangle\,. \tag{4.20}$$

Analogous correlation functions can also be defined for the model (4.7). They are

$$\Gamma_p(\rho)=\langle e^{i(\theta(0)-\phi(0))}e^{-i(\theta(\rho)-\phi(\rho))}\rangle \tag{4.21}$$

and

$$\Gamma_{XY}(\rho)=\langle e^{i(\theta(0)+\phi(0))}e^{-i(\theta(\rho)+\phi(\rho))}\rangle\,. \tag{4.22}$$

Following the same steps as the Appendix one finds that the Coulomb-gas representation for the correlation function is

$$\Gamma_p(\rho)=\frac{1}{\mathbb{Z}}\sum_{m,n,q}{}' \exp\Bigg[-\frac{\pi^2}{2}J\sum_R m^2(R)-\frac{\pi^2}{2}J\sum_R n^2(R)+\pi^2K\sum_R m(R)n(R)-\frac{1}{2L}\sum_r q^2(r)$$
$$+\frac{\pi p^2}{2\pi J+2\pi K}\sum_r [q'(r)]^2+2\pi J\sum_{(R,R')} m(R)\ln\left[\frac{|R-R'|}{a}\right]m(R')$$
$$+2\pi J\sum_{(R,R')} n(R)\ln\left[\frac{|R-R'|}{a}\right]n(R')-2\pi K\sum_{\substack{R\neq R'\\ R,R'}} m(R)\ln\left[\frac{|R-R'|}{a}\right]n(R')$$
$$+ip\sum_{R,r}[m(R)-n(R)]\Theta(R-r)q'(r)+\frac{2p^2}{2\pi J+2\pi K}\sum_{(r,r')} q'(r)\ln\left[\frac{|r-r'|}{a}\right]q'(r')\Bigg] \tag{4.23}$$

and

$$\Gamma_{XY}(\rho)=\frac{1}{\mathbb{Z}}\sum_{m,n,q}{}' \exp\Bigg[-\frac{\pi^2}{2}J\sum_R m^2(R)-\frac{\pi^2}{2}J\sum_R n^2(R)+\pi^2K\sum_R m(R)n(R)$$
$$-\left[\frac{1}{2L}+\frac{\pi p^2}{2\pi J+2\pi K}\right]\sum_r q^2(r)+2\pi J\sum_{(R,R')} m(R)\ln\left[\frac{|R-R'|}{a}\right]m(R')$$
$$+2\pi J\sum_{(R,R')} n(R)\ln\left[\frac{|R-R'|}{a}\right]n(R')-2\pi K\sum_{R\neq R'} m(R)\ln\left[\frac{|R-R'|}{a}\right]n(R')$$
$$+ip\sum_{R,r}[m(R)-n(R)]\Theta(R-r)q(r)+i\sum_R[m(R)+n(R)][\Theta(R)-\Theta(R-\rho)]$$
$$+\frac{2p^2}{2\pi J+2\pi K}\sum_{(r,r')} q(r)\ln\left[\frac{|r-r'|}{a}\right]q(r')-\frac{1}{\pi J}\ln\left[\frac{\rho}{a}\right]\Bigg]\ . \tag{4.24}$$

In (4.23), $q'(r)$ is defined by

$$q'(r)=\begin{cases} q(r)+1/p, & r=0\\ q(r)-1/p, & r=\rho\\ q(r), & r\neq 0,\rho. \end{cases} \tag{4.25}$$

D. Recursion relations and the phase diagram

Recursion relations for the Hamiltonian (4.17) can be derived by means of Kosterlitz's position-space renormalization-group techniques. To apply his methods, we rewrite (4.17) as

$$\mathbb{Z}=\sum_{m,n,q}{}' \exp\Bigg[(\ln y)\sum_R m^2(R)+(\ln y)\sum_R n^2(R)+(\ln y_q)\sum_r q^2(r)+2\pi J\sum_{(R,R')} m(R)\ln\left[\frac{|R-R'|}{a}\right]m(R')$$
$$+2\pi J\sum_{(R,R')} n(R)\ln\left[\frac{|R-R'|}{a}\right]n(R')-2\pi K\sum_{\substack{R,R'\\ R\neq R'}} m(R)\ln\left[\frac{|R-R'|}{a}\right]n(R')$$
$$+ip\sum_{R,r}[m(R)-n(R)]\Theta(R-r)q(r)+2\pi J_q\sum_{(r,r')} q(r)\ln\left[\frac{|r-r'|}{a}\right]q(r')\Bigg]\ , \tag{4.26}$$

where y, y_q, and J_q are defined by

$$\ln y=-\frac{\pi^2}{2}J\ ,$$
$$\ln y_q=-\frac{1}{2L}-\frac{2p^2}{2\pi J+2\pi K}\ , \tag{4.27}$$

and

$$2\pi J_q=\frac{2p^2}{2\pi J+2\pi K}\ .$$

Since the constraints (4.27) can be relaxed by the renormalization-group transformation, we will treat y, y_q, and J_q as free parameters. Although it is not necessary to introduce a new parameter for the fugacity of q's, because one can vary it by changing L, it will be more convenient in the intermediate steps to use y_q.

The term $\sum_R m(R)n(R)$ in (4.17) serves as a fugacity of combined vortices—configurations in which the m-vortex and n-vortex reside on the same lattice site. For brevity, such configurations were left out in (4.26). All but one type of these combined vortices are found to be irrelevant in the entire range of couplings we are interested in. The only complex vortex which can be relevant consists of an m-vortex and an n-vortex of the same sign. We introduced a new parameter, $\ln\alpha$, for the fugacity of such a double vortex.

Now we can study the partition function (4.26) in the region $y,y_q,\alpha \ll 1$, where the dilute-gas approximation $m(R),n(R),q(r)=0,\pm1$ can be used. Using standard methods,[29,30,35] the following recursion relations are obtained:

$$\begin{aligned}
\frac{dy}{dl}&=(2-\pi J)y\ ,\\
\frac{dy_q}{dl}&=(2-\pi J_q)y_q\ ,\\
\frac{d\alpha}{dl}&=[1-\pi(J-K)]\alpha\ ,\\
\frac{dJ}{dl}&=-\pi((2\pi)^2(J^2+K^2)y^2+2(2\pi)^2(J-K)^2\alpha^2-p^2y_q^2)\ ,\\
\frac{dJ_q}{dl}&=-\pi((2\pi)^2J_q^2y_q^2-2p^2y^2)\ ,\\
\frac{dK}{dl}&=-\pi(2(2\pi)^2JKy^2+2(2\pi)^2(J-K)^2\alpha^2-p^2y_q^2)\ .
\end{aligned} \tag{4.28}$$

Of these six equations, the first three describe the variation of fugacities y, y_q, and α, and the last three of couplings J, J_q, and K, under renormalization. It is easy to show that

$$\frac{d}{dl}\left[2\pi J_q-\frac{2p^2}{2\pi J+2\pi K}\right]=0\ . \tag{4.29}$$

Therefore, the initial relationship (4.27) between J_q and J,K is preserved under renormalization.

It may seem more attractive to work with the more physical parameter L instead of y_q. The recursion relation for the latter is [using (4.27) and (4.28)]

$$\frac{dL}{dl}=(4-2\pi J_q)L^2\ . \tag{4.30}$$

We consider the first three equations for different values of p, to determine regimes of J,K for which the various fugacities are irrelevant. We find that y is irrelevant for $(\pi J)^{-1}<\frac{1}{2}$, α is irrelevant for $K/J<1-(\pi J)^{-1}$, and y_q is irrelevant for $K/J<(p^2/4)(\pi J)^{-1}-1$. These regimes give rise to the various regions of stability, as indicated in Fig. 6.

First consider the $p\geq 4$ case [Fig. 6(c)]. In region A, y and α are irrelevant, and y_q (or L) is relevant. We assume that the RG flows will proceed to $L\to\infty$, $y,\alpha\to 0$. For

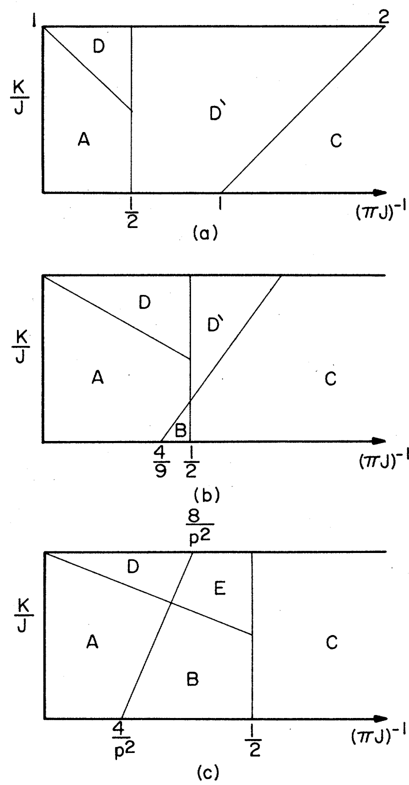

FIG. 6. Stability of the Gaussian plane $y=y_q=L=0$. (a) $p=2$, (b) $p=3$, and (c) $p\geq 4$.

this case one obtains the following behavior of the correlations:

$$\Gamma_{XY}(\rho)\sim\rho^{-1/\pi J_{\rm eff}},\quad \Gamma_p(\rho)\sim{\rm const} \tag{4.31}$$

for $\rho\to\infty$. Therefore, an initial Hamiltonian characterized by J,K in region A will have algebraic decay of the XY correlations and spontaneous breaking of the discrete symmetry.

In region B all these fugacities flow to zero, and thus algebraic decay of both correlation functions is expected:

$$\Gamma_{XY}(\rho)\sim\rho^{-1/\pi J_{\rm eff}},\quad \Gamma_p(\rho)\sim\rho^{-1/\pi(J_{\rm eff}+K_{\rm eff})}\ . \tag{4.32}$$

In this expression $J_{\rm eff},K_{\rm eff}$ denote the point in region B, reached (asymptotically) by the RG trajectory of some initial Hamiltonian, characterized by J,K and nonvanishing fugacities.

Since the line that separates region A from region B is the domain of attraction of Hamiltonians that lie on the phase boundary and on this line the exponents depend on $J_{\rm eff},K_{\rm eff}$, obviously the transition between phases of type (4.31) and (4.32) is nonuniversal. That is, Γ_{xy} decays with a power that depends on the initial Hamiltonian. However, the decay of $\Gamma_p(\rho)$, since it depends only on $J_{\rm eff}+K_{\rm eff}$, and this being a constant for the A/B boundary, is universal.

In region C, y is relevant, while y_q (e.g., L) is irrelevant. In this region both correlations decay exponentially. On the B/C boundary, $\pi J_{\rm eff}=2$ and, therefore, the index associated with Γ_{XY} is universal, whereas that corresponding to Γ_p is nonuniversal.

Other types of behavior, such as exponential decay of Γ_{XY} together with either (1) long-range order ($\Gamma_p \to$const), in region D, or (2) algebraic decay of Γ_p, in region E, are also possible. However, for the problem at hand one has initially $K=0$, and therefore regions A, B, and C are expected to serve as sinks of the various phases.

Thus, for $p \geq 4$ one expects for the model (4.7) to have three phases. At low temperatures the discrete symmetry is broken and $\Gamma_p \to$const, while Γ_{XY} decays algebraically. At high temperatures, in the disordered phase, both correlations decay exponentially. In the intermediate phase both correlation functions exhibit algebraic decay.

For $p=2$, in region A, again $\Gamma_p \to$const (Ising order) and Γ_{XY} is algebraic. In region D, $\Gamma_p \to$const and Γ_{XY} decays exponentially.

In region C the system is completely disordered. In between, in region D', since both y and y_q are relevant, correlations cannot be estimated directly. We consider recursion for $J-K$:

$$\frac{d}{dl}(J-K) = -\pi(2\pi)^2(J-K)^2 y^2 ,$$

and note that, in regions D' and C, $J-K \to 0$ with renormalization. Thus, concentrating on the $J=K$ subspace, we note that the partition function (4.17) becomes

$$\mathbb{Z} = \left[\prod_R \sum_{n(R)=-\infty}^{\infty}\right] \sum_{\substack{M(R),\\ q(r)}}' \exp\left[-\frac{\pi^2}{2} J \sum_R M^2(R) - \left[\frac{1}{2L}+\frac{1}{J}\right]\sum_r q^2(r) + 2\pi J \sum_{(R,R')} M(R) \ln\left[\frac{|R-R'|}{a}\right] M(R') \right.$$
$$\left. + i2\sum_{R,r} M(R)\Theta(R-r) q(r) + \frac{4}{2\pi J}\sum_{(r,r')} q(r) \ln\left[\frac{|r-r'|}{a}\right] q(r')\right], \tag{4.33}$$

where $M(R)=m(R)-n(R)$. Notice that, for $J-K=0$, the constraint $\sum_R n(R)=0$ is relaxed. Additional terms in the exponent of (4.33) that could possibly be generated by the RG transformation are

$$(\ln y)\sum_R n^2(R) + (\ln y')\sum_R n(R) M(R) . \tag{4.34}$$

Summation over n will generate only a fugacity term for $M^2(R)$, and we obtain

$$\mathbb{Z} = \sum_{\substack{M(R),\\ q(r)}}' \exp\left[(\ln y'')\sum_R M^2(R) - \left[\frac{1}{2L}+\frac{1}{J}\right]\sum_r q^2(r) + 2\pi J \sum_{(R,R')} M(R)\ln\left[\frac{|R-R'|}{a}\right] M(R')\right.$$
$$\left. + i2\sum_{R,r} M(R)\Theta(R-r) q(r) + \frac{4}{2\pi J}\sum_{(r,r')} q(r)\ln\left[\frac{|r-r'|}{a}\right] q(r')\right] . \tag{4.35}$$

This is recognizable as the partition function of an Ising model[34] with possibly incorrect fugacities. However, for any value of the fugacities, the transition in model (4.35) is governed by an Ising fixed point. For $J-K=0$ the correlation function $\Gamma_{XY}(\rho)$ is disordered for all J, whereas $\Gamma_2(\rho)$ becomes identical to the Ising correlation function, which decays to a constant for J greater then some J_c and decays exponentially for $J<J_c$. This is what happens on the $J=K$ subspace; since points with projection on D' flow to this subspace, we expect that D' contains two regions; one with Ising order, the other with disorder (same phase as C), separated by a line of Ising-like transitions.

This line (or, more precisely, the hypersurface it represents) may be connected to the low-(K/J) part of Fig. 6(a) in different ways.

(1) It reaches $K=0$ with $(\pi J)^{-1} > \frac{1}{2}$. This means a sequence of two transitions. Starting from the completely disordered phase, an Ising-like transition occurs first, followed at lower temperature by a transition below which Γ_{XY} decays algebraically.

(2) The Ising line in D' reaches the boundary of A with $K>0$. Then, depending on the trajectory of the Hamiltonian as the temperature is varied, either the phase structure described above, or a single transition, from completely disordered to ordered, can occur. This transition will, in general, be first order; only in a special case (of lower codimensionality) can it be continuous. To see that the transition is first order, note that one passes from region A, with finite Ising order, directly to the disordered region in D', and therefore the Ising order undergoes a discontinuity.

Thus we conclude that either a single transition (first order or continuous), or two transitions, is (are) expected, with an intermediate phase of *Ising order and XY disorder.*

However, referring back to Sec. II C, we note that there we concluded that an intermediate phase, if present, will have *XY order and Ising disorder.* Thus exclusion of the two-transition scenario for the $p=2$, $n=2$ model seems to be justified.

For $p=3$ similar arguments apply for the region D' [Fig. 6(b)]. In the most general case, one may have one of the following possibilities:

(1) Two transitions: intermediate phase (B) with both correlations algebraic.

(2) Single first-order transition.

(3) Two transitions: intermediate phase (D or D') with $\Gamma_3 \to$const and Γ_{XY} exponentially decaying.

V. SUMMARY AND DISCUSSION

In this work we studied spin systems whose low-temperature phase has two kinds of order. That associated with a spontaneously broken discrete $Z(2)$ symmetry, and the characteristics of the low-temperature phase of a system with $O(n)$ symmetry.

Important examples of such systems are fully frustrated two-dimensional XY models on triangular and square lattices and XY helimagnets. By symmetry arguments, we identify the order parameter, appropriate for transitions in these systems, and construct the most general LGW Hamiltonian. We study this Hamiltonian by means of an exact renormalization group in dimensions $d=4-\epsilon$ and $d=2+\epsilon$. In $d=4-\epsilon$ we find a single first-order transition.

Analysis of the LGW Hamiltonian in $2+\epsilon$ dimensions shows that one flows to a microscopic model (3.9), that contains $O(n)$ and Ising-type variables on each site. For $n>2$ such models will have only a single Ising transition in $d=2$. For $n=2$ this microscopic model can be viewed as a limiting case of a more general class of double-layer models. In these latter models each site has two XY-type variables, θ_i,ϕ_i, with ferromagnetic coupling within, and quadrupolar $\cos[2(\theta_i-\phi_i)]$-type coupling between layers. Therefore, we expect that analysis of such a double-layer model will yield results that are relevant to the original problem, i.e., the fully frustrated XY model in $d=2$.

The double-layer model is generalized to interlayer couplings of the form $\cos[p(\theta_i-\phi_i)]$. It is mapped onto a Coulomb-gas problem, to which standard RG techniques are applied. The resulting recursion relations are analyzed, and the following results are obtained.

For $p^2>8$ the low-temperature and disordered phases are separated by an intermediate massless phase. In this phase correlations associated with both the discrete and XY-type symmetry decay algebraically. Phase transitions in the $p^2>8$ case will be of Kosterlitz-Thouless type with unobservable essential singularities in the specific heat. Some exponents that characterize the decay of the correlation function at the transition are nonuniversal. For $p=2$ (the most interesting case) we predict a single transition. Our analysis cannot resolve whether this transition should be first order or continuous.

Note added in proof. Derivations of the Landau Hamiltonian for some of the systems discussed by us have appeared in recent papers by M. Y. Choi and S. Doniach (unpublished), T. C. Halsey (unpublished), D. H. Lee (unpublished), J. D. Joannopoulos (unpublished), and J. W. Negele and D. P. Landau (unpublished).

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APPENDIX

The Coulomb-gas representation of the partition sum (4.15), with $Z(m,n,q)$ given by (4.16), is obtained by performing summations (integrations) over $m(r,r')$, $n(r,r')$, $\theta(r)$, and $\phi(r)$. To do this, on each lattice site r we introduce integer variables $k(r)$ and $l(r)$. On the horizontal bonds of each layer we replace $m(r,r')$ by $k(r')-k(r')$ and $n(r,r')$ by $l(r')-l(r)$, while on the vertical bonds we define new variables $\overline{m}(r,r')$ and $\overline{n}(r,r')$ by $m(r,r')=\overline{m}(r,r')-k(r)+k(r')$ and $n(r,r')=\overline{n}(r,r')-l(r)+l(r')$. Now the summation over $m(r,r')$ and $n(r,r')$ can be replaced by the summation over $\overline{m}(r,r')$, $\overline{n}(r,r')$, $k(r)$, and $l(r)$. We have

$$\begin{aligned}\mathbb{Z}(m,n,q)=\sum_{\substack{\overline{m}(r,r'),\\ \overline{n}(r,r')}}&\left[\prod_r\sum_{k(r)}\int_{-\pi}^{\pi}d\theta(r)\right]\left[\prod_r\sum_{l(r)}\int_{-\pi}^{\pi}d\phi(r)\right]\left[\prod_R\delta_{m(R),\overline{m}(r_1,r_2)-\overline{m}(r_3,r_4)}\delta_{n(R),\overline{n}(r_1,r_2)-\overline{n}(r_3,r_4)}\right]\\
&\times\exp\Bigg[-\frac{J}{2}\sum_{\langle r,r'\rangle}^{=}[\theta(r)+2\pi k(r)-\theta(r')-2\pi k(r')]^2-\frac{J}{2}\sum_{\langle r,r'\rangle}^{=}[\phi(r)+2\pi l(r)-\phi(r')-2\pi l(r')]^2\\
&\quad+K\sum_{\langle r,r'\rangle}^{=}[\theta(r)+2\pi k(r)-\theta(r')-2\pi k(r')][\phi(r)+2\pi l(r)-\phi(r')-2\pi l(r')]\\
&\quad-\frac{J}{2}\sum_{\langle r,r'\rangle}^{\downarrow}[\theta(r)+2\pi k(r)-\theta(r')-2\pi k(r')-2\pi\overline{m}(r,r')]^2\\
&\quad-\frac{J}{2}\sum_{\langle r,r'\rangle}^{\downarrow}[\phi(r)+2\pi l(r)-\phi(r')-2\pi l(r')-2\pi\overline{n}(r,r')]^2\\
&\quad+K\sum_{\langle r,r'\rangle}^{\downarrow}[\theta(r)+2\pi k(r)-\theta(r')-2\pi k(r')-2\pi\overline{m}(r,r')]\\
&\qquad\times[\phi(r)+2\pi l(r)-\phi(r')-2\pi l(r')-2\pi\overline{n}(r,r')]+ip\sum_r q(r)[\theta(r)-\phi(r)]\Bigg]\,.\end{aligned}\tag{A1}$$

Here, $\overset{|}{\sum}$ denotes the sum over vertical bonds and $\overline{\sum}$ denotes the sum over horizontal bonds. Note that the integrand in (A1) does not depend on $\theta(r)$, $\phi(r)$, $k(r)$, and $l(r)$ separately, but only on the combinations $\theta(r)+2\pi k(r)$ and $\phi(r)+2\pi l(r)$. This property can be used to extend the range of integration by making the replacements

$$\begin{aligned} &\theta(r)+2\pi k(r)\to\bar\theta(r), \quad \sum_{k(r)}\int_{-\pi}^{\pi} d\theta(r)\to\int_{-\infty}^{\infty} d\bar\theta(r) ,\\ &\phi(r)+2\pi l(r)\to\bar\phi(r), \quad \sum_{l(r)}\int_{-\pi}^{\pi} d\phi(r)\to\int_{-\infty}^{\infty} d\bar\phi(r) . \end{aligned} \tag{A2}$$

At this point the integrals in (A1) reduce to the Gaussian form,

$$\begin{aligned} \left[\prod_r\int_{-\infty}^{\infty} d\bar\theta(r)d\bar\phi(r)\right]\exp\Bigg[&-\frac{J}{2}\overline{\sum_{\langle r,r'\rangle}}[\bar\theta(r)-\bar\theta(r')]^2-\frac{J}{2}\overline{\sum_{\langle r,r'\rangle}}[\bar\phi(r)-\bar\phi(r')]^2\\ &+K\overline{\sum_{\langle r,r'\rangle}}[\bar\theta(r)-\bar\theta(r')][\bar\phi(r)-\bar\phi(r')]+2\pi\overset{|}{\sum_{\langle r,r'\rangle}}[\bar\theta(r)-\bar\theta(r')][J\overline{m}(r,r')-K\overline{n}(r,r')]\\ &+2\pi\overset{|}{\sum_{\langle r,r'\rangle}}[\bar\phi(r)-\bar\phi(r')][J\overline{n}(r,r')-K\overline{m}(r,r')]+ip\sum_r q(r)[\theta(r)-\phi(r)]\Bigg] . \end{aligned} \tag{A3}$$

Symbolically, this can be written as

$$\int D\overline{\Theta}\,D\overline{\Phi}\exp\left[-\frac{J}{2}\overline{\Theta}\mathbf{G}^{-1}\overline{\Theta}+K\overline{\Theta}\mathbf{G}^{-1}\overline{\Phi}-\frac{J}{2}\overline{\Phi}\mathbf{G}^{-1}\Phi+A\overline{\Theta}+B\overline{\Phi}+C\right] , \tag{A4}$$

where the variables $\bar\theta(r)$ and $\bar\phi(r)$ are combined into the vectors $\overline{\Theta}(r)=(\bar\theta(r_1),\bar\theta(r_2),\ldots)$ and $\overline{\Phi}(r)=(\bar\phi(r_1),\bar\phi(r_2),\ldots)$. $\mathbf{G}^{-1}$ is the matrix of the quadratic form $\sum_{\langle r,r'\rangle}[\bar\theta(r)-\bar\theta(r')]^2$. From (A4) it is easy to see that shift of the $\overline{\Theta}$'s, $\overline{\Theta}(r)\to\overline{\Theta}(r)-(K/J)\overline{\Phi}(r)$, will decouple integration over $\overline{\Theta}$ from integration over $\overline{\Phi}$. Indeed, performing this shift in (A3), we arrive, after some algebraic rearrangement of the expression in the exponent, at decoupled Gaussian integrals,

$$\begin{aligned} \left[\prod_r\int_{-\infty}^{\infty} d\bar\theta(r)d\bar\phi(r)\right]\exp\Bigg\{ &-\frac{J}{2}\sum_{\langle r,r'\rangle}[\bar\theta(r)-\bar\theta(r')]^2-\frac{J}{2}\left[1-\frac{K^2}{J^2}\right]\sum_{\langle r,r'\rangle}[\bar\phi(r)-\bar\phi(r')]^2\\ &+\sum_r\bar\theta(r)\left[2\pi J\left[\overline{m}(r,r+\hat y)-\overline{m}(r-\hat y,r)-\frac{K}{J}\overline{n}(r,r+\hat y)+\frac{K}{J}\overline{n}(r-\hat y,r)\right]+ipq(r)\right]\\ &+\sum_r\bar\phi(r)\left[2\pi J\left[1-\frac{K^2}{J^2}\right]\left[\overline{n}(r,r+\hat y)+\frac{K}{J}\overline{n}(r-\hat y,r)\right]-ip\left[1-\frac{K}{J}\right]q(r)\right]\Bigg\} . \end{aligned} \tag{A5}$$

These integrals can be immediately evaluated by the standard method of Gaussian integration, giving, after some rearrangement of the terms in the exponent,

$$\begin{aligned} \mathbb{Z}(m,n,q)=\sum_{\substack{\overline{m}(r,r'),\\ \overline{n}(r,r')}}&\left[\prod_R\delta_{m(R),\overline{m}(r_1,r_2)-\overline{m}(r_3,r_4)}\delta_{n(R),\overline{n}(r_1,r_2)-\overline{n}(r_3,r_4)}\right]\\ \times\exp\Bigg[&\frac{p^2}{\pi J+\pi K}\sum_{r,r'}q(r)\mathbf{G}(r-r')q(r')+ip\sum_{r,r'}[\overline{m}(r,r+\hat y)-\overline{m}(r-\hat y,r)]\mathbf{G}(r-r')q(r')\\ &-ip\sum_{r,r'}[\overline{n}(r,r+\hat y)-\overline{n}(r-\hat y,r)]\mathbf{G}(r-r')q(r')\\ &-\pi J\sum_{r,r'}[\overline{m}(r,r+\hat y)-\overline{m}(r-\hat y,r)]\mathbf{G}(r-r')[\overline{m}(r',r'+\hat y)-\overline{m}(r'-\hat y,r')]\\ &-\pi J\sum_{r,r'}[\overline{n}(r,r+\hat y)-\overline{n}(r-\hat y,r)]\mathbf{G}(r-r')[\overline{n}(r',r'+\hat y)-\overline{n}(r'-\hat y,r')]\\ &+2\pi K\sum_{r,r'}[\overline{m}(r,r+\hat y)-\overline{m}(r-\hat y,r)]\mathbf{G}(r-r')[\overline{n}(r',r'+\hat y)-\overline{n}(r'-\hat y,r')]\\ &-2\pi^2 J\overset{|}{\sum}\left[\overline{m}^{\,2}(r,r')-2\frac{K}{J}\overline{m}(r,r')\overline{n}(r,r')+\overline{n}^{\,2}(r,r')\right]\Bigg] . \end{aligned} \tag{A6}$$

Since there is equal number of summation variables and Kronecker δ functions, we can solve for the $\overline{m}(r,r')$ and $\overline{n}(r,r')$ in terms of $m(R)$ and $n(R)$ in the form

$$\overline{m}(r,r+\hat{y})=-\sum_{j=0}^{\infty} m(r+\tfrac{1}{2}\hat{y}-(j-\tfrac{1}{2})\hat{x}),\quad \overline{n}(r,r+\hat{y})=-\sum_{j=0}^{\infty} n(r+\tfrac{1}{2}\hat{y}-(j-\tfrac{1}{2}\hat{x}) . \tag{A7}$$

The last step is to substitute (A7) into the exponent of (A6), and to simplify the resulting expression. We will carry out the calculation for the term

$$-ip\sum_{r,r'}[\overline{m}(r,r+\hat{y})-\overline{m}(r-\hat{y},r)]\mathbf{G}(r-r')q(r') . \tag{A8}$$

We wish to calculate

$$\sum_{r}[\overline{m}(r,r+\hat{y})-\overline{m}(r-\hat{y},r)]\mathbf{G}(r-r') . \tag{A9}$$

Substituting here (A7) and rearranging the summations, we obtain

$$\sum_{r}\sum_{j=0}^{\infty} m(r+\tfrac{1}{2}\hat{y}-(j-\tfrac{1}{2})\hat{x})[\mathbf{G}(r-r')-\mathbf{G}(r-\hat{y}-r')] , \tag{A10}$$

replacing the summation over r by the summation over $R=r+\frac{1}{2}\hat{x}-\frac{1}{2}\hat{y}$,

$$\sum_{R} m(R)\sum_{j=0}^{\infty}[\mathbf{G}(R+\tfrac{1}{2}\hat{y}+(j-\tfrac{1}{2})\hat{x}-r')-\mathbf{G}(R-\tfrac{1}{2}\hat{y}+(j-\tfrac{1}{2})\hat{x}-r')] . \tag{A11}$$

In a continuum notation, the sum over j reads

$$\int \partial_y \mathbf{G}(R-r')dR , \tag{A12}$$

where the path of integration runs from R to infinity in the positive x direction. Since $\mathbf{G}(R)$ can be well approximated by $\ln(R/a)$, this integral can be evaluated using the Cauchy-Riemann relation $\partial_y \ln(R-r')=-\partial_x \Theta(R-r')$, where $\Theta(R)=\tan^{-1}(y/x)$. Finally, we obtain, for (A8),

$$ip\sum_{R,r'} m(R)\Theta(R-r')q(r') . \tag{A13}$$

Other terms in the exponent of (A6) can be evaluated similarly, and the partition function (4.15) becomes

$$\begin{aligned}\mathbb{Z}=\sum_{m,n,q}\exp\Bigg[&-\frac{1}{2L}\sum_{r}q^2(r)+\pi J\sum_{R,R'} m(R)\mathbf{G}(R-R')m(R')+\pi J\sum_{R,R'} n(R)\mathbf{G}(R-R')n(R')\\ &-2\pi K\sum_{R,R'} m(R)\mathbf{G}(R-R')n(R')+ip\sum_{R,r'}[m(R)-n(R)]\Theta(R-r')q(r')\\ &+\frac{p^2}{2\pi J+2\pi K}\sum_{r,r'} q(r)\mathbf{G}(r-r')q(r')\Bigg] .\end{aligned} \tag{A14}$$

Here, $\mathbf{G}(R)$ is the lattice Green's function, defined by[36]

$$\mathbf{G}(R)=-2\pi\int_{-\pi}^{\pi}\frac{dq_x}{2\pi}\int_{-\pi}^{\pi}\frac{dq_y}{2\pi}\frac{e^{iqR}}{4-2\cos q_x-2\cos q_y} . \tag{A15}$$

It is easy to see that the Green's function $\mathbf{G}(R-R')$ diverges for $R=R'$. To isolate divergences, let us split $\mathbf{G}(R)$ into two parts,

$$\mathbf{G}(R)=\mathbf{G}(0)+\mathbf{G}'(R) , \tag{A16}$$

where $\mathbf{G}'(R)$ is the finite quantity,

$$\mathbf{G}'(R)=2\pi\int_{-\pi}^{\pi}\frac{dq_x}{2\pi}\int_{-\pi}^{\pi}\frac{dq_y}{2\pi}\frac{1-e^{iqR}}{4-2\cos q_x-2\cos q_y} , \tag{A17}$$

and $\mathbf{G}(0)$ is the infinite constant,

$$\mathbf{G}(0)=-2\pi\int_{-\pi}^{\pi}\frac{dq_x}{2\pi}\int_{-\pi}^{\pi}\frac{dq_y}{2\pi}\frac{1}{4-2\cos q_x-2\cos q_y} . \tag{A18}$$

Substitution of (A16) into (A14) yields

$$\mathbb{Z} = \sum_{m,n,q} \exp\left\{ \mathbf{G}(0) \left[\pi J \left[\sum_R m(R) \right]^2 - 2\pi K \left[\sum_R m(R) \right] \left[\sum_R n(R) \right] + \pi J \left[\sum_R n(R) \right]^2 + \frac{p^2}{2\pi J + 2\pi K} \left[\sum_r q(r) \right]^2 \right] \right\}$$
$$\times \exp\left[-\frac{1}{2L} \sum_r q^2(r) + 2\pi J \sum_{(R,R')} m(R)\mathbf{G}'(R-R')m(R') + 2\pi J \sum_{(R,R')} n(R)\mathbf{G}'(R-R')n(R') - 2\pi K \sum_{\substack{R,R' \\ R\neq R'}} m(R)\mathbf{G}'(R-R')n(R') + ip \sum_{R,r'} [m(R)-n(R)]\Theta(R-r')q(r') + \frac{2p^2}{2\pi J + 2\pi K} \sum_{(r,r')} q(r)\mathbf{G}'(r-r')q(r') \right], \tag{A19}$$

where $\sum_{(R,R')}$ denotes a summation over all pairs of lattice sites with each pair counted only once. Since $\mathbf{G}(0) \sim -\infty$, only terms with a vanishing expression in the large square brackets contribute to the partition function. If $0 \le K < J$, this expression vanishes if three neutrality conditions are satisfied, namely

$$\sum_R m(R) = 0, \quad \sum_R n(R) = 0, \quad \sum_r q(r) = 0 . \tag{A20}$$

For $K = J$ only two conditions remain, namely

$$\sum_R m(R) - n(R) = 0, \quad \sum_r q(r) = 0 . \tag{A21}$$

Therefore we can write

$$\mathbb{Z} = \sum_{m,n,q}{}' \exp\left[-\frac{1}{2L} \sum_r q^2(r) + 2\pi J \sum_{(R,R')} m(R)\mathbf{G}'(R-R')m(R') + 2\pi J \sum_{(R,R')} n(R)\mathbf{G}'(R-R')n(R') - 2\pi K \sum_{\substack{R,R' \\ R\neq R'}} m(R)\mathbf{G}'(R-R')n(R') + ip \sum_{R,r'} [m(R)-n(R)]\Theta(R-r')q(r') + \frac{2p^2}{2\pi J + 2\pi K} \sum_{(r,r')} q(r)\mathbf{G}'(r-r')q(r') \right], \tag{A22}$$

where the prime on the summation means summation over configurations satisfying condition (A20) if $0 \le K < J$, or condition (A21) if $K = J$. Approximating $\mathbf{G}'(R-R')$ by

$$\mathbf{G}'(R-R') \approx \ln\left[\frac{|R-R'|}{a} \right] + \frac{\pi}{2}, \tag{A23}$$

where a is the lattice spacing, we obtain expression (4.17) for the partition sum.

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