Normal part of (H,T) phase diagram of a superconductor: Possible applications to proximity systems

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It is shown that for weak superconductivity in a uniform field (the Eilenberger Green's function $|f| \ll 1$) microscopic Eilenberger equations reduce to the linear equation $\Pi^2 F = k^2 F$, where $\mathbf{\Pi}$ is the gauge-invariant gradient and F is the average of the function f over the Fermi surface. This equation holds in uniform fields for any impurity concentration and applies to H_{c2} and H_{c3} problems, to fluctuations of superconductivity at $T > T_c(H)$, as well as to various situations in proximity systems such as the superconductivity induced deep in the normal metal, the critical temperature, and the upper critical field of these systems. The parameter k^2 is to be determined self-consistently for each problem. The field and temperature dependence of $k^2(H,T)$ is obtained for a "moderately dirty" case. At a certain curve which starts at the zero-field T_c and is situated in the normal part of (H,T) phase diagram, k^2 is zero.

I. INTRODUCTION

The quasiclassical Eilenberger's equations of superconductivity, suitable for a great many inhomogeneous problems, read:[1]

$$\tau \mathbf{v}\cdot\mathbf{\Pi} f = g(F + 2\tau\Delta) - (G + 2\omega\tau) f \ , \quad (1)$$

$$\frac{\Delta}{2\pi T}\ln\frac{T_c}{T} = \sum_{\omega > 0}\left[\frac{\Delta}{\omega} - F\right] . \quad (2)$$

Here $f(\mathbf{r},\omega,\mathbf{v})$ and $g(\mathbf{r},\omega,\mathbf{v})$ are the Gor'kov Green's functions integrated over the energy; $\omega = \pi T(2n+1)$ is the Matsubara frequency with $n = 0,1,2,\ldots$. We adopt hereafter the system of units where $\hbar, c$, and k_B are unity; for the reader's convenience some resulting expressions are given in common units.

Further, $\mathbf{v}$ is the Fermi velocity, $\tau = l/v$ is the scattering time for nonmagnetic impurities; and l is the mean free path; only the s scattering is taken into account. The gauge-invariant gradient $\mathbf{\Pi} = \nabla - 2ei\mathbf{A}$ and $\mathbf{A}$ is the vector potential; one can also write $\mathbf{\Pi} = \nabla + 2\pi i \mathbf{A}/\phi_0$ with $\phi_0 = \pi/|e|$ the flux quantum.

The function $g = [1 - f(\mathbf{v}) f^*(-\mathbf{v})]^{1/2}$; $F(\mathbf{r},\omega)$ and $G(\mathbf{r},\omega)$ are the averages $\langle f(\mathbf{r},\omega,\mathbf{v})\rangle$ and $\langle g(\mathbf{r},\omega,\mathbf{v})\rangle$ over all $\mathbf{v}$ directions on the Fermi sphere. The pair potential $\Delta(\mathbf{r})$ depends upon position $\mathbf{r}$ exclusively as with a BCS superconductor with weak coupling. The equation for the current density along with Maxwell's equations, complete the Eilenberger system; we shall not use them in this paper.

In the normal phase $f = 0$ and $g = 1$. If the superconductivity is weak [e.g., for fluctuations of the superconducting (S) phase at $T > T_c$, or when the S phase starts to nucleate, or deep in the normal part of a proximity system], the function f is small ($|f| \ll 1$). Still, the function $g = 1$ in the approximation linear in f. Equation (1) then can be linearized. Introducing for convenience

$$\mathbf{l} = \mathbf{v}\tau, \quad \tilde{F} = F + 2\tau\Delta, \quad \beta = 1 + 2\omega\tau \ , \quad (3)$$

we have instead of Eq. (1):

$$\mathbf{l}\cdot\mathbf{\Pi} f = \tilde{F} - \beta f \ . \quad (4)$$

It has been shown[2] that in *the absence of a magnetic field* deep in the normal part of a proximity system (N), the superconductivity can be described by the equation[3]

$$\Pi^2 F = k^2 F \ , \quad (5)$$

where "the coherence length of the normal metal" $k^{-1}(T,l)$ is obtained from the self-consistency equation (2):

$$\frac{1}{2\pi T}\ln\frac{T_c}{T} = \sum_{\omega}\left[\frac{1}{\omega} - \frac{2\tau \tanh^{-1}(lk/\beta)}{kl - \tanh^{-1}(kl/\beta)}\right] . \quad (6)$$

All sums over ω are extended hereafter to $\omega > 0$. Although Eq. (5) is of the second order, it is more convenient than the first order Eq. (4) because F is $\mathbf{v}$ independent.

In the dirty limit Eq. (5), with $k^2(T)$ properly chosen, holds deep in the N metal in the presence of a magnetic field too (see, e.g., Ref. 2). This has been shown using the dirty-limit version of Eilenberger's theory, which is due to Usadel.[4] However, the conditions under which the Usadel equations are valid, are quite restrictive: Only the terms of order $(lk)^{1/2}$ are retained; the terms of the order lk are already neglected.[5] In fact, these conditions are met only in extremely dirty materials such as amorphous metals.

We show in this paper that Eq. (5) can be used to describe weak superconductivity (understood as indicated above) in a uniform magnetic field for any mean free path l. We also obtain $k^2(H,T,l)$ for moderately dirty materials.

II. WEAK SUPERCONDUCTIVITY IN UNIFORM FIELD

Equations (4) and (2), in fact, form a closed system for this case. We rewrite Eq. (4) in the form

$$f=(\beta+\mathbf{l}\cdot\mathbf{\Pi})^{-1}\tilde{F}=\int_0^\infty d\rho\, e^{-\rho(\beta+\mathbf{l}\cdot\mathbf{\Pi})}\tilde{F} \tag{7}$$

and take the average $\langle\ \rangle$ to obtain an equation for F:

$$F=\int_0^\infty d\rho\, e^{-\rho\beta}\langle e^{-\rho\mathbf{l}\cdot\mathbf{\Pi}}\tilde{F}\rangle \ . \tag{8}$$

We now consider solutions of Eq. (5) along with $\Pi^2\Delta=k^2\Delta$ and $\Pi^2\tilde{F}=k^2\tilde{F}$ as the ansatz for Eq. (8). The whole problem would have been resolved if, with the help of this ansatz, we were able to show that $F(\mathbf{r},\omega)$ is proportional to $\Delta(\mathbf{r})$: $F(\mathbf{r},\omega)=\Delta(\mathbf{r})\phi(\omega)$ with some ϕ dependent on ω exclusively. Then the position dependent $\Delta(\mathbf{r})$ would have canceled out of the self-consistency equation (2), leaving an equation for the parameter $k(H,T)$. Our formal procedure, in fact, is the same as that of the $H_{c2}(T)$ problem.[2]

To proceed, let us introduce Cartesian coordinates so that $\mathbf{H}=H\hat{z}$. The averaging in Eq. (8) turns out to be simpler when working with operators $\Pi^{\pm}=\Pi_x\pm i\Pi_y$, rather than with $\Pi_{x,y}$. One can easily verify that the commutator

$$[\Pi^+,\Pi^-]=4\pi H/\phi_0\equiv 2q^2 \ , \tag{9}$$

i.e., it is a number. In terms of $\Pi^{\pm}$, Eq. (5) reads

$$\Pi^+\Pi^-\tilde{F}=(k^2+q^2)\tilde{F} \ . \tag{10}$$

Now, $\mathbf{l}\cdot\mathbf{\Pi}=(l^-\Pi^++l^+\Pi^-)/2$ with $l^{\pm}=l_x\pm il_y$. Then the exponential operator in Eq. (8) can be transformed:

$$\exp(-\rho\mathbf{l}\cdot\mathbf{\Pi})=\exp(-\tfrac{1}{2}\rho l^-\Pi^+)\exp(-\tfrac{1}{2}\rho l^+\Pi^-)\times\exp(-\tfrac{1}{4}\rho^2 l^+l^-q^2) \ ,$$

where the known property of exponential operators and the commutator (9) have been used.

We replace now all the exponentials with their power series and use the spherical angles θ,ϕ at the Fermi sphere with the polar axis z. Then $l^{\pm}=l\sin\theta\exp(\pm i\phi)$. Taking the average

$$\langle\ \rangle=\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\phi/4\pi\cdots$$

in Eq. (8), we first integrate over ϕ to exclude all products $(l^+)^i(l^-)^j$ with $i\neq j$. After simple algebra we obtain

$$\langle\exp(-\rho\mathbf{l}\cdot\mathbf{\Pi})\tilde{F}\rangle=\sum_{m,j=0}^{\infty}\frac{(-q^2)^j}{(m!)^2j!}(\tfrac{1}{2}\rho l)^{2m+2j}\times\langle\sin^{2m+2j}\theta\rangle(\Pi^+)^m(\Pi^-)^m\tilde{F} \ . \tag{11}$$

Here

$$\langle\sin^{2m+2j}\theta\rangle=(2m+2j)!!/(2m+2j+1)!! \ .$$

Now substitute Eq. (11) in Eq. (8) and integrate over ρ to obtain

$$F=\frac{1}{\beta}\sum_{m,j=0}^{\infty}\frac{(-q^2)^j}{j!(2m+2j+1)}\left[\frac{(m+j)!}{m!}\right]^2\left[\frac{l}{\beta}\right]^{2m+2j}\times(\Pi^+)^m(\Pi^-)^m\tilde{F} \ . \tag{12}$$

By repetitive use of the commutator (9) and Eq. (10), one can show that

$$(\Pi^+)^m(\Pi^-)^m\tilde{F}=\tilde{F}\prod_{i=1}^{m}[k^2+(2i-1)q^2] \tag{13}$$

for $m>0$. We have now instead of Eq. (12):

$$F=\frac{\tilde{F}}{\beta}S(k,H,\omega,l) \ ,$$
$$S=\sum_{m,j=0}^{\infty}\frac{(-q^2)^j}{j!(2m+2j+1)}\left[\frac{(m+j)!}{m!}\right]^2\left[\frac{l}{\beta}\right]^{2m+2j}\times\prod_{i=1}^{m}[k^2+(2i-1)q^2] \ . \tag{14}$$

The sum S is $\mathbf{r}$ independent, if the field is uniform. Recalling the definition of $\tilde{F}$, one obtains the proportionality of F and Δ we are looking for:

$$F(\mathbf{r},\omega)=\Delta(\mathbf{r})\frac{2\tau S}{\beta-S} \ . \tag{15}$$

This is to be substituted in the self-consistency equation (2). The latter yields, after $\Delta(\mathbf{r})$ is canceled out, an equation for $k(H,T)$. Thus, our ansatz (5) indeed solves Eqs. (8) and (2), provided the k is found from

$$\frac{1}{2\pi T}\ln\frac{T_c}{T}=\sum_{\omega}\left[\frac{1}{\omega}-\frac{2\tau}{\beta/S-1}\right] \ . \tag{16}$$

This equation is rather cumbersome to deal with, in general.

III. UPPER CRITICAL FIELD

It is instructive for our purposes to consider this known problem in the context of the method proposed. Also, we obtain here H_{c2} in moderately dirty materials which might be useful.

The upper critical field is known to correspond to the lowest eigenvalue of Eq. (5), for which a finite solution exists on the whole xy plane.[6,7] It is easy to see that at the lowest "Landau-level" $k^2=-2\pi H/\phi_0=-q^2$. Denoting $q_c^2=2\pi H_{c2}/\phi_0$, we obtain[8] at once from Eq. (14):

$$S(H_{c2})=\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}j!\left[\frac{lq_c}{\beta}\right]^{2j} \ . \tag{17}$$

This series has been discussed in Ref. 7. Here $lq_c/\beta<lq_c\sim l/\xi$ (with ξ being the coherence length) so that the series converges rapidly for dirty materials. Keeping only $j=0,1,2$ terms, we have

$$S=1-\frac{l^2q_c^2}{3\beta^2}+\frac{2l^4q_c^4}{5\beta^4} \ , \tag{18}$$

if

$$(lq_c/\beta)^4<(lq_c)^4=(l/\xi)^4\ll 1 \ . \tag{19}$$

The dirty limit corresponds to the first two terms in Eq. (18); expression (18), therefore, describes a "moderately dirty" metal. The restriction (19) defines, in fact, the term "moderately dirty." With the rare exception of extremely clean metals, most superconductors in practice may be classified as moderately dirty.

To obtain an equation for $H_{c2}(T)=\phi_0 q_c^2/2\pi$ we invert the series (18) and substitute the result in Eq. (16):

$$\frac{1}{2\pi T}\ln\frac{T_c}{T}=\sum_{\omega}\left[\frac{1}{\omega}-\frac{1}{\omega+Dq_c^2/2\beta-13Dl^2q_c^4/30\beta^3}\right]. \tag{20}$$

Here $D=l^2/3\tau$ is the diffusion coefficient.

IV. ZERO FIELD

In the absence of the field we set $q=0$ in Eq. (14). Then only the $j=0$ term remains, and we have

$$S=\sum_{m=0}^{\infty}\frac{(kl/\beta)^{2m}}{2m+1}=\frac{\beta}{k_0 l}\tanh^{-1}\left[\frac{k_0 l}{\beta}\right],$$
$$k_0\equiv|k(0,T)|\ , \tag{21}$$

along with the convergence condition $(k_0 l/\beta)_{\max}<1$ or

$$k_0<\frac{1}{l}+\frac{1}{\xi_T},\ \ \xi_T=\frac{v}{2\pi T}\ . \tag{22}$$

($\xi_T=\hbar v/2\pi k_B T$ in common units.) For $T>T_c$, we substitute Eq. (21) in Eq. (16) to obtain Eq. (6) for the coherence length k_0^{-1} of the normal metal.[2]

At first sight, there is no point in considering Eqs. (16) and (21) for $T<T_c$: Our assumption about $|f|\ll 1$ and $g=1$ does not hold in the S phase, so Eilenberger's Eq. (1) could not be linearized and Eq. (5) could not be valid. This, however, is not always the case in proximity systems.

Consider an $S-N$ bilayer or multilayer. The critical temperature $T_{c,\text{pr}}$ of *the proximity system* differs from both T_{cS} and T_{cN}. Near $T_{c,\text{pr}}$, $|F|\ll 1$ and $G=1$ in both S and N parts of the system. The derivatives $\Pi\sim k_{S,N}$ are not small, because $T_{c,\text{pr}}$, in general, is not close to either T_{cS} or T_{cN}. Then our linearization procedure is applicable. (Near T_{cS}, e.g., one should have kept not only the terms linear in F_S, because $\Pi^2 F_S\sim F_S/\xi_S^2\ll F_S$.) Nucleation of superconductivity is described by Eq. (5): $\nabla^2 F_{S,N}=k_{S,N}^2 F_{S,N}$, where the $k_{S,N}^2$ are found from Eqs. (16) and (21) with T_c replaced by $T_{cS,N}$, respectively.

The function $k_N(0,T)$ has been discussed in detail in Ref. 2 for any mean free path. We only note here that for any real k [within restriction (22)] all terms in the sum (6) are negative.[9] Therefore, Eq. (6) cannot have real solutions k_S for $T<T_{cS}$. It does have imaginary solutions. Replacing $k_S(0,T)$ with ik_{S0}, we obtain for k_{S0}

$$\frac{1}{2\pi T}\ln\frac{T_{cS}}{T}=\sum_{\omega}\left[\frac{1}{\omega}-\frac{2\tau\tan^{-1}(lk_{S0}/\beta)}{lk_{S0}-\tan^{-1}(lk_{S0}/\beta)}\right] \tag{23}$$

and for F_S: $\nabla^2 F_S=-k_{S0}^2 F_S$.

In the dirty limit $lk_{S0}/\beta=x\ll 1$, $\tan^{-1}x=x-x^3/3+\cdots$, and Eq. (23) reduces readily to

$$\frac{1}{2\pi T}\ln\frac{T_c}{T}=\sum_{\omega}\left[\frac{1}{\omega}-\frac{1}{\omega+Dk_{S0}^2/2}\right]. \tag{24}$$

This coincides with the equation for $q_c^2=2\pi H_{c2}/\phi_0$ in the dirty limit [see, e.g., Refs. 7 or 2, or take the limit $l\to 0$ in our Eq. (20)]. Thus, in this case k_{S0}^{-1} coincides with the coherence length of a dirty superconductor, defined as $[\phi_0/2\pi H_{c2}(T)]^{1/2}$. We shall see that this is a particular case of the general result: In the *dirty limit* k^2 is H independent.

However, in general $k_{S0}^2\equiv -k_S^2(0,T)$ defined in Eq. (23) differs substantially from $\xi_S^{-2}=2\pi H_{c2}/\phi_0$ unless we are in the dirty limit or in the Ginzburg-Landau (GL) domain. As $T\to 0$ in the clean limit, $k_{S0}=2.72\Delta_0/\hbar v$ in common units (see Appendix A), while $(2\pi H_{c2}/\phi_0)^{1/2}=3.70\Delta_0/\hbar v$ (see, e.g., Ref. 7). Therefore, in this limit $k_{S0}\xi_S\,|_{T=0}=0.73$. For the "moderately dirty" situation, the result (which can be obtained with the help of the next section) reads $k_{S0}\xi_S\,|_{T=0}=1-0.9l/\xi_S(0)$, with $\xi_S(0)=(\hbar D/\Delta_0)^{1/2}$ being the coherence length at $T=0$ in the dirty limit. Thus, in general, the coherence length $\xi_S(T)$ related to H_{c2} cannot be used in finding the critical temperature of a proximity system in zero field. For any l_S, except the dirty limit, one should use $k_{S0}(T)$ of Eq. (23).

V. PARAMETER $k^2(H,T)$ FOR MODERATELY DIRTY MATERIAL

We consider now the sum (14) and keep in it only the terms with $m+j=0,1,2$:

$$S=1+\frac{l^2k^2}{3\beta^2}+\frac{l^4}{5\beta^4}(k^4+q^4)\ . \tag{25}$$

The truncation is justified if the last terms retained are small:

$$(lk/\beta)^4\ll 1 \text{ and } (lq/\beta)^4\ll 1\ . \tag{26}$$

Replacing here β with $\beta_{\min}=1+l/\xi_T$, we obtain the restrictions

$$\frac{k^4}{5}\left[\frac{1}{l}+\frac{1}{\xi_T}\right]^{-4}\ll 1,\ \ \frac{q^4}{5}\left[\frac{1}{l}+\frac{1}{\xi_T}\right]^{-4}\ll 1\ . \tag{27}$$

If the last term in Eq. (25) is, e.g., on the order 10^{-2}, the accuracy of this equation is, in fact, higher: the neglected terms are of the order $(lk/\beta)^6\sim 10^{-3}$. It is also worth noting, that the definition of the term "moderately dirty" includes a restriction imposed upon the magnetic field.[10]

With the same accuracy,

$$S^{-1}=1-\frac{l^2k^2}{3\beta^2}-\frac{l^4}{5\beta^4}(\tfrac{4}{9}k^4+q^4)\ . \tag{28}$$

Substitute this in Eq. (16) to obtain for $k^2(H,T)$

$$\frac{1}{2\pi T}\ln\frac{T_c}{T}=\sum_{\omega}\left\{\frac{1}{\omega}-\left[\omega-\frac{l^2k^2}{6\tau\beta}-\frac{l^4}{10\tau\beta^3}\left[\frac{4}{9}k^4+q^4\right]\right]^{-1}\right\}. \tag{29}$$

This can be solved numerically for $k^2(H,T)$.

To make progress analytically, we observe that the solution k^2 of Eq. (29) is a function of q^4 (k^2 could not depend upon odd powers of H). Further, in zero field $k^2(0,T)$ satisfies Eq. (29) where one sets $q=0$. Having in mind conditions (26), we look for a solution $k^2(q,T)$ of the form

$$k^2(H,T)=k^2(0,T)-\gamma(l,T)l^2q^4 \tag{30}$$

with a dimensionless γ. Substitute this in Eq. (29) and compare with the equation for $k^2(0,T)$ to obtain γ within the requirements (26):

$$\gamma(l,T)=\frac{3\sum\beta^{-3}(\omega-Dk_0^2/2\beta)^{-2}}{5\sum\beta^{-1}(\omega-Dk_0^2/2\beta)^{-2}}\ . \tag{31}$$

An interesting feature of the results (30) and (31) is that $k^2(H,T)$ vanishes at the field

$$H_0(T)=\frac{\phi_0 k(0,T)}{2\pi l}\gamma^{-1/2}\ . \tag{32}$$

The whole curve $H_0(T)$ is situated in the region $T>T_c$, where $k(0,T)$ is real. This curve is sketched in Fig. 1.

The very existence of a curve at the (H,T) phase diagram along which $k^2=0$ should not be a surprise. Indeed, k^2 is positive on the T axis to the right of T_c, vanishes at T_c, and is negative (and equal to $-2\pi H/\phi_0$) along the $H_{c2}(T)$ curve. Being a continuous function of H and T, $k^2(H,T)$ must turn zero at a curve which starts at T_c.

If $T<T_c$, parameter k^2 can also be given in terms of H_{c2} (Appendix B):

$$k^2(H,T)=k^2(0,T)\left[1-\frac{H^2}{H_{c2}^2(T)}\right]-\frac{2\pi H^2}{\phi_0 H_{c2}(T)}\ . \tag{33}$$

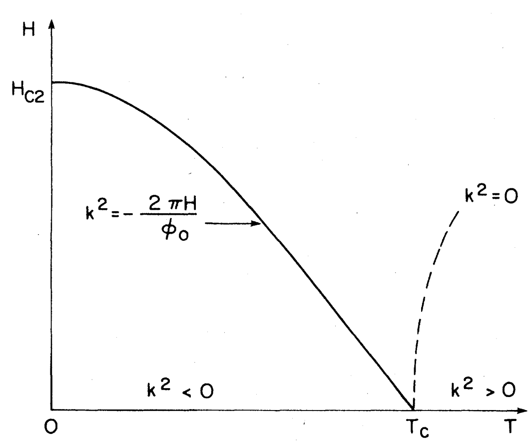

FIG. 1. (H,T) phase diagram of a superconductor. Parameter $k^2(H,T)$ vanishes at the dashed line.

In the domain $T>T_c$, the field $H_0(T)$ is useful in representing the H dependence of k^2. Take γ from Eq. (32) and substitute it in Eq. (30) to obtain

$$k^2(H,T)=k^2(0,T)\left[1-\frac{H^2}{H_0^2(T)}\right]. \tag{34}$$

VI. NORMAL METAL WITH $T_c=0$

The case $T_c=0$ is of interest due to the fact that Cu is often used as the N part in proximity systems. The net electron-electron interaction in Cu is still unknown; it is either a weak attractive one with a very low T_c,[11] or even repulsive.[12] Nevertheless the case $T_c=0$ is representative and simple formally, because in the absence of an interaction $\Delta=0$, and, in our notations $\widetilde{F}=F$. Then either of Eqs. (14) or (15) yield $\beta=S$, which gives $k^2(H,T;\omega)$.

The ω dependence of k^2 is a specific feature of the case $T_c=0$. It means that $F(\omega)$'s deep in the normal metal attenuate with characteristic lengths depending upon ω. One can show (see, e.g., Ref. 2) that the deepest penetration corresponds to $\omega_{\min}=\pi T$. We have, therefore,

$$(\beta-S)_{\omega=\pi T}=0\ . \tag{35}$$

In the field absence, we take S from Eq. (21) to obtain the known result for k_0:[13,2]

$$k_0 l=\tanh^{-1}(k_0 l/\beta_0),\quad \beta_0=1+2\pi T\tau\ . \tag{36}$$

For the moderately dirty situation defined by the inequalities (26) and (27), we use S of Eq. (25). Also, we introduce the T-dependent impurity parameter

$$\lambda_T=1/2\pi T\tau\ , \tag{37}$$

or $\hbar/2\pi k_B T\tau$ in common units. [Note, that the common definition $\lambda=1/2\pi T_c\tau$ would have been useless in the case $T_c=0$. With the definition (37), the same Cu sample being "clean" at some T, becomes "dirty" at low enough temperatures.] Equation (35) now reduces to

$$\frac{1}{\lambda_T}=\frac{l^2k^2}{3\beta_0^2}+\frac{l^4}{5\beta_0^4}(k^4+q^4)\ ,$$

which yields

$$k^2(H,T)=\frac{3}{l^2\lambda_T}\beta_0^2-\frac{3}{5}\left[\frac{2\pi Hl}{\phi_0\beta_0}\right]^2. \tag{38}$$

In particular, $k^2=0$ if

$$H_0=\frac{\phi_0}{2\pi l^2}\left[1+\frac{1}{\lambda_T}\right]^2\left[\frac{5}{\lambda_T}\right]^{1/2}. \tag{39}$$

At low temperatures ($\lambda_T\gg1$), $H_0\sim T^{1/2}$, while at "high" T's, for which $\lambda_T\ll1$, $H_0\sim T^{5/2}$.

VII. THIRD CRITICAL FIELD

The problem of H_{c3} has been solved in the dirty limit by Saint-James and De Gennes, who considered the linearized GL equation $\Pi^2\psi=-\xi^{-2}\psi$ for the order parameter ψ, subject to boundary condition $\partial\psi/\partial\mathbf{n}=0$ for the normal derivative at the free surface of a superconductor.[14] It has been shown that $H_{c3}=1.695\phi_0/2\pi\xi^2$. In the general case, the H_{c3} problem is quite difficult (see Ref. 15 for the case $T=0$ and Ref. 16 for $T\lesssim T_c$ in the clean limit).

Given the parameter $k^2(H,T,l)$, the problem can be easily approached for any l and T. Notice that Eq. (5) is exactly analogous to the linearized GL equation. Then, impose the condition $\partial F/\partial\mathbf{n}=0$ at the free boundary and replace ξ^{-2} with $-k^2$ in the result of Saint James and De Gennes:

$$H=-1.695\phi_0 k^2(H,T,l)/2\pi\ . \qquad (40)$$

One obtains $H_{c3}(T,l)$ solving this with respect to H.

In the dirty limit, k^2 is H independent and we have T-independent ratio $H_{c3}/H_{c2}=1.695$. The first correction to the dirty limit is obtained with the help of Eq. (30) for moderately dirty metals:

$$H_{c3}/H_{c2}=1.695[1+1.124l^2/\xi^2(T,l)]\ , \qquad (41)$$

where $\xi^2=2\pi H_{c2}/\phi_0$.

VIII. PROXIMITY SYSTEMS

In Sec. IV we have discussed the problem of the critical temperature $T_{c,\mathrm{pr}}$ of a proximity system. In fact, the equation explored by Werthamer[17,18] in solving the $T_{c,\mathrm{pr}}$ problem for the *dirty limit*, is just the same as Eq. (5). Although Werthamer's function [call it $F_W(x)$] differs from our $F(x,\omega)$, these two are closely related: $F_W(x)=2\pi TN(0)\sum_\omega F(x,\omega)$. It is easy to see that both approaches yield the same $T_{c,\mathrm{pr}}$ of the *dirty* proximity system, if De Gennes' boundary conditions[18] of continuity for $F_W(x)/N(0)$ and $DF'_W(x)$ at the interface are replaced in our case by the continuity of $F(x,\omega)$ and $DN(0)F'(x,\omega)$.[19]

The new result of Secs. II and IV is that the same Eq. (5) can be used to find $T_{c,\mathrm{pr}}$ for any impurity concentration in S and N. Also, we derived equations for the parameters $k_S(T,l)$ and $k_N(T,l)$ that should be used in this problem. In particular, we have shown that K_{S0} in zero field differs substantially from $\xi_S^{-1}=(2\pi H_{c2}/\phi_0)^{1/2}$, as long as the dirty-limit conditions are not satisfied; the GL domain ($T_{c,\mathrm{pr}}\lesssim T_{cS}$) is another exception to this rule.

It is difficult, however, to compare the theory proposed with experimental data on $T_{c,\mathrm{pr}}$. One of the difficulties lies in the incomplete theory: The boundary conditions for F's at the S-N interface are still to be formulated. Recently Zaitsev[20] obtained the boundary conditions which are to be imposed upon the Eilenberger's functions f in terms of the transmission and reflection coefficients of the interface. The latter are not usually known; they are certainly very sensitive to each particular sample preparation procedure.

The relevant equations for the problem of H_{c2} of proximity systems are, in fact, the same Eq. (5) written for S and N. The new result obtained here is the H dependence of both k_S and k_N, which is quite strong except in the GL domain or in the dirty limit. As in the $T_{c,\mathrm{pr}}$ problem, the difficulty associated with the boundary conditions impairs the chance for the direct comparison between the theory and the data. One might try to extract the correct form of the boundary conditions from the measured $T_{c,\mathrm{pr}}$ at zero field, and then apply it in the $H_{c2}(T)$ calculation. Effort is in progress along these lines.

Some caution should be exercised, however, in applying results obtained for a homogeneous case to proximity systems. The point is that in the derivation of $S(k^2,H)$ needed to evaluate $k^2(H,T)$, we assumed that F is z independent (z being the coordinate along the field direction). This is still the case if $\mathbf{H}$ is parallel to the N-S interface. However, for an arbitrary orientation of $\mathbf{H}$ with respect to the interface, one has to take the z dependence of F explicitly into account. As a result, the parameter k^2 might depend not only upon the field value, but, in general, upon the relative orientation of the field and the interface as well. As an example, we have considered the H_{c2} problem for a proximity SN multilayer with $\mathbf{H}$ normal to the layers. It turned out that the $\mathbf{H}$ direction enters the series $S(k^2,H)$ only in terms of the order k^6l^6. In other words, all formulas of Sec. V for moderately dirty metals can be used regardless of the field orientation.

In conclusion, we indicate yet another possible application of the $k^2(H,T,l)$ obtained in this work. This is the fluctuating superconductivity in the S part of a proximity system in the temperature domain $T_{c,\mathrm{pr}}(H)<T<T_{cS}(H)$, when the whole system is, in fact, normal. The problem should not be much different from that already solved in the GL domain,[21] because Eq. (5) is formally identical to the linearized GL equation for the order parameter.

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APPENDIX A

In the clean limit $l\to\infty,\beta\simeq 2\omega\tau$, and $lk/\beta=vk/2\omega$. Equation (23) assumes the form

$$\sum_\omega\left[\frac{1}{(\omega^2+\Delta_0^2)^{1/2}}-\frac{2}{vk}\tan^{-1}\left[\frac{vk}{2\omega}\right]\right]=0\ , \qquad (\mathrm{A1})$$

where $\Delta_0(T)$ is the BCS gap:

$$\frac{1}{2\pi T}\ln\frac{T_c}{T}=\sum_\omega\left[\frac{1}{\omega}-\frac{1}{(\omega^2+\Delta_0^2)^{1/2}}\right]\ .$$

At $T=0$, the sum over ω in Eq. (A1) can be replaced by the integral $(2\pi T)^{-1}\int d\omega$ from zero to the Debye frequency ω_D. Integration yields

$$\ln\frac{2\omega_D}{\Delta_0}=\frac{2\omega_D}{kv}\tan^{-1}\left[\frac{kv}{2\omega_D}\right]+\frac{1}{2}\ln\left[1+\frac{4\omega_D^2}{k^2v^2}\right].$$

Assuming $kv \ll 2\omega_D$, we obtain $k=2.72\Delta_0/v$, so that the assumption checks: $kv/2\omega_D \sim \Delta_0/\omega_D \ll 1$.

APPENDIX B

Formula (32) gives $H_0(T)$ in terms of the known $k(0,T)$ and $\gamma(l,T)$. One can also evaluate $H_0(T)$ directly from the self-consistency equation (29) by setting there $k=0$:

$$\frac{1}{2\pi T}\ln\frac{T}{T_c}=\sum_{\omega}\frac{l^4q_0^4}{\omega\beta^4}\left[\frac{10\omega\tau}{\beta}-\frac{l^4q_0^4}{\beta^4}\right]^{-1},$$
$$q_0^2=2\pi H/\phi_0 . \tag{B1}$$

The second term in the parenthesis can be neglected in the domain (26) where our theory is valid. For the same reason, all terms under the sum (B1) are small. Therefore, $\ln(T/T_c)$ is small too, and Eq. (B1) holds, in fact, if $T-T_c \ll T_c$. Replacing T on the right side of Eq. (B1) with T_c, we obtain after simple algebra:

$$H_0=\frac{\phi_0}{2\pi l^2}\left[\frac{5\pi\tau}{\hbar\eta}\right]^{1/2}(T-T_c)^{1/2} , \tag{B2}$$

$$\eta=\lambda^3\sum_{n=0}^{\infty}(2n+1)^{-2}(2n+1+\lambda)^{-3} , \tag{B3}$$

with the impurity parameter $\lambda=(2\pi T_c\tau)^{-1}$ ($\hbar/2\pi k_B T_c\tau$ in common units). In the dirty limit $\eta=\pi^2/8$; in the clean case $\eta=\lambda^3\sum_n(2n+1)^{-5}=1.0045\lambda^3$.

We notice that the curve $H_0(T)$ in the dirty limit reduces to the almost straight vertical line due to the large prefactor $\phi_0/2\pi l^2$. This means that the H dependence of k^2 vanishes as $l\to 0$.

In the clean case we rearrange Eq. (B2):

$$H_{0,\mathrm{clean}}=1.578\frac{\phi_0}{2\pi\xi_{T_c}^2}\left[\frac{T}{T_c}-1\right]^{1/2} \tag{B4}$$

with $\xi_{T_c}=v/2\pi T_c$ ($\hbar v/2\pi K_B T_c$ in conventional units). The applicability domain for this result is quite narrow. The second of conditions (27) reads here: $H_0 \ll \phi_0/2\pi\xi_{T_c}^2$ which, in turn, translates in a quite narrow temperature domain. [In fact, the truncated series (25) with $k=0$ can be used in the clean case, only if the field is small.]

Equation (30) gives $k^2(H,T)$ in terms of $k^2(0,T)$ and γ. It might be useful to have $k^2(H,T)$ in terms of other quantities. Assume, e.g., that $H_{c2}(T)$ is known for $T<T_c$. Then we have from Eq. (30):

$$-q_c^2=k_0^2-\gamma l^2q_c^4,\quad q_c^2=2\pi H_{c2}/\phi_0,\quad k_0\equiv k(0,T) \tag{B5}$$

(at H_{c2}, $k^2=-q_c^2$). This gives $\gamma(T)$ in terms of H_{c2} and k_0^2. We obtain now Eq. (33) or in terms of γ and H_{c2}:

$$k^2(H,T)=-\frac{2\pi H_{c2}(T)}{\phi_0}+\gamma(T)\left[\frac{2\pi l}{\phi_0}\right]^2[H_{c2}^2(T)-H^2] . \tag{B6}$$

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[9]To prove this, observe that $kl/\beta<\tanh^{-1}(kl/\beta)$ or $kl/\tanh^{-1}(kl/\beta)-1<2\omega\tau$.

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