PHYSICAL REVIEW B VOLUME 32, NUMBER 2 15 JULY 1985

Brief Reports

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Scaling of conductivities in the fractional quantum Hall effect

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(Received 5 November 1984)

We suggest that Hall steps in the fractional quantum Hall effect are physically similar to those in the ordinary quantum Hall effect. This proposition leads to a simple scaling diagram containing a new type of fixed point, which we identify with the destruction of fractional states by disorder.

The purpose of this paper is to propose the scaling diagram for the quantum Hall effect illustrated in Fig. 1. We are not presenting a rigorous or complete theory, but rather arguing on general grounds that *if* a scaling theory of the effect exists and *if* it may be projected onto the $\sigma_{xx}-\sigma_{xy}$ plane in a meaningful way, then the simplest picture consistent with existing experiments[1] has the topology shown in Fig. 1. The purpose of this "minimal" scaling picture is to provide a conceptual framework for future research on electron transport in this system. Its physical content is that all quantum Hall steps are equivalent and that fractional quantum Hall plateaus disappear with increasing disorder by narrowing continuously, in rough analogy to the continuous closing of a superconducting gap with increasing temperature. In addition, the plateaus are nested, so that the complete destruction of any plateau implies the previous destruction of all those deriving from it hierarchically.

The first step leading to Fig. 1 is the assumption that scal-

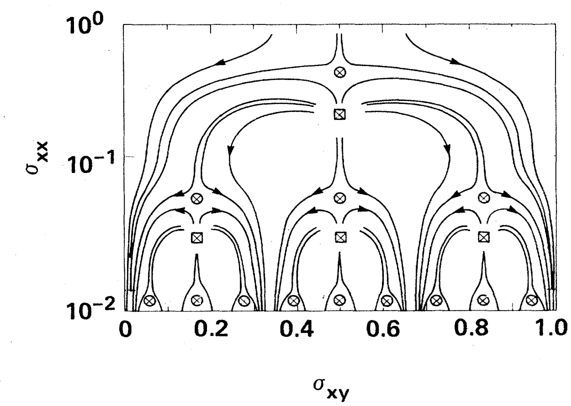

FIG. 1. Scaling diagram for the fractional quantum Hall effect generated by Eqs. (8) and (9). The units of conductivity are e^2/h.

ing theories[2,3] of the quantum Hall effect in the absence of Coulomb interactions, based on recent field theoretical work of three of us,[4,5] are topologically correct. An example of such a theory is illustrated in Fig. 2. σ_{xx} and σ_{xy} are the only relevant variables in this theory, so that the issue of projectability is moot. We leave open the possibility that the correct theory of the ordinary quantum Hall effect has the strength of the Coulomb interaction as a third relevant variable[6] and must be projected onto the $\sigma_{xx}-\sigma_{xy}$ plane to look like Fig. 2. This theory is specifically a description of the phase transition that occurs as the Fermi level of a two-dimensional electron gas in a strong magnetic field is raised through the center of a Landau level. If the system is infinite, its parallel conductivity σ_{xx} remains zero, except over a range of measure zero, while its Hall conductivity σ_{xy} jumps abruptly from zero to e^2/h. If the system is very small, localization does not occur, and the conductivities maintain their mean-field values, given by

$$\sigma_{xx}^{(0)}=\frac{\rho\epsilon e^2\tau}{m}\frac{1}{1+(\omega_c\tau)^2} \tag{1}$$

and

$$\sigma_{xy}^{(0)}=\frac{nec}{H_0}-\frac{\sigma_{xx}^{(0)}}{\omega_c\tau}\ , \tag{2}$$

where ρ is the Fermi surface density of states, ϵ is the Fermi energy, n is the electron density, and τ is a suitable elastic collision time, as the Fermi level is raised. The transition between these two behaviors as the length scale L of the sample is increased is given by[7]

$$\frac{\partial\sigma_{xx}}{\partial\ln(L)}=\beta_{xx} \tag{3}$$

and

$$\frac{\partial\sigma_{xy}}{\partial\ln(L)}=\beta_{xy}\ , \tag{4}$$

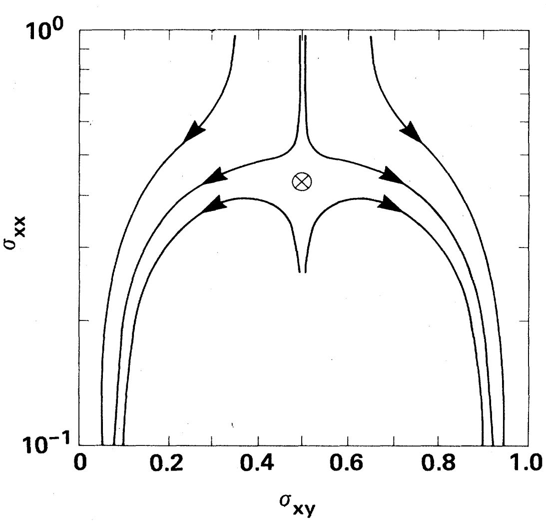

FIG. 2. Scaling diagram for the ordinary quantum Hall effect generated by Eqs. (5) and (6) in the asymptotic region. The units of conductivity are e^2/h.

where β_{xx} and β_{xy} are given asymptotically ($\sigma_{xx}\rightarrow\infty$) by[2]

$$\beta_{xx}=-\frac{1}{8\pi^2\sigma_{xx}}-\tilde{D}_0\sigma_{xx}^3\cos(2\pi\sigma_{xy})e^{-4\pi\sigma_{xx}} \tag{5}$$

and

$$\beta_{xy}=-\tilde{D}_0\sigma_{xx}^3\sin(2\pi\sigma_{xy})e^{-4\pi\sigma_{xx}}\ , \tag{6}$$

with $\tilde{D}_0$ a possibly nonuniversal number thought to be[2] of order $(4\pi)^2$ when the disordering potential is Gaussian white noise. The units of σ_{xx} and σ_{xy} are taken to be e^2/h. It is helpful to think of the starting values in the limit of $\tau\rightarrow\infty$, when $\sigma_{xy}^{(0)}$ represents the electron density and $\sigma_{xx}^{(0)}$ represents the amount of disorder. Thus, the quantum Hall jump in a large sample occurs when the Fermi level is raised to make $\sigma_{xy}^{(0)}=\frac{1}{2}$, or when the Landau level is half full. Figure 2 was generated using Eqs. (5) and (6) down to the vicinity of the fixed point, denoted by a circle in Fig. 2, where they need not be valid. Well below the fixed point, they are invalid.

The second step leading to Fig. 1 is the assumption that the transition between fractional quantum Hall states[8] as the Fermi level is raised or lowered is physically similar to the transition between ordinary quantum Hall states in the same circumstance. This similarity may be understood in the following way. Imagine two adjacent Landau levels with the Fermi level between them, so that one is full and one is empty. The full Landau level is the hole band and the empty one is the electron band. In the presence of disorder, these broaden and merge into a continuum of localized states interrupted by two "measure zero" extended-state bands, one each for electrons and holes. If the Fermi energy is raised sufficiently so that it passes through the electron-extended-state band, the Hall conductance jumps discontinuously by $+e^2/h$. Similarly, if it is lowered through the hole-extended-state band, the Hall conductance jumps by $-e^2/h$. The transition between an ordinary quantum Hall state and its neighbor is caused by the passage of the Fermi level through a band of extended electrons or holes. The analogous phenomenon in the fractional quantum Hall effect would be the passage of the quasi-Fermi level through a band of extended quasielectrons or quasiholes. Our present understanding of the $\frac{1}{3}$ state is that it is characterized by bands of quasielectrons and quasiholes analogous to adjacent Landau levels.[8] That this analogy holds true for the localization physics, in addition to the density of states, is suggested by one's ability to observe the fractional quantum Hall effect at all. The gauge argument[9] identifying the quantum of Hall conductance with the quasiparticle charge will not work unless charge added to the system or removed from it is trapped on localized quasiparticle states. If the transitions $0\rightarrow\frac{1}{3}e^2/h$, $\frac{1}{3}e^2/h\rightarrow\frac{2}{3}e^2/h$, and $\frac{2}{3}e^2/h\rightarrow e^2/h$ are all equivalent to the transition $0\rightarrow e^2/h$, then we must assign to each a fixed point like that in Fig. 2. The assumption that the ordinary and fractional quantum Hall steps are described by renormalization-group flows with similar fixed points allows for the possibility of different exponents at different levels of the hierarchy. We have made the physically sensible, but arbitrary, assumption that the location on the σ_{xx} scale of the fixed point at a given stage in the hierarchy is controlled by the dimensionless parameter $(e^*)^2/\gamma$, where e^* is the relevant quasiparticle charge and γ is an appropriate dimensionless measure of disorder.

The final step leading to Fig. 1 is the observation that the

$\frac{1}{3}e^2/h \rightarrow \frac{2}{3}e^2/h$ circle is topologically incompatible with the $0 \rightarrow e^2/h$ circle without the introduction of a new, totally repulsive fixed point to separate them. This fixed point, denoted by a box in Fig. 1, corresponds physically to the phase transition in which increasing disorder destroys the $\frac{1}{3}$ effect. That this is the case is shown in more detail in Fig. 3. The horizontal eigenflows out of the box, which may be idealized as flowing into the $0 \rightarrow \frac{1}{3}e^2/h$ and $\frac{2}{3}e^2/h \rightarrow e^2/h$ circles, bound a basin of attraction for the $\frac{1}{3}$ state. If the sample is sufficiently dirty so that the starting value $\sigma_{xx}^{(0)}$ lies above this line, the $\frac{1}{3}$ state cannot be reached. If the starting point is below this line, the $\frac{1}{3}$ state cannot be avoided. Immediately above the $0 \rightarrow \frac{1}{3}e^2/h$ circle, the line is vertical and is analogous physically to the vertical eigenflow line above the $0 \rightarrow e^2/h$ circle. Crossing it by changing $\sigma_{xy}^{(0)}$ is tantamount to moving the Fermi level through a narrow extended-state band in the quasiparticle density of states. The curvature of this line at larger values of $\sigma_{xx}^{(0)}$ and its disappearance into the box imply that the quasihole-extended-state band floats upward in energy with increasing disorder, as does the extended-state band in the center of a Landau level in the ordinary quantum Hall effect,[10] and eventually collides with the quasielectron-extended-state band. The box is thus a gap-closing transition, but for a mobility gap rather than a real gap. The symmetry of this diagram about $\sigma_{xy} = \frac{1}{2}$ has the important implication that the quasielectron-extended-state band does not float, insofar as the starting value of σ_{xy} reflects the electron density solely. The minimum number of boxes needed is one. It is conceivable that the real system has more than one.

It is presently believed that a devil's staircase of fractional quantum Hall states exists,[11,12] at each level of which a new fractional state is formed from quasiparticles of the previous one. For example, the $\frac{2}{7}$ and $\frac{2}{5}$ states observed experimentally are thought to be condensations of charge $\frac{1}{3}e$ quasiholes and quasielectrons, respectively, added to the $\frac{1}{3}$ state. The validity of this picture implies that each level of the hierarchy should be contingent on the existence of the previous one,[11] and therefore that the box describing the destruction of a daughter state should lie within the basin of attraction of the parent. This is consistent with the current belief that the energy gaps of the hierarchical states decrease rapidly [roughly as (fraction denominator)$^{-2.5}$],[12,13] and thus that less disorder should be required to annihilate a daughter state than to annihilate a parent. Given this restriction, the generalization of our ideas to describe the full hierarchy of fractional quantum Hall states is straightforward. In Fig. 1 we have included bifurcations down to the level of $\frac{1}{5}$, $\frac{2}{7}$, $\frac{2}{5}$, $\frac{5}{7}$, and $\frac{4}{5}$. The $\frac{1}{5}$ and $\frac{4}{5}$ states, which have not been shown conclusively to exist, do not derive hierarchically from the $\frac{1}{3}$ and $\frac{2}{3}$ states, and thus need not be contingent upon them. We have shown them contingent in Fig. 1 because we believe this to be the case when the repulsive forces between the electrons are Coulombic. The β function actually used to generate Fig. 1 is

$$\beta_{xx} = -\frac{1}{8\pi^2\sigma_{xx}} - (4\pi)^2\sigma_{xx}^3 \times [\cos(2\pi\sigma_{xy})e^{-4\pi\sigma_{xx}} + (3)^8\cos(6\pi\sigma_{xy})e^{-(3)^2 4\pi\sigma_{xx}} + \cdots] \quad (7)$$

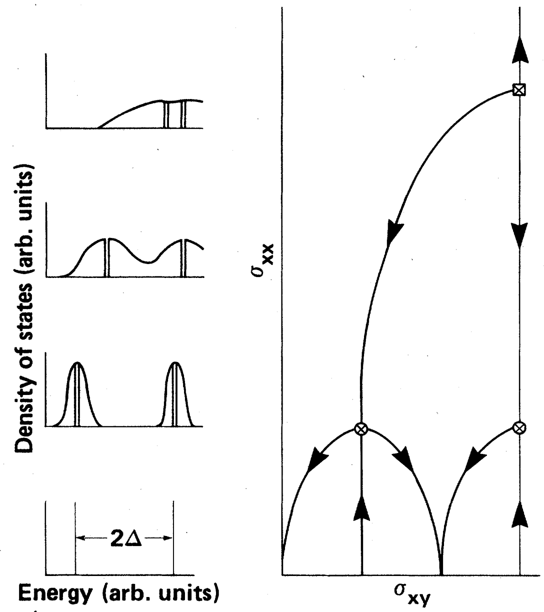

FIG. 3. Illustration of the closing of the mobility gap of the $\frac{1}{3}$ state with increasing disorder. A larger starting value for σ_{xx} implies more disorder, more broadening of the quasiparticle density of states (shown on the left), and a smaller quasiparticle mobility gap.

and

$$\beta_{xy} = -(4\pi)^2\sigma_{xx}^3 \times [\sin(2\pi\sigma_{xx})e^{-4\pi\sigma_{xx}} + (3)^9\cos(6\pi\sigma_{xy})e^{-(3)^2 4\pi\sigma_{xx}} + \cdots] \ . \quad (8)$$

It is *ad hoc* and should be understood as an illustration only.

The ideas presented here may be tested experimentally in a number of ways. A series of measurements of the energy gap of the $\frac{1}{3}$ state by the activation energy method[14] for increasingly dirty samples would verify our picture of a continuous gap closing, as would measurements of the plateau width. Conductivity measured at finite temperature or at nonzero frequency can test the effects of finite size. We suspect strongly that the partial resurrection of the $\frac{1}{3}$ plateau at finite frequency reported by McFadden *et al.*[15] is an effect of the box.

We would like to thank the Aspen Center for Physics, where most of this work was performed. The contribution of R.B.L. was performed under the auspices of the U. S. Department of Energy under Contract No. W-7405-Eng-48. M.L.C. was supported by National Science Foundation Grant No. DMR-83-19024 and by the Director, Materials Science Division of the U. S. Department of Energy under Contract No. DE-AC03-76SF00098. S.B.L. was supported by the U.S. Department of Energy Grant No. DE-AC02-76ER03130.A013TASK*A*. J.M.K. was supported by National Science Foundation Grant No. DMR-83-16893. A.M.M.P. was supported by U. S. Department of Energy Grant No. DE-AC02-76ER02220 and U. S. Office of Naval Research Grant No. N00014-80-C-0657.

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