# Small correction to the quantization of Hall conductance due to current-current interactions and charge redistribution

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The self-consistent Hall potential, current, charge, and magnetic-field distributions of an interacting electron gas with filled Landau levels in a thin strip are calculated. A Hartree approximation recently described by MacDonald, Rice, and Brinkman [Phys. Rev. B **28**, 3648 (1983)] is extended to include the thickness of the strip and the effect of current-current interactions. It is found that the weighting of the Hall potential, current, and charge distributions toward the edges of the strip increases with decreasing thickness. This effect is due to the magnetic field generated by the current itself. The resulting self-consistent global charge and magnetic-field inhomogeneities lead to a nonvanishing correction to the quantization of the Hall conductance, which is nonlinear in the total current. For sample parameters corresponding to actual quantum Hall experiments the calculated correction is extremely small (one part in $10^{11}$).

## I. INTRODUCTION

The experimental discovery of the quantum Hall effect[1] has stimulated much theoretical work on two-dimensional electron systems subject to strong magnetic fields.[2] In a recent article[3] the shape of the Hall potential has been calculated for an ideal two-dimensional electron gas with completely filled Landau levels. The result is that the charge, current, and Hall voltage distributions are weighted toward the sample edges with a decay length into the bulk which depends on sample size and magnetic-field strength.

Here, we extend the work of Ref. 3 to a strip with nonzero depth.[4] We further include the effect of the magnetic field $\mathbf{B}_j$ which is created by the current distribution itself. The numerical solution of the appropriate self-consistent equation will show that the effect of this additional small field is not negligible in certain cases. Furthermore, owing to $\mathbf{B}_j$ and the redistribution of charge, the Hall conductance shows a small deviation from the quantized value, which will be esimated. As in Ref. 3 we shall restrict ourselves to a system with $n$ fully occupied Landau levels.

We consider a system of electrons confined to the strip

$$-L_x/2 \leq x \leq L_x/2,\ \ -a/2 \leq z \leq a/2 \quad (a << L_x),$$
$$0 \leq y \leq L_y \quad (L_y >> L_x) \ .$$

In the presence of a constant magnetic field $B_0$ in the $z$ direction and an imposed total current $I$ in the $y$ direction the electron system reacts by a redistribution of charge and current leading to a Hall field $\mathbf{E}$ such that, in classical terms, the Lorentz force is compensated at each point inside the strip. We define

$$\mathbf{E} = \mathrm{grad}(V)/e \ ,$$

where $V$ is the potential energy and $-e$ the charge of an electron. Our task is to calculate $V$ given $\mathbf{B}_0$ and $I$.

## II. SELF-CONSISTENT EQUATION FOR THE HALL POTENTIAL

In the following we neglect the small $z$ variations of $V$ and $\mathbf{B}_j$ inside the strip. These fields will be represented by their values at $z=0$. The total magnetic field is therefore

$$\mathbf{B}(x) = [0,0,B(x)] = [0,0,B_0 + B_j(x)] \ . \tag{1}$$

Assuming $V(x)$ slowly varying[3] then in the neighborhood of an arbitrary coordinate $x_0$ the one-particle Schrödinger equation locally describes an electron in a homogeneous electric and magnetic field. Therefore, in the appropriate gauge, the $x$-dependent parts of the local solutions are Hermite polynomials multiplied by a factor which decreases exponentially with the distance from their centers $x_p$ (with a decay length of the order of a cyclotron orbit), where[3]

$$x_p = hcp/[L_y eB(x_0)] - mc^2 V'(x_0)/[e^2 B^2(x_0)] \ . \tag{2}$$

In actual quantum Hall experiments about $10^4$–$10^5$ orbital centers $x_p$ are contained within two cyclotron orbits $\lambda$ ($\lambda \approx 100$ Å $<< L_x$). Therefore, for a system with $n$ completely filled Landau levels one can express the two-dimensional electron density $\sigma(x)$ at $x = x_p$ by the number $n$ of occupied Landau orbital centers $x_p$ per unit surface

$$\sigma(x_p) = n/(x_{p+1} - x_p)L_y \ . \tag{3}$$

The three-dimensional electron density will be approximated by $\sigma(x)/a$.

From (2) we obtain an expression for $x_{p+1} - x_p$ which we develop around $x_p$. Keeping the constant and linear terms and putting the resulting expression for $x_{p+1} - x_p$ into (3), we obtain

$$\sigma(x_p) = neB(x_p)/(hc) + nB'(x_p)(p+1)/[L_y B(x_p)] + (nmc/eh)[V''(x_p)/B(x_p) - 2V'(x_p)B'(x_p)/B^2(x_p)] \ . \tag{4}$$

If the current $I$ is zero $B'$, $V'$, and $V''$ vanish, and $\sigma(x)$ becomes a constant $\sigma_0$ equal to $neB_0/hc$. The difference between $\sigma(x)$ and $\sigma_0$ is denoted by $\delta\sigma(x)$.

 

Approximating $p+1$ in (4) by $p$ (since $p_{\max} >> 1$), expressing $p$ in terms of $x_p$ by relation (2), and going over to continuous $x_p$ we obtain

$$\delta\sigma(x) = (ne/hc)B_j(x) + (ne/hc)xB_j'(x) + (n^2 m/h^2\sigma_0)V''(x) \ . \tag{5}$$

Here, we have neglected terms nonlinear in $I$.

The three-dimensional current distribution $[0, j(x), 0]$ across the strip is given by

$$j(x) = \frac{c\,\sigma(x)V'(x)}{aB(x)} \ . \tag{6}$$

[An orbital centered at $x_p$ contributes a total current $cV'(x_p)/B(x_p)$ in the $y$ direction.] The magnetic field $[0, 0, B_j(x)]$ generated by this current distribution is given by

$$B_j(x) = -\frac{2}{c}\int_{-a/2}^{a/2}(dz)\int_{-L_x/2}^{L_x/2}\frac{(x-u)j(u)}{(x-u)^2+z^2}du \ . \tag{7}$$

Approximating $\sigma(x)$ by $\sigma_0$ and $B(x)$ by $B_0$ in relation (6) one obtains

$$B_j(x) = \frac{-4en}{hca}W(x,L_x) \ , \tag{8}$$

where

$$W(x,L_x) = \pi V(x) - V(-L_x/2)\arctan[0.5a/(x+L_x/2)] + V(L_x/2)\arctan[0.5a/(x-L_x/2)] - 0.5a\int_{-L_x/2}^{L_x/2}\frac{V(y)}{(x-y)^2+a^2/4}dy \ . \tag{9}$$

The potential $V(x)$ is generated by the three-dimensional charge density $-e\,\delta\sigma(x)/a$ (the homogeneous part $-e\sigma_0/a$ is counterbalanced by the positively charged background). The solution of the Poisson equation has the form

$$V(x) = (-2e^2/a\epsilon)\int_{-a/2}^{a/2}dz\int_{-L_x/2}^{L_x/2}\ln[(x-u)^2+z^2]^{1/2}\delta\sigma(u)du \ , \tag{10}$$

where $\epsilon$ is the dielectric constant in the strip and $\delta\sigma$ is given as a function of $V(x)$ by (5), (8), and (9). Performing the $z$ integration and expressing lengths in units of $L_x/2$ we obtain the final equation

$$V(x) = \frac{4e^2n^2L_x}{h^2c^2a\epsilon}\int_{-1}^{1}F(x,y)[W(y,2)+yW'(y,2)]dy - \frac{4e^2n^2m}{h^2\sigma_0 L_x}\int_{-1}^{1}F(x,y)V''(y)dy \ , \tag{11}$$

where

$$F(x,y) = 0.5\ln[a^2/4+(x-y)^2] - 1 + (2/a)|x-y|\arctan(0.5a/|x-y|) \ . \tag{12}$$

Equation (11) is linear in $V$. It has to be solved under the subsidiary condition of charge neutrality, which can be satisfied by imposing

$$V(x) = -V(-x) \ , \tag{13}$$

whence

$$\delta\sigma(x) = -\delta\sigma(-x) \ , \tag{14}$$

and

$$B_j(x) = -B_j(x) \ . \tag{15}$$

The normalization of $V(x)$ is obtained by integrating Eq. (6) whence, using (13)–(15) and neglecting terms nonlinear in $I$,

$$V(L_x/2) = \frac{Ih}{2ne} \ . \tag{16}$$

The first integral in Eq. (11) is due to the field $\mathbf{B}_j(x)$. If this term is neglected and if in (12) $a$ tends to zero one recovers Eq. (15) of Ref. 3 (where $\epsilon=1$). The numerical solution of Ref. 3 shows a flattening of $V(x)$ in the middle of the strip and an increase of its steepness toward the edges. This behavior is the more pronounced the smaller the prefactor of the $V''$ term is. The effect on $V(x)$ of the new terms appearing in (11) [which are due to the field $\mathbf{B}_j(x)$] can be qualitatively understood as follows: For positive $x$ the current $\mathbf{j}(x)$ generates a field $\mathbf{B}_j(x)$, which is opposite to the applied field and whose absolute value increases with $x$. Therefore, the distances between adjacent orbital centers $x_p$ are increased. This means that the $B_j$ terms alone would give a negative $\delta\sigma(x)$ for positive $x$. However, $\delta\sigma(x)$ must be positive for positive $x$ in order to create an electric field, which locally counterbalances the Lorentz force. Therefore, in order to compensate the decrease of $\delta\sigma(x)$ caused by the $B_j$ terms, the $V''$ terms must increase, i.e., the curvature of $V(x)$ increases if the $B_j$ terms are taken into account.

Equation (11) has been solved numerically using a method similar to that of Ref. 3 [the double integrals appearing in (11) due to $W(y,2)$ have been reduced to a single integral during the numerical process].

## III. RESULTS AND DISCUSSION

The shape of the $V(x)$ curve depends on the parameters $L_x$, $a$, $m$, $\sigma_0$, $\epsilon$, $n$. Figure 1 shows numerical solutions of Eq. (11) for parameter values $L_x=0.06$ cm, $m/\sigma_0=0.615\times10^{-39}$ g cm$^2$ (e.g., $m=0.0676m_e$, $\sigma_0=10^{11}$ cm$^{-2}$), $n^2/\epsilon=1.24$ (e.g., $n=4$, $\epsilon=12.9$) and $n^2/\epsilon=0.0775$ (e.g., $n=1$, $\epsilon=12.9$), and different depths $a$. It illustrates the importance of the $B_j(x)$ terms in Eq. (11). For instance, for $n^2/\epsilon=1.24$ and $a=100$ Å the $B_j$ terms give a lowering of $V(x)$ by a factor of about 10 in the major part of the interval $0<x<L_x/2$. One can also see that the influence of the $B_j$ terms increases with increasing $L_x/a$ and also with increasing $n$. The latter is due to the fact that $B_j$ constitutes a higher fraction of the total field if $n$ increases.

In Fig. 2 the behavior of $j(x)$, $\delta\sigma(x)$, and $B_j(x)$ is shown for $n^2/\epsilon=0.0775$ and $a=100$ Å. For a current $I=6\times10^{-10}$ A (corresponding to $10^{-6}$ A m$^{-1}$) we obtain here $V(L_x/2)=1.2\times10^{-18}$ erg, and $\delta\sigma(L_x/2)\approx10^{-5}\sigma_0$. The average potential energy per particle (resulting from the

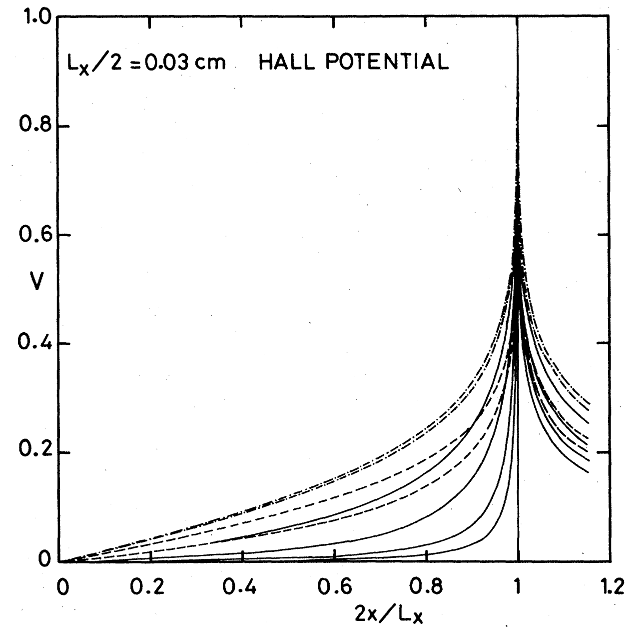


FIG. 1. Hall potential $V(x)$ (in units of $Ih/2ne$) of a strip of width $L_x$ and thickness $a$. The continuous lines represent solutions of Eq. (11) for the parameter values $n^2/\epsilon = 1.24$ and $m/\sigma_0 = 0.615\times10^{-39}$ g cm$^2$ (see text) and $a = 1000, 300, 100, 50$ Å (from top to bottom). The dot and dashed lines represent the corresponding solutions of Eq. (11), where the terms due to the magnetic field $B_j(x)$ have been neglected, for $a = 1000$ Å (upper line), $a = 50$ Å (lower line); the solutions for $a = 100$ Å and $a = 300$ Å nearly coincide with the solution for $a = 50$ Å in this case. The dashed lines correspond to $n^2/\epsilon = 0.0775$ and $a = 100$ Å, where in the upper dashed line the $B_j$ terms in Eq. (11) have been neglected. Note that $V(x)$ is antisymmetric with respect to $x = 0$.

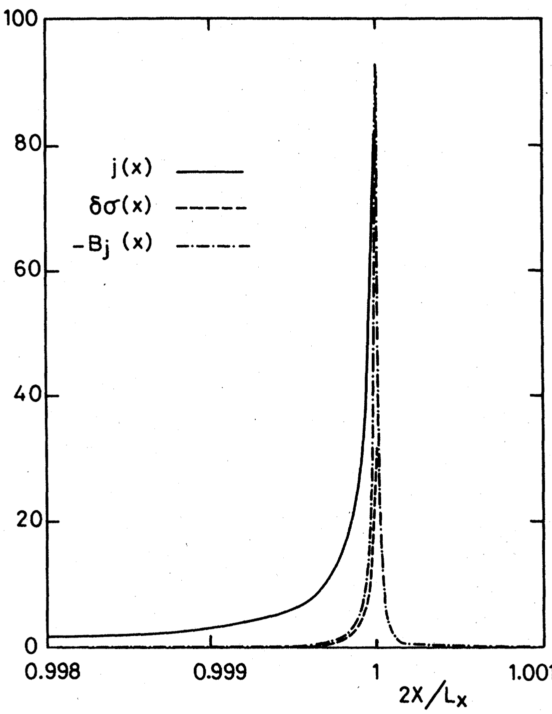


FIG. 2. Current distribution $j(x)$ [in units of $10^3\times(ne/ha)\times V(L_x/2)$], inhomogeneous part of the electron density $\delta\sigma(x)$ [in units of $10^3\times(\epsilon/2e^2)V(L_x/2)$], and inhomogeneous part of the magnetic field $B_j(x)$ [units of $10^{-3}(4en/hca)V(L_x/2)$] for $n^2/\epsilon = 0.0775$ and thickness $a = 100$ Å. Figure 2 corresponds to the lower dashed line in Fig. 1 [$j(x)$ is symmetric, $\delta\sigma(x)$ and $B_j(x)$ are antisymmetric with respect to $x = 0$].

antisymmetric redistribution of charge) is $1.2\times10^{-27}$ erg, and the average kinetic energy is $4\times10^{-29}$ erg. These values have to be compared with $\hbar\omega_c = 1.1\times10^{-14}$ erg ($n = 1$).

From (6), using (13)–(15) the total current can be expressed as

$$I = a\int_{-L_x/2}^{L_x/2} j(x)dx = (ne^2/h)(1+\Delta)V_H \ , \tag{17}$$

where

$$V_H = 2V(L_x/2)/e \tag{18}$$

is the Hall voltage across the strip and

$$\Delta = [-hc/(ne^2V_HB_0^2)]\int_{-L_x/2}^{L_x/2} \delta\sigma(x)V'(x)B_j(x)dx \ . \tag{19}$$

In Eq. (19) terms cubic in $V_H$ are neglected. $\Delta$ is quadratic in $V_H$, since $\delta\sigma(x)$, $V'(x)$, and $B_j(x)$ each are proportional to $V(L_x/2)$, but their shape is independent of the value of $V(L_x/2)$ [see (5), (8), (9), (11)]. Therefore, using (16) and (18) we can write

$$\Delta = \Delta' I^2 \ . \tag{20}$$

Through the solution of Eq. (11) $\Delta'$ depends on all the parameters of the problem except on $I$. The Hall conductance $\sigma_H$ becomes

$$\sigma_H = -I/V_H = -(ne^2/h)(1+\Delta), \quad \Delta > 0 \ . \tag{21}$$

As an indication we mention that the values of $\Delta'$ corresponding to the numerical calculations of Fig. 1 are contained in the range $(0.2\text{–}4)\times10^{-21}$ (sec/esu)$^2$ for $\sigma_0 = 10^{11}$ cm$^{-2}$. For $I = 3\times10^{-5}$ A (i.e., 0.05 A m$^{-1}$ for $L_x = 0.06$ cm) this would give a correction $-(ne^2/h)(0.16\text{–}3.2)\times10^{-11}$ to the quantized value $ne^2/h$ of the Hall conductance. This is below the limit of accuracy ($10^{-7}$–$10^{-8}$) of actual quantum Hall effect measurements.

In view of applications of the quantum Hall effect[5] all possible corrections to the quantized value must be known. The correction discussed here (which is nonlinear in the total current $I$) is a consequence of the magnetic field of the current and of the global self-consistent charge inhomogeneity across the sample resulting from the Coulomb interaction among the electrons. It is present even in ideal systems with fully occupied Landau levels. (If the magnetic field of the current and the self-consistent redistribution of charge are neglected, all corrections nonlinear in the electric field vanish, even in the presence of an aperiodic substrate potential.[6])

Finally we should like to give some arguments about how the inclusion of impurities might modify our results. Consider the case where insulating regions (at least several cyclotron radii apart) are regularly distributed in the system, so that new edges are created. Equation (11) now applies in each interval in the $x$ direction between neighboring insulators. Since $V(x)$ has the same value on both sides of an insulating region, $V(x)$ is a succession of ascending ripples over the sample width $L_x$; i.e., on a macroscopic scale the potential is a linear function of $x$. The current $j(x)$ is composed of curves shaped according to Fig. 2, fit together by intervals of zero current (due to the insulating regions). Therefore, on a macroscopic scale, the current distribution is a constant. If the insulating regions become narrower, the electrons in the formerly independent conducting intervals will be more and more correlated electrostatically across the insulators. In the limit of pointlike impurities $V(x)$ and $j(x)$ should take forms similar to those shown in Figs. 1 and 2. In the samples actually used for quantum Hall experiments the range of the impurity potentials is about 50–100 Å.[7] This should lead to an intermediate behavior.

In summary, we expect that the strong weighting toward the edges of the ideal $V(x)$ and $j(x)$ curves will be attenuated according to the number and nature of the impurities.


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