PHYSICAL REVIEW B VOLUME 31, NUMBER 12 15 JUNE 1985

Ultrasonic attenuation in strongly disordered electronic systems

B. Gabriel Kotliar
Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, California 93106

T. V. Ramakrishnan
Indian Institute of Science, Bangalore 560012, India

(Received 5 November 1984)

The attenuation of long-wavelength phonons due to their interaction with electronic excitations in disordered systems is investigated here. Lattice strain couples to electronic stress, and thus ultrasonic attenuation measures electronic viscosity. The enhancement and critical divergence of electronic viscosity due to localization effects is calculated for the first time. Experimental consequences for the anomalous increase of ultrasonic attenuation in disordered metals close to the metal-insulator transition are discussed. In the localized regime, the appropriate model is one of *electronic* two-level systems (TLS's) coupled to phonons. The TLS consists of a pair of states with one localized state occupied and the other unoccupied. The density of such low-excitation-energy TLS's is nonzero due to long-range Coulomb interactions. The question of whether these could be significant low-energy excitations in glasses is touched upon.

I. INTRODUCTION

In this paper we investigate the effects of electronic disorder on the propagation of long-wavelength phonons.[1] Ultrasonic propagation and attenuation in metallic and insulating glasses have been analyzed in terms of coupling with low-lying two-level configurational excitations in such systems (Anderson, Halperin, and Varma[2(a)] and Phillips[2(b)]). However, electronic effects on ultrasonic propagation have not been investigated for such systems and others where the nature of electronic states is strongly affected by disorder.

We discuss this problem here, considering two regimes in detail, namely the metal with increasing disorder as it approaches the metal-insulator transition, and the Anderson insulator with exponentially localized states near the Fermi energy. There is a well-developed theory for the weakly disordered metal regime, due to Pippard,[3] Tsuneto,[4] Schmid,[5] and others (see Sec. II below). Here we investigate effects not considered by these authors and connected with quantum interference or localization. Since sound couples to electronic stress, its attenuation measures stress-stress correlations or electronic viscosity which diverges as the electrons localize. This leads to characteristic anomalous enhancement of the attenuation, in particular to a new critical exponent δ for its frequency dependence, i.e., $\alpha \simeq \omega^{2-\delta}$, near the metal-insulator transition ($\delta = 0$ in the absence of localization effects). The exponent δ is calculated to be $(\epsilon/2)$ to lowest order in ϵ for a system in $2+\epsilon$ dimensions.

There has been no calculation of electronic effects on ultrasonic attenuation in an Anderson insulator. In Sec. III we discuss this regime using the idea, due to Mott,[6] that low-frequency transport and dissipative behavior in these systems is determined by pairs of states, one occupied and the other unoccupied, with energy difference of order $\hbar\omega$ or $k_B T$, and with spatial separation $r \gg \xi_{\text{loc}}$, the localization length. We calculate the contribution of such pairs to relaxational absorption of sound, and show that inclusion of Coulomb or Efros-Shklovskii[7] types of effects leads to characteristic enhancement of attenuation. The attenuation depends *linearly* on frequency in this case. A physical description of these results is given in terms of electronic two-level systems. Upon including Coulomb interactions, one finds that these have a nonvanishing density for small energy differences Ω between the two states. These two-level systems have other properties, such as saturation effects, which also are similar to the AHV-Phillips (AHV denotes Anderson, Halperin, and Varma) configurational two-level systems.

In the calculation of the ultrasonic attenuation we use the model Hamiltonian derived by Tsuneto[4] and by Schmid[5] to describe the long-wavelength properties of the electron-phonon system. It describes the electron system in a frame co-moving with the lattice. Since the random scatterers are embedded in the lattice and move with it, electron scattering is elastic and the electronic distribution relaxes to equilibrium in this moving frame. The interaction between electrons and the lattice in this frame is due to nonuniformity of lattice displacement (lattice strain) which produces electronic stress. Tsuneto showed that one can go over to this frame by a canonical transformation. Physical quantities such as Green's functions, being traces, are invariant under such a canonical transformation. Furthermore, this is also the natural framework for the localized regime, since, to a first approximation, the localized electronic states follow the long-wavelength lattice wave. Thus, physical effects in both metallic and localized regimes can be discussed in a single formal framework, described in detail in Ref. 5 for the metallic case.

The Hamiltonian after transformation is given by

$$H = H_e + H_{\text{e-ph}} + H_{\text{ph}} ,$$

where

$$H_e=\sum_{\sigma}\int d^d r\,\psi_\sigma^\dagger(\mathbf{r})\left[-\frac{\nabla^2}{2m^*}+v(\mathbf{r})\right]\psi_\sigma(\mathbf{r})$$
$$+\tfrac{1}{2}\sum_{\sigma,\sigma'}\int d^d r\,d^d r'\,\psi_\sigma^\dagger(\mathbf{r})\psi_{\sigma'}^\dagger(\mathbf{r}')\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\psi_{\sigma'}(\mathbf{r}')\psi_\sigma(\mathbf{r})\,, \tag{1}$$

$$H_{e\text{-ph}}=\sum_{\alpha,\beta}\int d^d r\,\tau_{\alpha\beta}(\boldsymbol{\nabla}_\alpha U_\beta)\,, \tag{2a}$$

with

$$\tau_{\alpha\beta}=\frac{1}{4m^*}\sum_{\sigma}(\boldsymbol{\nabla}-\boldsymbol{\nabla}')_\alpha(\boldsymbol{\nabla}-\boldsymbol{\nabla}')_\beta\psi_\sigma^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r}')\Bigg|_{\mathbf{r}=\mathbf{r}'} \tag{2b}$$

and

$$H_{\text{ph}}=\sum_{\alpha,\beta}\int e_{\alpha\beta}\boldsymbol{\nabla}_\alpha U_\beta(\mathbf{r})\boldsymbol{\nabla}_\alpha U_\beta(\mathbf{r})d\mathbf{r}\,. \tag{3}$$

In Eqs. (1) and (2b), $\psi_\sigma^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r})$ are electron-field creation and annihilation operators, respectively, $U_\beta(\mathbf{r})$ is the phonon displacement field,

$$\mathbf{U}(\mathbf{r})=\frac{1}{\sqrt{V}}\sum_{qb}\frac{\mathbf{e}_b(\mathbf{q})}{\sqrt{2\rho\omega_b(\mathbf{q})}}[B_b(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}+B_b^\dagger(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{r}}]\,,$$

and $v(\mathbf{r})$ is a random potential, conveniently chosen Gaussian with short-range correlations,

$$\langle v(\mathbf{r})v(\mathbf{r}')\rangle=\frac{\delta(\mathbf{r}-\mathbf{r}')}{2\pi g(\mu)\tau}\,, \tag{4}$$

where $g(\mu)$ is the density of states per unit volume at the Fermi energy and τ is the plane-wave lifetime. These assumptions about v(r) are particularly convenient for discussing the metallic regime. For the insulating regime, we shall work with the exact (localized) eigenstates and eigenvalues of H_e. The term $H_{e\text{-ph}}$ describes the interaction between the elastic strain and the electronic stress or kinetic-energy fluctuations. The term H_{ph} is the phonon Hamiltonian in an elastic continuum or Debye approximation, $e_{\alpha\beta}$ are elastic constants, and ρ is the mass density. The Coulomb interaction is contained in H_e. Since only the long-wavelength part of the interaction is relevant, its effect will be treated in the random-phase approximation on the metallic side. On the insulating side, the absence of screening of the Coulomb interaction leads to qualitatively new effects as discussed by Efros and Shklovskii;[7] the results and ideas relevant to ultrasonic attenuation will be mentioned in Sec. III.

II. METALLIC REGIME

The ultrasonic attenuation is obtained from the imaginary part of the polarization operator of the phonon Green's function. It is defined by

$$\mathscr{D}^{-1}(\omega,\mathbf{q},b)=\mathscr{D}_0^{-1}(\omega,\mathbf{q},b)-\pi(\omega,\mathbf{q},b)\,. \tag{5}$$

We write

$$\mathscr{D}_0^{-1}(\omega,\mathbf{q},b)=-\frac{\omega^2+[\omega_b(\mathbf{q})]^2}{[\omega_b(\mathbf{q})]^2}\,, \tag{6}$$

for the noninteracting phonon Green's function. The symbol b stands for a polarization index ($b=l,t$ for longitudinal and transverse waves respectively) while $\omega_b(q)=c_b q$ gives the bare phonon dispersion relation.

We calculate the phonon polarization operator π, neglecting phonon vertex corrections to second order in $1/k_F l$. We use the standard diagrammatic technique to average over the random potential, impurity scattering being represented by a dotted line as in Fig. 1(a). The electron-phonon vertex corresponds [taking into account Eq. (2)] to a factor

$$\Gamma_0=-\frac{i}{m\sqrt{\rho\omega(q)}}\mathbf{p}\cdot\mathbf{q}\,\mathbf{e}_b\cdot\mathbf{p}\,. \tag{7}$$

$\mathbf{e}_b(q)$ is a polarization unit vector. This vertex is represented in Fig. 1(b). We include the effect of the Coulomb long-range force by summing density-density insertions (bubble diagrams) to infinite order. This is graphically represented in Fig. 2. The cross-hatched bubble representing the full phonon polarization (including Coulomb interaction lines) is expressed in terms of irreducible density-density, stress-stress, and density-stress correlation functions (parallel-hatched bubbles).

Denoting the irreducible (with respect to the Coulomb interaction) density-density, stress-density, and stress-stress correlation functions by G_{dd}, G_{ds}, and G_{ss}, we find (see Fig. 2)

$$\pi=G_{ss}+\frac{(G_{ds})^2(4\pi/q^2)e^2}{1-e^2(4\pi/q^2)G_{dd}}\,. \tag{8}$$

We now show that this equation simplifies for long wavelengths (small q) due to perfect screening in a metal.

On very general grounds, G_{dd} assumes the form

$$G_{dd}=\frac{dn}{d\mu}\frac{Dq^2}{-i\omega+Dq^2} \tag{9}$$

in the long-wavelength limit, where $dn/d\mu$ is the compressibility.

We will be interested in the small-q regime, so that the factor $(4\pi e^2/q^2)G_{dd}$ has the value K^2/q^2 for $\omega=qv_s\ll Dq^2$ and equals $DK^2/i\omega$ for $\omega\gg Dq^2$. Here, K^2 is the inverse screening length squared, i.e.,

$$4\pi e^2\frac{dn}{d\mu}=K^2\,.$$

Clearly, both limiting values are much larger than unity for low frequencies and long wavelengths, i.e., $\omega<\omega_D$ and $q\ll K$, so that $(4\pi e^2/q^2)G_{dd}\gg 1$. At the mobility edge and for small frequency, i.e., $v_s q=\omega<\omega_D/10$, we enter the dynamic regime, i.e., $Dq^2<\omega$ and

$$\frac{D(q,\omega)}{\omega}\simeq\left[\frac{dn}{d\mu}\omega\right]^{-2/3}.$$

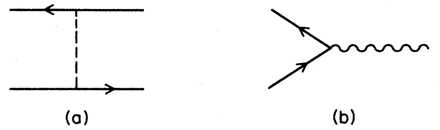

FIG. 1. (a) Impurity line; (b) electron-phonon vertex.

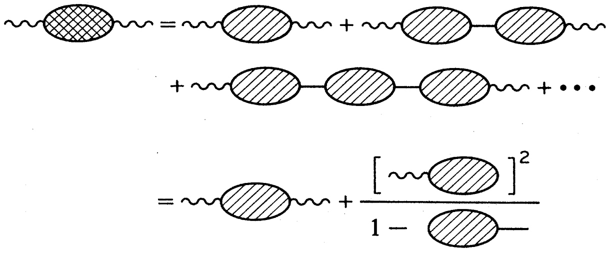

FIG. 2. Random-phase-approximation resummation of the Coulomb interaction. The solid lines are Coulomb lines. The cross-hatched bubble representing the full phonon polarization operator is expressed in terms of irreducible (in terms of the Coulomb line) correlation functions (parallel-hatched bubbles).

We have

$$\frac{4\pi e^2}{q^2}\frac{D(\omega)q^2}{\omega} \sim \left[\frac{\varepsilon_F}{\omega}\right]^{2/3} .$$

This number is again very large. The simple conclusion is that, at very low phonon frequencies, screening is perfect, so that one has, to a very good approximation,

$$\pi = G_{ss} - (G_{ds})^2/G_{dd} \ . \tag{10}$$

We now discuss the polarizability π, based on Eq. (10), for transverse and longitudinal phonons.

A. Transverse phonons

Owing to the symmetry of the tensor vertex in the shear case, $G_{ds}=0$. This describes the fact that with zero range scattering shear or transverse fluctuations do not couple to longitudinal or density fluctuations. Furthermore, to lowest order in $(k_F l)^{-1}$, G_{ss} is given by the process shown in Fig. 3(a) describing short-range shear fluctuations in an electron gas. The electron and hole lines are propagators in the presence of random scatterers. Diffusion or ladder corrections of the sort shown in Fig. 3(b) vanish. Physically, this has to do with the fact that under collisions shear stress has no conserved component, unlike longitudinal stress or density fluctuations. We find, in the long-wavelength limit,

$$\mathrm{Im}\Pi = -\frac{g(\mu)p_F^4(\omega\tau)}{m^2\rho c_t^2}\frac{1}{d(d+2)} \ , \tag{11}$$

where d is the spatial dimensionality, ρ the mass density, c_t the transverse sound velocity, and p_F the Fermi momentum. The linear dependence on ω arises only from the density of particle-hole excitations. In a clean electron gas the latter varies as ω/qv_F, this being replaced by $\omega\tau$ in the present case for $ql \ll 1$. The result (11) leads to the following expression for attenuation, first obtained by Pippard using semiclassical arguments:

$$\alpha_t(\omega) = -\frac{\omega\,\mathrm{Im}\Pi}{c_t} = \frac{\omega^2\tau g(\mu)p_F^4}{m^2\rho c_T^2 d(d+2)} \tag{12a}$$

$$\equiv \frac{\alpha_{0t}\omega^2}{d(d+2)} \ . \tag{12b}$$

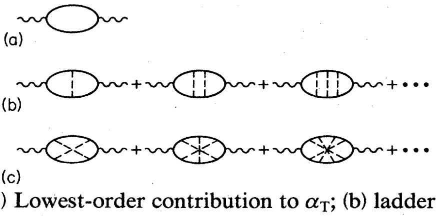

FIG. 3. (a) Lowest-order contribution to α_T; (b) ladder corrections; (c) maximally crossed diagrams.

We notice that the attenuation (i) varies as ω^2, i.e., one power of frequency higher than the real part so that the Goldstone mode is well defined, (ii) has a small coefficient $(m/M)^{1/2}$, and (iii) is smaller, the smaller the collision time τ, or the conductivity $\sigma = ne^2\tau/m$. The first and last results are true only if quantum-interference or localization effects are neglected.

Quantum-interference effects occur to higher order in $(k_F l)^{-1}$. For transverse attenuation, only the single maximally crossed diagram set, Eq. (3c), contributes. Calculation of the graphs listed in Fig. 3(c) in the limit of vanishing external momenta but finite frequency $(Dq^2 \ll \omega)$ gives, in the weakly localized regime, the zero temperature

$$\alpha_t = \begin{cases} \dfrac{\alpha_{0t}\omega^2}{8}\left[1+\dfrac{1}{2\pi k_F l}\ln\dfrac{1}{\omega\tau}\right], & d=2 \qquad (13a) \\[2ex] \dfrac{\alpha_{0t}\omega^2}{15}\left[1-\dfrac{1}{\pi k_F l}\sqrt{\omega\tau}\right], & d=3 \ . \qquad (13b) \end{cases}$$

We note that the condition $Dq^2 \ll \omega$ is satisfied if $ql \ll v_S/v_F$, e.g., for frequencies less than $\omega_D^2/\varepsilon_F(k_F l)$.

We notice that localization effects increase the attenuation. Singular backscattering reinforces stress fluctuations while it reduces current flow. As electron diffusion slows down, the electron gas becomes more rigid against shear deformation. This effect is dynamic, i.e., frequency dependent. The effect of the same backscattering process on conductivity is to reduce it in a scale-dependent way. Formally, the difference is due to the fact that, while backscattering reverses the current, which is a vector, it does not change the stress, which is a symmetric tensor.

Repeating the calculation at nonzero temperature, one finds

$$\alpha_t = \begin{cases} \dfrac{\alpha_{0t}\omega^2}{8}\left[1+\dfrac{1}{2\pi k_F l}\ln\dfrac{\tau_{\mathrm{in}}(T)}{\tau}\right], & d=2 \qquad (14a) \\[2ex] \dfrac{\alpha_{0t}\omega^2}{15}\left[1-\dfrac{1}{\pi k_F l}\left[\dfrac{\tau}{\tau_{\mathrm{in}}(T)}\right]^{1/2}\right], & d=3 \ . \qquad (14b) \end{cases}$$

Formulas (14a) and (14b) are appropriate when $\omega\tau_{\mathrm{in}}(T) \ll 1$. The lifetime $\tau_{\mathrm{in}}(T)$ is the particle-particle propagator lifetime that scales like $\tau_{\mathrm{in}}(T) \sim 1/T^p$. In three-dimensions, $p=\frac{3}{2}$ for electron-electron interaction.

The behavior near the mobility edge can be calculated in an $\epsilon = d-2$ expansion. Assuming that the ultrasonic attenuation at the mobility edge scales, one can exponentiate the series in Eq. (13a).

The mobility edge in $2+\epsilon$ dimension occurs[8] when

$$1/2\pi k_F l=\epsilon/2 \ .$$

Then

$$\alpha_t=(\alpha_{0t}/8)\omega^{2-\delta} \tag{15a}$$

at the mobility edge, with

$$\delta=\epsilon/2+O(\epsilon^2) \ . \tag{15b}$$

Therefore, close to the mobility edge, the frequency dependence of the ultrasonic attenuation will cross over from an ω^2 dependence at high frequencies to an $\omega^{2-\delta}$ dependence at low frequency with $\delta>0$. This reflects an enhancement of the absorption due to localization effects.

At nonzero temperatures such that $\tau_{\text{in}}\omega \ll 1$ and $l_{\text{in}} \ll \xi_{\text{loc}}$, our theory predicts a characteristic change in the temperature dependence of α,

$$\alpha_t \approx \alpha_{0t}\omega^2[\tau_{\text{in}}(T)]^{\delta} \ . \tag{16}$$

Explicit calculation shows that the first corrections to α_{T} due to diffusion-enhanced electron interaction vanish. Hence, α_{T} is sensitive to localization effects only. This is to be contrasted with other transport coefficients, such as the conductivity, that are comparably affected by localization and diffusion-enhanced electron-electron interactions.[8]

B. Longitudinal attenuation

In the calculation of the longitudinal attenuation, G_{ds} is nonvanishing. As Schmid has shown, the subtraction of G_{ds}^2/G_{dd} in Eq. (10) cancels the diffusion enhancement of the electron-phonon vertex. The graphs entering the calculation are shown in Fig. 4. We write

$$G_{ss}=ig(\mu)\left[\frac{p_F^2}{m\sqrt{\rho}c_l}\right]^2\left[\frac{3}{d(d+2)}+\frac{3i\omega\tau}{d(d+2)}+\frac{i\omega\tau}{d^2(Dq^2-i\omega)}\left[1+\omega\tau-\frac{3Dq^2}{d+2}\right]^2\right] , \tag{17a}$$

$$\frac{G_{ds}^2}{G_{dd}}=ig(\mu)\left[\frac{p_F^2}{m\sqrt{\rho}c_l}\right]^2\frac{1}{d^2(Dq^2-i\omega)} \ . \tag{17b}$$

For small ω and q (i.e., $\omega\tau \ll 1$ and $ql \ll 1$), one finds that the contributions $\sim 1/(Dq^2-i\omega)$ in (17a) and (17b) cancel one another, giving the classical result for the longitudinal attenuation,

$$\alpha_l=\frac{\omega^2\tau g(\mu)}{c_\rho}\left[\frac{p_F^2}{m^*\sqrt{\rho(\mu)}c_{\text{L}}}\right]^2\frac{2(d-1)}{d^2(d+2)} \ . \tag{18}$$

Thus, as pointed out by Schmid, there is no diffusion enhancement of the electron-phonon vertex or of the attenuation, a consequence of relaxation in the moving frame and of perfect screening.

To calculate localization corrections to the longitudinal attenuation coefficient, one must calculate only diagrams with tensor vertices. The contribution of these diagrams to G_{sd} and G_{dd} turns out to be nonsingular in the limit $q \to 0$, i.e., of relative order q^2. The relevant diagrams are shown in Fig. 5.

The result at zero temperature is

$$\alpha_l=\begin{cases}\alpha_{0l}\omega^2\dfrac{2}{45}\left[1+\dfrac{1}{2\pi k_F l}\ln\dfrac{1}{\omega\tau}\right], & d=2 \qquad (19a)\\[2ex] \alpha_{0l}\omega^2\dfrac{2}{45}\left[1+\dfrac{1}{\pi k_F l}\sqrt{\omega\tau}\right], & d=3 \ . \qquad (19b)\end{cases}$$

where

$$\alpha_{0l}=\frac{g(\mu)\tau}{c_{\text{L}}}(p_F^2/m^*\sqrt{\rho}c_{\text{L}})^2 \ .$$

At finite temperatures, ω should be replaced by τ_{in} as in Eqs. (14).

Assuming scale invariance at the mobility edge, $1/2\pi k_F l=\epsilon/2$, one can calculate the scaling of α_{L} as a function of frequency in a $2+\epsilon$ expansion,

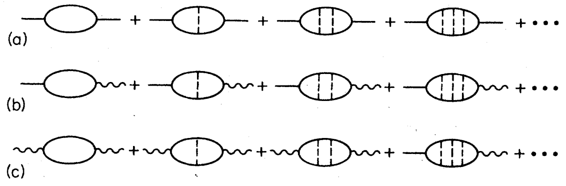

FIG. 4. Graphs contributing, to lowest order in $1/k_F l$, to the longitudinal attenuation. (a) Density-density correlations G_{dd}; (b) stress-density correlation G_{ds}; (c) stress-stress correlation G_{ss}.

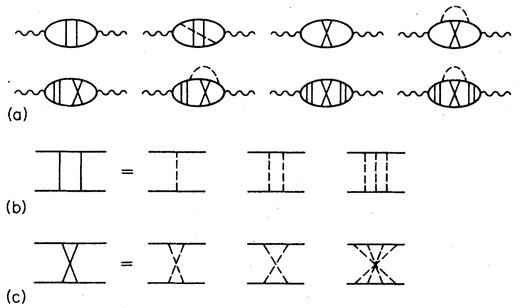

FIG. 5. (a) Diagrams of order $(1/k_F l)^2$ entering the calculation of the longitudinal attenuation. The ladder and cross-ladder resummations are defined in (b) and (c), respectively.

$$\alpha_l \simeq \omega^{2-\epsilon/2} \, , \tag{20}$$

at the mobility edge. At finite temperatures, such that $\omega\tau_{\text{in}}(T) \ll 1$,

$$\alpha_l \simeq \omega^2(\tau_{\text{in}}(T))^{\epsilon/2} \, . \tag{21}$$

Thus, within localization theory to leading order in ϵ, α_{L} and α_{T} scale with the same exponent, but a calculation shows that unlike α_{T}, α_{L} is indeed affected by electron-electron interactions.

C. Experimental consequences

Ultrasonic attenuation has been described above in two limiting cases, namely at critical disorder and when disorder is weak. Using standard scaling ideas, we describe the expected behavior in the broad, experimentally accessible intermediate regime. We consider three-dimensional systems here, extrapolating the $(2+\epsilon)$-expansion results to $\epsilon = 1$ $(d=3)$. Suppose that deviation from critical disorder r_c is characterized by $(r_c - r)/r_c$, where r is the actual disorder. [Experimentally, $(r_c - r)/r_c$ can be changed in many ways, e.g., by changing the stoichiometry or electron density, or even by applying external stress.] The system is then characterized by a correlation length $\xi \simeq l(1-r/r_c)^{-1}$, where l is the mean free path. Electron diffusion is nonclassical for smaller length scales, or, equivalently, for frequencies

$$\omega > \omega_c \sim D(\xi)\xi^{-2} = \frac{2\varepsilon_F}{3}(1/k_F l)(1-r/r_c)^3 \, .$$

The regime of classical diffusion $(\omega < \omega_c)$ shrinks rapidly to zero as $r \rightarrow r_c$. Correspondingly, at $T=0$, ultrasonic attenuation has the critical form of Eq. (15) for $\omega > \omega_c$, and crosses over for $\omega < \omega_c$ to a Pippard-like behavior, i.e.,

$$\alpha_t(\omega) \begin{cases} \simeq [\alpha_{0t}/d(d+2)]\omega^2(\omega_c\tau)^{-1/2} \ \text{ for } \omega \ll \omega_c \, , & \text{(22a)} \\ \simeq [\alpha_{0t}/d(d+2)]\omega^2(1-r/r_c)^{-3/2} \ \text{ for } \omega \ll \omega_c \, . & \text{(22b)} \end{cases}$$

At nonzero temperatures, inelastic collisions characterized by a time $\tau_{\text{in}}(T)$ or a rate $\Gamma_{\text{in}}(T)$ terminate scale-dependent diffusion, so that, effectively, ω, in Eqs. (15) and (22), is ω or $\Gamma_{\text{in}}(T)$, whichever is larger. A variety of behavior is possible as a function of the three frequency variables ω, ω_c, and $\Gamma_{\text{in}}(T)$. Experimentally, the directly accessible quantities are ω, the disorder r, and the temperature T. The latter two affect ω_c and $\Gamma_{\text{in}}(T)$, respectively. The natural scales of these frequencies are very different. Phonon frequencies of interest here are rather low, $\omega \sim \omega_D \sqrt{m/M}$, so that $\omega \gtrsim Dq^2$. The inelastic width Γ_{in} depends on the dominant scattering mechanism; for example, for electron-electron collisions, $\Gamma_{\text{in}} \sim \varepsilon_F(T/\varepsilon_F)^{3/2}$ in three dimensions. At sufficiently low temperatures (e.g., $T \lesssim 1$ K), ω can become larger than Γ_{in}. Thus, the $\omega, \Gamma_{\text{in}}$ crossover is achieved by cooling. The disorder frequency scale $\omega_c \sim \varepsilon_F(1-r/r_c)^3$is characterized by an electronic energy, but close to critical disorder, one can have $\omega \gtrsim \omega_c$. This is the critical regime.

In the critical regime where $\max(\omega, \Gamma_{\text{in}}) > \omega_c$, the attenuation is strongly enhanced. At very low temperatures where $\omega > \Gamma_{\text{in}}$, the attenuation varies as $\omega^{3/2}$ [Eq. (15)]. Upon increasing the temperature so that $\omega \lesssim \Gamma_{\text{in}}$, the Pippard form ω^2 is expected, but with a large temperature-dependent coefficient $\Gamma_{\text{in}}(T)^{-1/2}$. Outside the critical regime, but for strong disorder, i.e., for $\omega < \omega_c \ll \varepsilon_F$, the attenuation varies as ω^2 at zero temperature, but with a large coefficient [Eq. (22)]. As temperature increases, the attenuation decreases from this value as $\Gamma_{\text{in}}(T)^{1/2}$. Finally, for weak disorder, one has the forms (13b) and (14b).

There are presently no experiments on ultrasonic attenuation in strongly disordered three-dimensional metallic systems close to critical disorder, and it would therefore be of interest to verify the predictions made here of strong temperature dependence and characteristic large enhancement in the ultrasonic attenuation, particularly for shear waves (which are unaffected by interaction effects to leading order).

Sound propagation in two classes of disordered metallic systems, namely metallic glasses and doped degenerate semiconductors, has been studied for a long time, and attenuation anomalies are known to exist in them. However, for different reasons, neither is a good example of the type discussed above. Metallic glasses are not sufficiently disordered to be close to the metal-insulator transition, and furthermore, effects due to tunneling or two-level states coupled to phonons and to electron-hole excitations complicate the picture. In doped semiconductors, e.g., Ge or Si doped with shallow-level impurities, the most important absorption mechanism is due to lifting of valley degeneracy by applied shear stress.[9] This changes the relative population of electrons in different valleys. Intervalley electron scattering relaxes this disbalance and leads to sound absorption. This mechanism can be described by a term coupling external shear strain to the electron-number difference $\delta n_i - \delta n_j$, where i and j are valley indices. The symmetry dependence of ultrasonic attenuation (e.g., only c_{44} is affected by Ge) confirms the dominance of this mechanism in heavily doped semiconductors. The coupling of external strain and electronic stress fluctuations discussed in this paper is an additional intravalley term. It is much smaller than the intervalley term because of its strong dependence on electron density [from Eq. (12), we see that it varies as $n^{5/3}$], whereas the intervalley term varies as $n^{1/3}$. The extra $n^{4/3}$ comes from the electron-phonon coupling, which varies as $p_F^2 \sim n^{2/3}$, the attenuation being proportional to the square of this coupling. Experimentally, Sb-doped Ge shows an anomalous increase in attenuation near the metal-insulator transition, especially as temperature is lowered.[10] This could be due to localization influencing intervalley relaxation.

III. ATTENUATION IN THE INSULATING REGIME

A good physical picture of the electronic states on the insulating side of the mobility edge is provided by Mott's[6] resonant two-level system. With slight modifications,[7,11] it can be extended to include the effect of the intersite long-range Coulomb repulsion responsible for the

Coulomb pseudogap. In this section we calculate the contribution of these electronic two-level systems to the absorption of the ultrasound. The theory is, in many ways, similar to the standard theory of ultrasonic attenuation in glasses,[1,2] but we stress the characteristic dependence on the localization length and the sensitivity to the Coulomb pseudogap. A similar picture was used by Bhatt and Ramakrishnan[12] to analyze the scaling of the frequency-dependent conductivity on the insulating side near the mobility edge.

The electron-phonon coupling in the moving frame [Eq. (2)] can be expressed in terms of creation and annihilation operators for single-particle energy eigenstates,

$$H_{e\text{-ph}}=\sum_{\alpha,\beta}\int d\mathbf{r}\,\nabla_\alpha U_\beta(r)\tau_{\alpha\beta}\ . \tag{23}$$

The quantity $\tau_{\alpha\beta}$ is the electronic stress tensor given by

$$\tau_{\alpha\beta}(\mathbf{r})=\frac{1}{m^*}\sum_{m,n}c_m^\dagger c_n\lambda_{\alpha\beta}^{mn}(\mathbf{r})\ , \tag{24a}$$

where

$$\lambda_{\alpha\beta}^{mn}(\mathbf{r})=\tfrac{1}{4}(\nabla-\nabla')_\alpha(\nabla-\nabla')_\beta[\phi_m^*(\mathbf{r})\phi_n(\mathbf{r}')]_{\mathbf{r}=\mathbf{r}'}\ . \tag{24b}$$

The ϕ_m are the eigenstates of the one-particle Hamiltonian

$$H=(-\nabla^2/2m^*+v)\ .$$

In the pair approximation, we consider explicitly only a pair of states m and n (see Ref. 11 for a discussion of its validity). To calculate the absorption due to relaxation, we consider the diagonal part of $\tau_{\alpha\beta}(\mathbf{r})$:

$$\tau_{\alpha\beta}(\mathbf{r})=\frac{1}{2m^*}[\lambda_{\alpha\beta}^{mm}(\mathbf{r})+\lambda_{\alpha\beta}^{nn}(\mathbf{r})][c_m^\dagger c_m+c_n^\dagger c_n]+\frac{1}{2m^*}[\lambda_{\alpha\beta}^{mm}(\mathbf{r})-\lambda_{\alpha\beta}^{nn}(\mathbf{r})](c_m^\dagger c_m-c_n^\dagger c_n)\ . \tag{25}$$

In the pair approximation, $c_m^\dagger c_m+c_n^\dagger c_n=1$, so that only the second term need be kept. The electronic stress is proportional to the population difference between the pair of states. From Eqs. (23) and (25) we see that an external strain causes an additional population difference, which, in the Boltzmann transport-equation approximation, is given by

$$\langle N_m\rangle-\langle N_n\rangle=\frac{1}{m^*}iq_\alpha u_\beta\frac{\lambda_{\alpha\beta}^{mm}(\mathbf{q})-\lambda_{\alpha\beta}^{nn}(\mathbf{q})}{1+i\omega\tau_{mn}}\,\frac{1}{(k_BT)\cosh^2[(E_m-E_n)/k_BT]}\ . \tag{26}$$

Here, E_m-E_n is the energy difference between the two states and τ_{mn}^{-1} is the transition rate from one to another. This population difference leads to a change in the energy of the system. The ultrasonic attenuation can be calculated from the ratio of the energy absorbed from the field to the energy of the sound wave,

$$\alpha_{mn}=\frac{1}{\rho c_b^3}\frac{1}{(m^*)^2}\{[\lambda^{mm}(\mathbf{q})-\lambda^{nn}(\mathbf{q})]_b\}^2\left[\frac{\omega^2\tau}{1+\omega^2\tau^2}\right]\frac{1}{k_BT}\,\frac{1}{4\cosh^2[(E_m-E_n)/k_BT)]}\ , \tag{27}$$

where ρ is the mass density and c_b is the velocity of sound with polarization b. To obtain the total attenuation, one must sum over the distribution of electronic two-level systems.

The two-level systems are characterized by an energy splitting $\Omega=E_m-E_n$ and a spatial separation $r_{mn}=r$, both of which are randomly and independently distributed. The relaxation (i.e., transition from state m to state n) is due, for example, to phonon-induced electron transfer, so that

$$1/\tau_{mn}=\nu_{\text{ph}}\exp(-r_{mn}/2\xi)\ , \tag{28}$$

where ν_{ph} is the characteristic phonon frequency and ξ is the localization length, the latter assumed to be nearly the same for both states m and n. The exponentially small matrix element dominates the rate. However, the prefactor ν_{ph} could also be a sensitive function of Ω. For example, if the decay involves emission of low-frequency phonons (i.e., $E_m-E_n\ll\hbar\omega_D$), a phonon density-of-states factor $\Omega^2=(E_m-E_n)^2$ will be present in ν_{ph}.[13] The distribution function of the electronic two-level systems is given by[11,12]

$$F(\Omega,r)=\begin{cases}g^2(\mu)(\Omega+e^2/\kappa r), & \text{for } e^2/\kappa r\gg\Delta\ , \quad (29\text{a})\\[4pt] \dfrac{g^2(\mu)}{\Delta^4}(\Omega+e^2/\kappa r)^{2d-1}\ , & \text{for } e^2/\kappa r\ll\Delta\ , \quad (29\text{b})\end{cases}$$

where $g(\mu)$ is the density of electronic states per unit energy per unit volume, κ the dielectric constant, and Δ the Coulomb pseudogap, i.e., $\Delta=[e^6g(\mu)/\kappa^3]^{1/2}$. Equation (29b) is valid for long hops such that $r_{mn}>e^2/\kappa\Delta$. The second term in Eq. (29a) is due to the long-range Coulomb interaction. In its absence, the density of pair states with one of them occupied and the other unoccupied is proportional to the energy difference Ω as in any noninteracting Fermi system (i.e., one must have $E_m>\mu$, $E_n<\mu$, and $E_m-E_n=\Omega$). However, with Coulomb interactions, since having both states occupied costs an extra energy $e^2/\kappa r$, the pair is single occupied as long as E_m and E_n are within $e^2/\kappa r$ of the Fermi energy. Coulomb interactions lead to a finite density of two-level systems in the limit of small energy difference Ω. Equation (29b) is the modification in the pair density of states due to the Coulomb pseudogap in the single-particle density of states, i.e., the fact that for $|E-\mu|<\Delta$,

$$g_{\text{eff}}(E)=g(\mu)[(E-\mu)^2/\Delta^2]\ .$$

Using Eq. (29) while summing over states in Eq. (27), integrating over Ω and r, and noting that the main contribution to relaxation comes from $\omega\tau(r)\simeq1$ or from the region $r=r_\omega\simeq2\xi\ln(\nu_{\text{ph}}/\omega)$, one finds, with logarithmic accuracy, that

$$\alpha(\omega)=\pi\left[\frac{\omega}{v_S}\right]\frac{1}{\rho v_S^2}\left[\frac{\xi}{2}\right]^2 g(\mu)^2\frac{e^2}{\kappa}\langle\,|\lambda^{mm}-\lambda^{nn}|_b^2\,\rangle_{\rm av}\ln\left[\frac{\nu_{\rm ph}}{\omega}\right]\quad\text{for }\frac{e^2}{\kappa\xi}\ln(\nu_{\rm ph}/\omega)\gg\Delta\ , \tag{30}$$

i.e., for

$$e\xi g(\mu)^{1/2}\kappa^{-1/2}\ln(\nu_{\rm ph}/\omega)\ll 1\ .$$

In the opposite limit, i.e., for

$$e\xi g(\mu)^{1/2}\kappa^{-1/2}\ln(\nu_{\rm ph}/\omega)\gg 1\ ,$$

one has

$$\alpha(\omega)\simeq\left[\frac{\omega}{v_S}\right]\frac{\kappa\xi}{\rho v_S^2}\langle\,|\lambda_{mm}-\lambda^{nn}|_b^2\,\rangle_{\rm av}\left[\xi\ln\left[\frac{\nu_{\rm ph}}{\omega}\right]\right]^{-3}\ . \tag{31}$$

Both the results (30) and (31) are for three-dimensional systems. The matrix elements $|\lambda^{mm}-\lambda^{nn}|_b^2$ for a strain component b are averaged over singly occupied pair states m and n with energy difference $\sim k_BT$. The most striking feature of Eqs. (30) and (31) is that the attenuation is proportional to frequency, so that the sound mode is not qualitatively well defined. In the absence of Coulomb interactions, the density of pair states goes linearly to zero with energy Ω, so that the factor $e^2/\kappa\xi\ln(\nu_{\rm ph}/\omega)$ in Eq. (30) is replaced by a typical $\Omega\sim k_BT$, and one has

$$\alpha(\omega)=\pi\left[\frac{\omega}{v_S}\right]\frac{k_BT}{\rho v_S^2}\left[\frac{\xi}{2}\right]^3 g(\mu)^2\,|\lambda^{mm}-\lambda^{nn}|_b^2\ . \tag{32}$$

Results similar to Eq. (30) for relaxational sound absorption from configurational two-level systems (TLS's) can be obtained for sufficiently low frequencies ω, i.e., for $\omega\tau\sim 1$, where $\tau^{-1}\simeq AT^3$ in the standard TLS-phonon model at low temperatures.[13] At temperatures and frequencies such that $\hbar\omega/k_BT\ll 1$ and $\omega/AT^3\sim 1$, relaxational ultrasonic attenuation has the form of Eq. (30).

We now discuss the above results. Clearly, there is a close functional similarity between electronic and configurational systems. Resonant and relaxational absorption, saturability, etc. are all similar. The electronic TLS's have a finite density because of Coulomb effects, this value [derivable from Eq. (29)] depending on single-particle density of states $g(\mu)$, localization length ξ, and Coulomb gap Δ. Their size is of order the localization length ξ times a factor logarithmic in frequency. The coupling to external strain is described by a poorly known matrix element $|\lambda^{mm}-\lambda^{nn}|_b$ which is similar to the deformation-potential coupling matrix element of the configurational TLS.

The second expression, Eq. (31), is appropriate when characteristic hopping distances are so large that the pair states lie in the Coulomb gap,

$$|E_m|,|E_n|\sim\frac{e^2}{\kappa\xi\ln(\nu_{\rm ph}/\omega)}<\Delta\ .$$

In this regime, which, according to Eq. (31a) is accessible if $g(\mu)$ is not small, $\alpha(\omega)$ depends *only* on the Coulomb energy content in a hopping volume. Explicit dependence on $g(\mu)$ drops out; the density depends mainly on localization length. For $\kappa=5$, $\ln(\nu_{\rm ph}/\omega)\simeq 3$, $\xi\simeq 10$ Å, and the density of electronic two-level systems in this limit, is 1.5×10^{20} states/eV cm^{-3} The obvious question is whether the observed characteristic low-temperature effects in glasses are due to configurational two-level systems, or to electronic ones, or to both. Simple estimates, e.g., for As_2Se_3 and homologous glassy semiconductors where $g(\mu)\sim(3–8)\times 10^{18}$ states/eV cm^{-3} and $\kappa\sim 5–6$, with localization length $\xi\sim 10$ Å, lead to an effective density of electronic two-level states which is about a factor of 100 less than $n_0\sim 10^{21}$/eV cm^{-3}, a typical number for configurational two-level systems. However, these estimates are very unreliable; for example, the states near the Fermi energy in chalcogenide semiconductors are close pairs (occupied as well as unoccupied). Not much is known about their density, size, and relaxation. It is however, clear that when $g(\mu)$ is sizable, e.g., in the range $10^{20}–10^{21}$ states/eV cm^{-3}, this mechanism can be very important. Not far from the mobility edge, even in amorphous semiconductors, one has $g(\mu)$ of this order, so that such systems and alloys like $Nb_{1-x}Si_x$ which have a metal-insulator transition would be good candidates for looking at electronic two-level—system effects on attenuation. The divergence of the ac conductivity prefactor[12] is a signature of Coulomb-interaction-dominated, electronic two-level systems, so that simultaneous measurements of the ac conductivity and ultrasonic attenuation (frequency dependence and size) can indicate clearly if this new kind of two-level system is a significant low-lying excitation in glassy insulators.

ACKNOWLEDGMENTS

We are thankful to E. Abrahams for discussions and to H. Fukuyama for pointing out Ref. 10 to us. One of us (T.V.R.) is thankful to colleagues at the Institute for Theoretical Physics (Santa Barbara), where this work was done, for their hospitality during his stay there. This material is based upon research supported by the National Science Foundation under Grant No. PHY-77-27084, supplemented by funds from the National Aeronautics and Space Administration.

---

[1] See, for example, *Amorphous Solids*, edited by W. A. Phillips (Springer, New York, 1981), Chaps. 6 and 7; *Metallic Glasses*, edited by H. J. Güntherodt (Springer, New York, 1981).

[2] (a) P. W. Anderson, B. I. Halperin, and C. M. Varma, Philos. Mag. **25**, 1 (1972); (b) W. A. Phillips, J. Low Temp. Phys. **7**, 351 (1972).

[3] A. B. Pippard, Philos. Mag. **46**, 1104 (1955).

[4] T. Tsuneto, Phys. Rev. **121**, 402 (1960).

[5] A. Schmid, Z. Phys. **59**, 421 (1973).

[6] See, for example, N. F. Mott and E. A. Davis, *Electronic Pro-*

cesses in Noncrystalline Materials (Clarendon, Oxford, 1979).

[7] A. L. Efros and B. I. Shklovskii, J. Phys. C **8**, L49 (1975).

[8] See, for example, P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. (to be published).

[9] R. W. Keyes, IBM J. Res. Dev. **5**, 266 (1961).

[10] H. Sakurai and K. Suzuki, J. Phys. Soc. Jpn. **52**, 4192 (1983).

[11] A. L. Efros, Philos. Mag. B **43**, 29 (1981).

[12] R. N. Bhatt and T. V. Ramakrishnan J. Phys. C **17**, L639 (1984).

[13] These considerations are well known in the theory of configurational two-level systems, where the transition rate $m \rightarrow n$ due to phonons was first calculated by J. Jäckle [Z. Phys. **357**, 212 (1972)], and is known to be $(\pi^3 n_0 \langle m^2 D^2 \rangle / 32\rho^2 c_b^3 \hbar^4) T^3$ in the limit $\hbar\omega \ll k_B T$, where N_0 is the density of TLS's and mD is the transition matrix element.