Optical absorption by a small conducting sphere: Bulk magnetoplasmons

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The absorption of electromagnetic radiation at the plasma frequency ω_p is calculated in the Rayleigh limit for a spherical, lossless plasma in a weak, uniform magnetic field $\mathbf{B}_0$. In the quasistatic approximation for the electromagnetic fields produced by the free oscillations of the plasma in the sphere, the general form of the Hamiltonian is derived. The plasma eigenmodes with a nonvanishing space-charge density are analyzed and only one of them is found to interact with the external fields. This mode, characterized by spherically symmetric space- and surface-charge densities, can be excited only by the ac magnetic field parallel to $\mathbf{B}_0$.

I. INTRODUCTION

The oscillations of a free-carrier magnetoplasma in small (semi)conducting samples have been extensively studied, both theoretically and experimentally (see, e.g., the references in our previous paper[1]).

In our previous paper[1] (hereafter denoted I) we examined the eigenmodes of the magnetoplasma in a small conduting sphere (or ellipsoid). Then we calculated the absorption of the external electromagnetic field due to interaction with these modes. We have concentrated on the case of the modes with vanishing (exactly or up to first order in the cyclotron- to plasma-frequency ratio ω_c/ω_p) space-charge density. However, the eigenmode analysis in I revealed that there exist oscillations with arbitrary space-charge density (inside the sphere), whose frequency, up to first order in ω_c/ω_p, equals ω_p. The analysis of these "bulk magnetoplasmons" and their interaction with the external electromagnetic field is the purpose of the present paper.

The resonance which we consider was observed in electron-energy-loss experiments.[2] In the optical experiments, however, it can only be excited in the presence of the external magnetic field (see Sec. VI) and, to our knowledge, has not yet been observed. The ω_p resonance in semiconductors occurs in the infrared spectral region, and in high-mobility materials it should be observable (provided it is separated from the optical phonon's energies). In metals it could be observable in the x-ray region, since in the visible and uv range the interband transitions dominate the absorption spectrum.

We make the same basic assumptions as in I. The radius of the sphere is assumed to be small compared to the wavelength of radiation of frequency ω_p, but large compared to the Thomas-Fermi screening length. The energy band of carriers is assumed to be spherical and parabolic. The jellium model for the plasma is adopted and the damping of the carrier motion is neglected.

In Sec. II we briefly quote (after I) the basic equations for our problem. In Sec. III we examine these equations for the frequency equal to ω_p up to first order in ω_c/ω_p. Within this accuracy, the ω_p modes are degenerate. Some additional conditions which enable us to choose them are derived in Sec. IV from the analysis of the magnetoplasma Hamiltonian. In Sec. V we specify the basis for the electric potential inside the sphere. The absorption at the ω_p frequency is calculated in Sec. VI and the (unique) absorbing mode is described. In Sec. VII some higher-order corrections in ω_c/ω_p to the basis equations are analyzed.

II. BASIC EQUATIONS

We consider a sphere of radius r_0 placed in a static uniform magnetic field $\mathbf{B}_0$. The sphere is filled with an ideal electron (or hole) plasma neutralized by a uniform nonmagnetic background with dielectric constant ϵ. The carriers of charge q and effective mass m^* have equilibrium concentration n.

Studying the collective oscillations of the free plasma within the jellium model, we can express the position of an oscillating carrier (an element of the plasma) as the sum of the equilibrium position $\mathbf{r}$ and a small displacement vector $\boldsymbol{\xi}(\mathbf{r},t)$ which is a continuous function of $\mathbf{r}$. The carriers produce the volume-charge density $\rho(\mathbf{r},t)=-4\pi nq\nabla\cdot\boldsymbol{\xi}(\mathbf{r},t)$ and the surface-charge density

$$\sigma(\mathbf{r}_0,t)=nq\hat{\mathbf{r}}_0\cdot\boldsymbol{\xi}(\mathbf{r}_0,t)\ .$$

These densities enter into two quasistatic Maxwell equations and two boundary conditions (at the surface of the sphere, i.e., for $\mathbf{r}=\mathbf{r}_0$, $\hat{\mathbf{r}}_0\equiv\mathbf{r}_0/r_0$) for the self-consistent electric field. Looking for the normal oscillations of the plasma, we assume a harmonic time dependence of the fields and displacements [e.g., $\boldsymbol{\xi}(\mathbf{r},t)=\boldsymbol{\xi}(\mathbf{r})e^{-i\omega t}$], and then these four equations reduce to the following closed system (see paper I):

$$\nabla^2\Phi^+(\mathbf{r})=0 , \tag{1}$$

$$(\omega^2-\omega_p^2-\omega_c^2)\nabla^2\Phi^-(\mathbf{r})+(\omega_p\omega_c/\omega)^2\frac{\partial^2\Phi^-(\mathbf{r})}{\partial z^2}=0 , \tag{2}$$

$$\Phi^+(\mathbf{r}_0)=\Phi^-(\mathbf{r}_0) , \tag{3}$$

$$(\omega^2-\omega_c^2)\mathbf{r}_0\cdot\nabla\Phi^+(\mathbf{r}_0)=\epsilon(\omega^2-\omega_p^2-\omega_c^2)\mathbf{r}_0\cdot\nabla\Phi^-(\mathbf{r}_0)+\epsilon(\omega_p\omega_c/\omega)^2\left[z\frac{\partial\Phi^-(\mathbf{r})}{\partial z}\right]\Bigg|_{\mathbf{r}=\mathbf{r}_0} + i\epsilon(\omega_p^2\omega_c/\omega)\left[y\frac{\partial\Phi^-(\mathbf{r})}{\partial x}-x\frac{\partial\Phi^-(\mathbf{r})}{\partial y}\right]\Bigg|_{\mathbf{r}=\mathbf{r}_0} . \tag{4}$$

In the derivation, the solution of linearized equation of a carrier motion (for $\omega\neq\omega_c$),

$$\boldsymbol{\xi}(\mathbf{r})=-\frac{q}{m^*(\omega^2-\omega_c^2)}\left[\nabla\phi^-(\mathbf{r})-i\frac{\omega_c}{\omega}\hat{\mathbf{z}}\times\nabla\phi^-(\mathbf{r})-\frac{\omega_c^2}{\omega^2}\hat{\mathbf{z}}[\hat{\mathbf{z}}\cdot\nabla\phi^-(\mathbf{r})]\right] , \tag{5}$$

was used. Here, $\phi^+(\mathbf{r})$ and $\phi^-(\mathbf{r})$ denote the (complex) electric-field-potential amplitudes inside and outside the sphere, respectively. $\mathbf{B}_0$ was chosen parallel to the z axis, $\omega_c=(qB_0/m^*c)$ and $\omega_p=(4\pi nq^2/m^*\epsilon)^{1/2}$.

The set of equations (1)—(4) determine the eigenfrequencies and the potentials. In our previous paper we considered the solutions with the space-charge density vanishing exactly or up to first order in ω_c/ω_p. In these cases Eq. (2) reduces to $\Delta\Phi^-=0$ and in the limit of the vanishing magnetic field $\mathbf{B}_0$ the frequencies tend either to zero (heliconlike modes) or to

$$\omega_l=[4\pi nq^2l/m^*(\epsilon l+l+1)]^{1/2},\quad l=1,2,\ldots$$

(surface magnetoplasmons).

The remaining class of solutions of Eqs. (1)—(4), i.e., the oscillations with arbitrary space-charge density, involves (see I) the modes with $\omega\to\omega_p$ when $\mathbf{B}_0\to 0$ (bulk magnetoplasmons). They will be studied in the following section in the case of a weak magnetic field $\mathbf{B}_0$, i.e., for $|\omega_c|/\omega_p\ll 1$.

III. BULK MAGNETOPLASMONS

Up to first order in ω_c/ω_p we can look for the solutions of Eqs. (1)—(4) in the form

$$\omega=\omega_p+\omega_1 , \tag{6}$$

$$\Phi^\pm(\mathbf{r})=\Phi_0^\pm(\mathbf{r})+\Phi_1^\pm(\mathbf{r}) , \tag{7}$$

where the $\Phi_0^\pm(\mathbf{r})$ are the potentials in the absence of the magnetic field, while ω_1 and $\Phi_1^\pm(\mathbf{r})$ are the corrections of first order in ω_c/ω_p. Inserting these expansions into the set (1)—(4), we obtain to zeroth order from (1) and (4),

$$\Phi_0^+(\mathbf{r})\equiv 0 , \tag{8}$$

and then, from (3),

$$\Phi_0^-(\mathbf{r}_0)\equiv 0 , \tag{9}$$

while $\Phi_0^-(\mathbf{r})$ inside the sphere remains arbitrary.

In the first order in ω_c/ω_p, Eqs. (1)—(4) yield, respectively,

$$\nabla^2\Phi_1^+(\mathbf{r})=0 , \tag{10}$$

$$\omega_1\nabla^2\Phi_0^-(\mathbf{r})=0 , \tag{11}$$

$$\Phi_1^+(\mathbf{r}_0)=\Phi_1^-(\mathbf{r}_0) , \tag{12}$$

$$r_0\frac{\partial\Phi_1^+(\mathbf{r})}{\partial r}\Bigg|_{\mathbf{r}=\mathbf{r}_0}=2r_0\epsilon\left[\frac{\omega_1}{\omega_p}\right]\frac{\partial\Phi_0^-(\mathbf{r})}{\partial r}\Bigg|_{\mathbf{r}=\mathbf{r}_0} - i\epsilon\left[\frac{\omega_c}{\omega_p}\right]\frac{\partial\Phi_0^-(\mathbf{r})}{\partial\phi}\Bigg|_{\mathbf{r}=\mathbf{r}_0} , \tag{13}$$

where the spherical coordinates were introduced. In the case $\omega_1\neq 0$ we obtain the Laplace equation for Φ_0^-, which, together with (9), implies that $\Phi_0^-(\mathbf{r})=0$. Owing to the linearity and homogeneity of the system (1)—(4), this means that $\Phi^-(\mathbf{r})\equiv 0$. Therefore there are no nontrivial solutions with $\omega_1\neq 0$. In the case $\omega_1=0$, we obtain, from (10)—(13), that $\Phi_1^\pm(\mathbf{r})$ fulfills the same conditions as $\Phi_0^\pm$, so that up to first order the total potential vanishes outside the sphere and on its surface, being arbitrary inside the sphere.

Some additional conditions for Φ_0^- and Φ_1^- can be obtained from Eq. (2) to second and third order in ω_c/ω_p, but we shall postpone discussion of them until Sec. VII. Up to first order in ω_c/ω_p, we deal with a degenerate case, i.e., all eigenmodes have the same frequency ω_p and arbitrary potential $\Phi^-(\mathbf{r})$ vanishing on the surface of the sphere.

The real (denoted by the subscript R), time-dependent potential for the considered oscillations can be written as

$$\Phi_R^-(\mathbf{r},t)=\tfrac{1}{2}\left[\sum_k\Phi_k(\mathbf{r})b_k(t)+\text{c.c.}\right] , \tag{14}$$

where $\Phi_k(\mathbf{r})$ form a complete set of smooth[3] functions defined inside the sphere and vanishing on its surface,

$$\Phi_k(\mathbf{r}_0)\equiv 0 , \tag{15}$$

while $b_k(t)=b_ke^{-i\omega_pt}$ are the time-dependent, complex amplitudes. According to Eq. (5) the displacement corresponding to the potential $\Phi_R^-(\mathbf{r},t)$ has the form, up to first order in ω_c/ω_p,

$$\xi_R(\mathbf{r},t)=\frac{1}{2}\left[\sum_k \frac{q}{m^*\omega_p^2}\left\{\nabla - i\frac{\omega_c}{\omega_p}\hat{\mathbf{z}}\times\nabla\right\}\times\Phi_k(\mathbf{r})b_k(t)+\text{c.c.}\right] . \quad (16)$$

If the functions $\Phi_k(\mathbf{r})$ are to describe different modes, they should satisfy some orthogonality conditions. We shall find these conditions by studying the Hamiltonian of the free (unperturbed) plasma.

IV. HAMILTONIAN

We shall now derive the general form of the Hamiltonian of the spherical magnetoplasma in the quasistatic approximation. The Lagrangian of the free plasma placed in the uniform magnetic field is

$$L_0=T-U+(nqB_0/2c)\times\int_{r\le r_0} d^3r\{\hat{\mathbf{z}}\times[\mathbf{r}+\xi_R(\mathbf{r},t)]\}\cdot\dot{\xi}_R(\mathbf{r},t) , \quad (17)$$

with kinetic energy

$$T=(nm^*/2)\int_{r\le r_0} d^3r\,\dot{\xi}_R^2(\mathbf{r},t) \quad (18)$$

and potential energy

$$U=(nq/2)\int_{r\le r_0} d^3r[\Phi_R^-(\mathbf{r}+\xi_R(\mathbf{r},t),t)-\Phi_R^-(\mathbf{r},t)] \cong (nq/2)\int_{r\le r_0} d^3r\,\xi_R(\mathbf{r},t)\cdot\nabla\Phi_R^-(\mathbf{r},t) . \quad (19)$$

We subtracted here the contribution of the positive background. In the second part of Eq. (19) only the terms up to second order in ξ_R were retained. The potential $\Phi_R^-(\mathbf{r},t)$ in the potential energy U should be expressed in terms of $\xi_R(\mathbf{r},t)$. This will be derived below for an arbitrary carrier displacement, using only the Maxwell equations.

The quasistatic approximation allows us to introduce the complex electric potentials $\Phi^{\pm}(\mathbf{r},t)$ which satisfy the Poisson equation

$$\nabla^2\Phi^-(\mathbf{r},t)=\frac{4\pi qn}{\epsilon}\nabla\cdot\xi(\mathbf{r},t) , \quad (20)$$

$$\nabla^2\Phi^+(\mathbf{r},t)=0 , \quad (21)$$

with the boundary conditions on the surface of the sphere

$$\Phi^+(\mathbf{r}_0,t)=\Phi^-(\mathbf{r}_0,t) , \quad (22)$$

$$\left.\frac{\partial\Phi^+(\mathbf{r},t)}{\partial r}\right|_{\mathbf{r}=\mathbf{r}_0}=\epsilon\left.\frac{\partial\Phi^-(\mathbf{r},t)}{\partial r}\right|_{\mathbf{r}=\mathbf{r}_0}-4\pi nq\hat{\mathbf{r}}_0\cdot\xi(\mathbf{r}_0,t) . \quad (23)$$

The general regular solutions of Eqs. (20) and (21) can be written as

$$\Phi^-(\mathbf{r},t)=-\frac{nq}{\epsilon}\int_{r'\le r_0} d^3r'\frac{\nabla'\cdot\xi(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}+\frac{q}{r_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}C_{lm}^-(t)(r/r_0)^l Y_{lm}(\theta,\phi) , \quad (24)$$

$$\Phi^+(\mathbf{r},t)=\frac{q}{r_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}C_{lm}^+(t)(r/r_0)^{-l-1}Y_{lm}(\theta,\phi) , \quad (25)$$

where the $C_{lm}^{\pm}(t)$ are the complex amplitudes and the Y_{lm} are the orthonormalized spherical harmonics. Inserting expressions (24) and (25) into Eqs. (22) and (23), we can express $C_{lm}^{\pm}(t)$ in terms of ξ and its derivatives. Using the well-known formula

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) , \quad (26)$$

where $r_<$ ($r_>$) denote the smaller (larger) of the lengths r and r', and noting that

$$\left.\left[\frac{\partial}{\partial r}\int_{r'\le r_0} d^3r'\,\xi(\mathbf{r}',t)\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]\right|_{\mathbf{r}=\mathbf{r}_0}=4\pi\hat{\mathbf{r}}_0\cdot\xi(\mathbf{r}_0,t)-4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l+1}{2l+1}r_0^{-l-2}Y_{lm}(\theta,\phi)\int_{r'\le r_0} d^3r'\,\xi(\mathbf{r}',t)\cdot\nabla'[r'^l Y_{lm}^*(\theta',\phi')] , \quad (27)$$

we obtain (after some manipulation) $C_{lm}^{\pm}(t)$ expressed only in terms of the displacement field $\xi(\mathbf{r},t)$ as

$$C_{lm}^-(t)=\frac{4\pi nr_0}{(2l+1)\epsilon}\left[\int d\Omega\,\mathbf{r}_0\cdot\xi(\mathbf{r}_0,t)Y_{lm}^*(\theta,\phi)+\frac{(l+1)(\epsilon-1)}{(l+1+\epsilon l)r_0^{l+1}}\int_{r\le r_0} d^3r\,\xi(\mathbf{r},t)\cdot\nabla[r^l Y_{lm}^*(\theta,\phi)]\right] . \quad (28)$$

We omit the expression for $C_{lm}^+(t)$ since it is not necessary for further calculation. Inserting Eq. (28) into Eq. (24) and taking the real part of the potential, we obtain the desired result,

$$\Phi_R^-(\mathbf{r},t)=\frac{nq}{\epsilon}\int_{r'\le r_0} d^3r'\,\xi_R(\mathbf{r}',t)\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}+\frac{4\pi nq(\epsilon-1)}{\epsilon}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l+1)}{(2l+1)(l+1+\epsilon l)}\frac{r^l}{r_0^{2l+1}}Y_{lm}(\theta,\phi)\int_{r'\le r_0} d^3r'\,\xi_R(\mathbf{r}',t)\cdot\nabla'[r'^l Y_{lm}^*(\theta',\phi')] , \quad (29)$$

where expansion (26) was used. Now the Hamiltonian can be derived from L_0 in a standard way (with the replacement of the derivatives by the functional derivatives) with the use of Eqs. (17)–(19). This leads to

$$H_0=\frac{n}{2m^*}\int_{r\le r_0}d^3r\left[\mathbf{p}_R(\mathbf{r},t)-\frac{m^*\omega_c}{2}\hat{\mathbf{z}}\times[\mathbf{r}+\boldsymbol{\xi}_R(\mathbf{r},t)]\right]^2+\frac{nm^*\omega_p^2}{8\pi}\int_{r\le r_0}d^3r\int_{r'\le r_0}d^3r'[\boldsymbol{\xi}_R(\mathbf{r},t)\cdot\nabla][\boldsymbol{\xi}_R(\mathbf{r}',t)\cdot\nabla']\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$
$$+\frac{nm^*\omega_p^2}{2}(\epsilon-1)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l+1}{(2l+1)(l+1+\epsilon l)r_0^{2l+1}}\left|\int_{r\le r_0}d^3r\,\boldsymbol{\xi}_R(\mathbf{r},t)\cdot\nabla[r^lY_{lm}(\theta,\phi)]\right|^2, \tag{30}$$

where $\mathbf{p}_R(\mathbf{r},t)$ is the canonical momentum conjugate to the displacement $\boldsymbol{\xi}_R(\mathbf{r},t)$, and the last two terms represent the potential energy U. The Hamilton equations for H_0 lead to the equation of motion consistent with Eq. (1) of I.

The general form of H_0 describes, in particular, the vanishing-space-charge modes analyzed in I, i.e., for $\nabla\cdot\boldsymbol{\xi}=0$ expression (30) can be easily reduced to formula (46) of I.

Let us now restrict ourselves to the case of the considered ω_p oscillations for which the condition $\Phi_R^+(\mathbf{r},t)\equiv 0$ is satisfied. This condition implies that $\Phi_R^-(\mathbf{r}_0,t)\equiv 0$, which, with the use of Eq. (29), turns out to be equivalent to

$$\int_{r\le r_0}d^3r\,\boldsymbol{\xi}_R(\mathbf{r},t)\cdot\nabla[r^lY_{lm}(\theta,\phi)]=0\ . \tag{31}$$

Here, again, expansion (26) was used. This restriction, imposed on the previously arbitrary displacement $\boldsymbol{\xi}(\mathbf{r},t)$, leads to the vanishing of the last term in the Hamiltonian H_0.

At this point we return to Eq. (16), which will now be treated as a transformation to new dynamical *variables* $b_k(t)$. Inserting Eq. (16) and the corresponding transformation for the momentum,

$$\mathbf{p}_R(\mathbf{r},t)=\frac{m^*\omega_c}{4}\hat{\mathbf{z}}\times\mathbf{r}-\frac{q}{2\omega_p}\sum_k\left[i\nabla+\frac{\omega_c}{2\omega_p}\hat{\mathbf{z}}\times\nabla\right]\Phi_k(\mathbf{r})b_k(t)+\text{c.c.}\ , \tag{32}$$

into Eq. (30) (with the last term dropped), we obtain H_0 expressed in terms of $b_k(t)$. Demanding that $b_k(t)$ represent the normal coordinates of the free plasma, i.e., demanding that H_0 assume the form

$$H_0=\sum_k\hbar\omega_p b_k^*(t)b_k(t)\ , \tag{33}$$

we obtain the desired orthogonality conditions for the $\Phi_k(\mathbf{r})$ functions

$$(\Phi_k\,|\,\Phi_{k'})\equiv\frac{\epsilon}{8\pi\hbar\omega_p}\int_{r\le r_0}d^3r\left[\nabla\Phi_k^*\cdot\nabla\Phi_{k'}-\frac{3i\omega_c}{2\omega_p}\nabla\Phi_k^*\cdot(\hat{\mathbf{z}}\times\nabla\Phi_{k'})\right]=\delta_{kk'}\ . \tag{34}$$

This equation holds within ω_c/ω_p accuracy. One can check, that for arbitrary smooth functions vanishing on the surface of the sphere, $(\Phi_k\,|\,\Phi_{k'})$ has all the properties of the scalar product, if $|\omega_c|/\omega_p\le\frac{2}{3}$. As we assume $|\omega_c|/\omega_p\ll 1$, this condition is obviously fulfilled.

The Φ_k basis, orthonormal in the sense of the scalar product (34), can be chosen in various ways. Some indication for the convenient choice of this basis can be obtained from the analysis of the interaction Hamiltonian.

Let us therefore consider our system placed in a weak electromagnetic radiation field oscillating with a frequency ω_r. The vector potential inside the sphere is

$$\mathbf{A}(\mathbf{r},t)=\mathbf{A}(\mathbf{r})\exp(-i\omega_r t)\ .$$

Choosing the vanishing–scalar-potential gauge for the radiation field, we obtain the full Hamiltonian in the form

$$H=H_0+H_1\ , \tag{35}$$

with the interaction Hamiltonian

$$H_1=-\frac{nq}{m^*c}\int_{r\le r_0}d^3r\{\mathbf{p}_R(\mathbf{r},t)+(m^*\omega_c/2)[\mathbf{r}+\boldsymbol{\xi}_R(\mathbf{r},t)]\times\hat{\mathbf{z}}\}\cdot\mathrm{Re}[\mathbf{A}(\mathbf{r})e^{-i\omega_r t}]\ , \tag{36}$$

where the terms quadratic in $\mathbf{A}$ were dropped as they describe the photon scattering, and not the (one-photon) absorption.[4] In Eq. (36) the term of order $(\xi_R/r_0)H_1$ was neglected, as was also done in I.

We are interested in frequencies ω_r close to the frequency of the modes ω_p. Therefore, due to the quasistatic condition $(r_0\omega_p/c)^2\ll 1$ [see Eq. (7) of I], the vector potential $\mathbf{A}(\mathbf{r})$ is slowly varying on the r_0 distance and can be expanded around the center of the sphere. As the Hamiltonian H_0 is correct up to first order in $r_0\omega_p/c$, H_1 can only be written with the same accuracy. Thus, expanding $\mathbf{A}(\mathbf{r})$ in Eq. (36), we must retain only the terms up to first order in $r_0\omega_r/c$. The coefficients of this expansion can be expressed by the radiation fields in the absence of the sphere: $\mathbf{E}^r(\mathbf{r})$ and $\mathbf{B}^r(\mathbf{r})$. We obtain (see I)

$$A_\mu(\mathbf{r}) = -\frac{3}{\epsilon+2}\frac{ic}{\omega_r}E^r_\mu(\mathbf{0}) - \tfrac{1}{2}\varepsilon_{\mu\nu\kappa}r_\nu B^r_\kappa(\mathbf{0}) - \frac{5}{2\epsilon+3}\frac{ic}{2\omega}[E^r_{\mu,\nu}(\mathbf{0})+E^r_{\nu,\mu}(\mathbf{0})]r_\nu \;, \tag{37}$$

where $\varepsilon_{\mu\nu\kappa}$ is the totally antisymmetric Levi-Civitá tensor and the comma denotes differentiation.

Now transformations (16) and (32) can be inserted into Hamiltonian (35), which for H_0 obviously leads to expression (33), while for H_1, after a simple calculation, we obtain

$$H_1 = -\frac{\epsilon\omega_c}{8\pi c}\mathrm{Re}[B^r_z(\mathbf{0})e^{-i\omega_r t}] \times \sum_k \left[\int_{r\le r_0} d^3r\,\Phi_k(\mathbf{r})\right] b_k(t) + \text{c.c.} \tag{38}$$

Condition (15) and Eq. (37) were used here.

Formula (38) for H_1 indicates that the only ω_p modes that can be excited by the external electromagnetic field are those with

$$\int_{r\le r_0} d^3r\,\Phi_k(\mathbf{r}) \neq 0 \;. \tag{39}$$

Searching for the most convenient choice of the Φ_k basis, we shall therefore try to minimize the number of the basis functions satisfying condition (39).

It should be noted that transformations (16) and (32) are, in general, incomplete, as only some normal coordinates $b_k(t)$ appear in these expressions, namely those corresponding to the ω_p modes. However, we observe that the utilization of the full transformation [containing, e.g., $b_{lm\tau}(t)$ corresponding to the $\omega_{lm\tau}$ modes of I] would lead to some additional terms in H_0 as well as in H_1, independent of our variables $b_k(t)$. These additive terms in the Hamiltonian clearly do not affect the calculation of the absorption by the ω_p modes, and for this purpose use of transformations (16) and (32) is justified.

V. DETERMINATION OF THE Φ_k FUNCTIONS

We shall now specify the potentials $\Phi_k(\mathbf{r})$ continuous (together with their gradients) inside the sphere and satisfying conditions (15) and (34).

Equation (15) suggests that it is convenient to separate the radial and the angular dependence in $\Phi_k(\mathbf{r})$. In view of the "interaction condition" (39), we shall start with a complete set of linearly independent functions of the form

$$\Phi_{nlm}(\mathbf{r}) = f_{nlm}(r)Y_{lm}(\theta,\phi) \;, \tag{40}$$

where, for every l,m, the functions $f_{nlm}(r)$ ($n=0,1,2,\ldots$) form a complete set of functions defined in the interval $0\le r\le r_0$, and vanishing at $r=r_0$. We note that the only Φ_{nlm} functions satisfying (39) are those with $l=m=0$. In order to obtain the Φ_k basis, we now must orthogonalize the set (40) (e.g., with the use of the Gramm-Schmidt procedure) in the sense of the scalar product (34).

Let us start with the orthogonalization of Φ_{n00} functions, for which the condition (34) reduces to

$$\frac{r_0\epsilon}{8\pi\hbar\omega_p}\int_0^1 dx\,x^2\frac{df_n(x)}{dx}\frac{df^*_{n'}(x)}{dx} = \delta_{n,n'} \;, \tag{41}$$

where we introduced the functions $f_n(x)\equiv f_{n00}(xr_0)$ $(0\le x\le 1)$. The continuity of ∇f_{n00} at $r=0$ and condition (15) give, respectively,

$$\left.\frac{df_n(x)}{dx}\right|_{x=0} = 0 \;, \tag{42}$$

$$f_n(1) = 0 \;. \tag{43}$$

Therefore we search for a set of functions $f_n(x)$, complete in the class C of differentiable functions vanishing for $x=1$ and with derivatives which vanish at $x=0$. The derivatives $df_n(x)/dx$ should be orthogonal with the weight $w(x)=x^2$. These functions can be found among the polynomials, and their construction is the following.

Let us consider a complete set of polynomials $G_m(x)$ ($m=0,1,2,\ldots$) defined for $-1\le x\le 1$, orthogonal with the weight $w(x)=x^2$. It can be easily shown that due to $w(-x)=w(x)$, the $G_m(x)$ satisfy the following condition (see Ref. 5):

$$G_m(-x) = (-1)^m G_m(x) \;. \tag{44}$$

We observe that, choosing only the antisymmetric functions $G_m(x)$ (i.e., with odd m), we obtain the set of polynomials complete in the interval [0,1] in the class of functions vanishing at $x=0$. Obviously, the orthogonality condition on [0,1],

$$\int_0^1 dx\,x^2 G_{2n+1}(x)G_{2n'+1}(x) \propto \delta_{n,n'} \;, \tag{45}$$

will be also satisfied. Now if we define

$$f_n(x) = \int_1^x dx\,G_{2n+1}(x), \quad 0\le x\le 1 \tag{46}$$

all the conditions (41)—(43) will be fulfilled. The completeness of the $f_n(x)$ functions [defined by (46)] in the class C follows from

$$g(x) = \int_1^x dx\frac{dg(x)}{dx} = \int_1^x dx\sum_n \alpha_n G_{2n+1}(x) = \sum_n \alpha_n f_n(x) \tag{47}$$

$g(x)$ here is an arbitrary function belonging to C. In the second equality the completeness of $G_{2n+1}(x)$ in the class of functions vanishing at $x=0$ was used. Thus we only have to determine the G_m functions.

We now utilize the theorem[5] that the complete set of polynomials G_m orthogonal on the interval $[-1,1]$ with the weight x^2 can be constructed from the complete set of polynomials ϕ_n on the interval [0,1] as

$$G_{2n}[x,-1,-1,u^2] \equiv \phi_n[x^2,0,1,u^{1/2}] \;, \tag{48}$$

$$G_{2n+1}[x,-1,1,u^2] \equiv x\phi_n[x^2,0,1,u^{3/2}] \;. \tag{49}$$

The first symbol in square brackets denotes the argument, the second and the third represent the interval, and the last denotes the weight. Being interested in antisymmetric G_m, we must find only the $\phi_n[y,0,1,u^{3/2}]$ functions, which turn out to be

$$\phi_n[y,0,1,u^{3/2}]=F(-n,n+\tfrac{5}{2},\tfrac{5}{2};y)\ . \qquad (50)$$

$F(a,b,c;y)$ is the hypergeometric function that , in (50), reduces to the Jacobi polynomial (see, e.g., Ref. 6). Inserting Eqs. (49) and (50) into (46), and using the relations

$$\frac{d}{dx}F(-n,n+a,c;x)=-\frac{n(n+a)}{c}\times F(-n+1,n+a+1,c+1;x)\ , \qquad (51)$$

$$F(-n-1,n+\tfrac{3}{2},\tfrac{3}{2};1)=0\ , \qquad (52)$$

we obtain

$$f_n(x)=-\frac{3(2n+\frac{5}{2})^{1/2}\Gamma(n+\frac{5}{2})}{(n+\frac{3}{2})(n+1)!\Gamma(\frac{5}{2})}\left[\frac{\pi\hbar\omega_p}{r_0\epsilon}\right]^{1/2}\times F(-n-1,n+\tfrac{3}{2},\tfrac{3}{2};x^2)\ . \qquad (53)$$

The normalizing factor was chosen to satisfy the orthogonality condition (41). This completes the determination of the first set of the Φ_k functions, namely those equal to $f_n(r/r_0)Y_{00}$.

Now we should continue our procedure by subsequent orthogonalization of the remaining Φ_{nlm} functions, i.e., those with $l\neq 0$, to all previously determined Φ_{n00} functions. However, it is easy to check that, for $l\neq 0$,

$$(\Phi_{n00}\,|\,\Phi_{nlm})=0\ . \qquad (54)$$

Therefore, the further orthogonalization procedure will mix only the Φ_{nlm} functions with $l\neq 0$. As all these functions have a vanishing space integral (39), the Φ_k functions, which are their linear combinations, also have this property, i.e., they represent the modes which do not interact with the external radiation. Therefore, their explicit construction is not necessary for calculating the absorption.

VI. ABSORPTION

The interaction Hamiltonian H_1, Eq. (38), can now be evaluated with the use of the Φ_{n00} functions obtained in the preceding section. It follows from the orthogonality condition for the Jacobi polynomials, with the use of

$$F(-1,\tfrac{3}{2},\tfrac{3}{2};x^2)=1-x^2\ , \qquad (55)$$

that the volume integral in (38) vanishes for all n except $n=0$. Thus,

$$H_1=\frac{r_0^3\epsilon\omega_c}{15c}\left[\frac{5\hbar\omega_p}{2r_0\epsilon}\right]^{1/2}\mathrm{Re}(B_z^r e^{-i\omega_r t})b_{000}(t)+\text{c.c.} \qquad (56)$$

Therefore, out of all the Φ_{n00} modes, only that with $n=0$ interacts with the external fields.

At this point the full Hamiltonian $H=H_0+H_1$ [with H_0 given by (33)] can be quantized by replacing the amplitudes $b_k(t)$ and $b_k^*(t)$ by the annihilation and creation operators b_k and $b_k^\dagger$, respectively. They satisfy the standard commutation rules for Bose operators.

The first-order perturbation calculus in H_1 shows that absorption occurs only if the radiation frequency is equal to the plasma frequency. The corresponding net power $P(\omega_r)$ absorbed by the sphere (i.e., with stimulated emission subtracted) is given by

$$P(\omega_r)=\frac{\pi\epsilon r_0^5\omega_p^2\omega_c^2}{180c^2}\,|B_z^r|^2\delta(\omega_r-\omega_p)\ . \qquad (57)$$

This result is temperature independent due to the partial cancellation of the plasmon occupation factors for transitions with photon absorption and stimulated emission. The power does not depend on $\hbar$, which indicates that (57) could be obtained classically (cf. paper I). This formula is also consistent with the adopted jellium model because it contains only charge and mass densities and there are no individual carrier parameters.

The absorption occurs due only to the ac magnetic field component parallel to $\mathbf{B}_0$ and vanishes for $\mathbf{B}_0=0$. We also observe that the power absorbed does not depend on ϵ. For the system of noninteracting spheres of a given total volume, the power increases with r_0. Assuming that $|\omega_c|/\omega_p=\frac{1}{4}$, $\epsilon(\omega_p r_0/c)^2=\frac{1}{4}$, and $\epsilon=10$, it turns out that the power given by (57) is of the same order as the power $P_{2m\tau}$ of I.

Let us now analyze the only mode which interacts with the external electromagnetic field, i.e., the one with $n=l=m=0$. From Eq. (40), with the use of (53) and (55), we obtain the potential inside the sphere as

$$\Phi_{000}(\mathbf{r})=-\left[\frac{5\hbar\omega_p}{2r_0\epsilon}\right]^{1/2}\left[1-\left[\frac{r}{r_0}\right]^2\right]\ , \qquad (58)$$

and from Eq. (16) the corresponding displacement

$$\xi_{000}(\mathbf{r},t)=\frac{2q}{m^*\omega_p^2}\left[\frac{5\hbar\omega_p}{2r_0^3\epsilon}\right]^{1/2}\left[\frac{r}{r_0}\right]\times\left[\mathbf{e}_r-i\frac{\omega_c}{\omega_p}\sin\theta\,\mathbf{e}_\phi\right]b_{000}e^{-i\omega_p t}\ . \qquad (59)$$

Therefore the potential of the mode is spherically symmetric and the corresponding electric field is radial. The trajectory of the carrier is elliptical in the plane tangent to the cone defined by $\theta=$const.

The space-charge density turns out to be uniform inside the sphere and is compensated by the surface-charge density, which also does not depend on the position. This is consistent with the fact that the electric field outside the sphere vanishes.

The magnetic dipole moment associated with the mode considered can be also calculated. Using the formula

$$\boldsymbol{\mu}_{000}(t)=\frac{1}{2c}\int_{r\le r_0}d^3r\,\mathbf{r}\times\mathbf{j}_{000}(\mathbf{r},t)\ , \qquad (60)$$

where the linearized current density is given by

$$\mathbf{j}_{000}(\mathbf{r},t)=-inq\omega_p\xi_{000}(\mathbf{r},t)\ , \qquad (61)$$

we obtain

$$\boldsymbol{\mu}_{000}(t)=-\frac{\epsilon r_0^3\omega_c}{15c}\left[\frac{5\hbar\omega_p}{2r_0\epsilon}\right]^{1/2}\mathbf{e}_z b_{000}(t)\ . \qquad (62)$$

Thus, the magnetic moment is parallel to $\mathbf{B}_0$. We also observe that the interaction Hamiltonian can be written as

$$H_1 = -\mathrm{Re}[\boldsymbol{\mu}_{000}(t)]\cdot\mathrm{Re}(\mathbf{B}^r e^{-i\omega_r t}) \ . \quad (63)$$

Therefore, as far as the interaction with the external electromagnetic field is concerned, the mode can be represented solely by its macroscopic magnetic dipole moment.

VII. HIGHER-ORDER CORRECTIONS

So far we have searched for the solutions of the system (1)—(4) considering only the zeroth- and first-order terms in ω_c/ω_p in the expressions for the frequencies and potentials [see (6) and (7)]. Higher-order terms were neglected in our equations [see (10)—(13)]. However, if these terms are taken into account, it turns out that Eq. (2), to second and third order in ω_c/ω_p, yields some additional conditions on Φ_0^- and Φ_1^-, namely

$$(2\omega_2\omega_p - \omega_c^2)\Delta\Phi_0^- + \omega_c^2 \frac{\partial^2\Phi_0^-}{\partial z^2} = 0 \ , \quad (64)$$

$$2\omega_3\omega_p\Delta\Phi_0^- + (2\omega_2\omega_p - \omega_c^2)\Delta\Phi_1^- + \omega_c^2 \frac{\partial^2\Phi_1^-}{\partial z^2} = 0 \ . \quad (65)$$

Here, ω_2 and ω_3 are the second- and third-order corrections to the frequency ω, i.e., $\omega \cong \omega_p + \omega_2 + \omega_3$. These equations should be satisfied by the eigenmode potentials and lift the degeneracy of the ω_p modes.

One can easily check that the Φ_{000} potential (58) fulfils Eqs. (64) and (65), provided that

$$\omega_2 = \omega_c^2/3\omega_p \ , \quad (66)$$

$$\omega_3 = 0 \ . \quad (67)$$

The remaining Φ_k potentials do not, in general, satisfy these equations. In particular, none of the Φ_{n00} $(n \neq 0)$ functions satisfies them. Therefore, they may not represent the correct eigenmode oscillations. Nevertheless, any of the correct eigenmode potentials must be a linear combination of the Φ_k functions (they form a complete set). Since all Φ_k except Φ_{000} have a vanishing space integral $\int d^3r\, \Phi_{nlm}$ and, therefore, do not yield any electromagnetic absorption [see Eq. (38)], all the correct eigenmodes except Φ_{n00} do not interact with the external radiation. Thus, the fact that our Φ_k functions do not, in general, fulfill Eqs. (64) and (65), does not influence the results for the power absorption.

For the purpose of analyzing other means of plasma excitation (e.g., the characteristic-energy-loss experiments), it would be interesting to find all eigenmode potentials satisfying (64) and (65), but this would seem a difficult task.

VIII. CONCLUSIONS

Considering the magnetoplasma oscillations in a small conducting sphere in our previous paper, paper I, we restricted ourselves to the case of the vanishing space-charge density (exactly or up to first order in ω_c/ω_p). The modes were classified into two groups: surface magnetoplasmons with the frequencies ω_{lm1} $(l=1,2,\ldots, m=-l,\ldots,l)$ and the heliconlike magnetoplasmons with the frequencies ω_{lm2} $[l=2,3,\ldots;\ m=-(l-1)\mathrm{sgn}q]$ (see paper I). Each of these modes was characterized by its macroscopic electric and/or magnetic multipole moment. In the adopted quasistatic approximation, nine of these modes (those with $l=1,2$) interact with the external radiation.

In the present paper we analyzed the oscillations with, in general, nonvanishing space-charge density (bulk magnetoplasmons). Within ω_c/ω_p accuracy their frequency equals ω_p. The only mode interacting with the external fields is the Φ_{000} mode possessing a magnetic dipole moment parallel to $\mathbf{B}_0$ and giving no electric field outside the sphere. We found that within third-order accuracy in ω_c/ω_p its frequency is $\omega_p[1+(\omega_c^2/3\omega_p^2)]$ [see (66) and (67)]. This mode can be excited only by the ac external magnetic field parallel to $\mathbf{B}_0$, and *only* if $\mathbf{B}_0 \neq \mathbf{0}$. We note that the modes considered in I can be excited by the ac uniform electric field (those with $l=1$) or by the ac uniform magnetic and ac linearly varying electric fields $(l=2)$. The detailed polarization rules are given in I.

The resonance at $\omega \cong \omega_p$ does not appear in the results of Ford, Furdyna, and Werner,[7] reproduced by Ford and Werner[8] in the Rayleigh limit of their general theory of the absorption by a gyrotropic sphere. The condition on the wave number inside (q) and outside (k) the sphere, $|q/k| \gg 1$, limiting the validity the analytical results of Ford and Werner, is not fulfilled for the ω_p modes [for the lossless plasma and $|\omega_c| \ll \omega_p$ we obtain $|q/k|^2 \cong \epsilon(\omega_c/\omega_p)^3$]. However, the numerical analysis of the general expressions of Ford and Werner should reveal the ω_p resonance.

Comparing the power absorption for the modes considered in both papers, we found that it is the strongest for the ω_{1m1} electric dipole resonance, while all other resonances, including the Φ_{n00} mode, yield weaker absorption. For reasonable values of the semiconducting sample parameters, we estimated that the electric dipole absorption peak should be about 20 times higher than other peaks. Nevertheless, the resonance at ω_r should be observable experimentally.

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