Magnetoplasma oscillations in a small conducting sphere

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The eigenmodes of a free-carrier lossless magnetoplasma in a small conducting sphere (or ellipsoid) are calculated and discussed in detail. In particular, we determine the motion of carriers, the charge densities, and the electric and magnetic moments associated with these modes of oscillation. The modes fall into three classes of surface, heliconlike, and bulk magnetoplasmons. The interaction of the modes with an external electromagnetic field is calculated and the expressions for the power absorption are given. Our eigenmode approach is more complete and gives a better physical insight into the nature of magnetoplasma oscillations in the Rayleigh limit than the study of magnetoplasma oscillations forced by an external electromagnetic field.

I. INTRODUCTION

Since the pioneering paper of Mie,[1] and especially in the last two decades, there has been considerable interest in the electronic plasma oscillations in small metallic spheres (see Ref. 2 and references therein). Relatively less effort was devoted to the study of the plasma oscillations in the presence of the external static magnetic field. In metals the effect of the magnetic field on high-frequency plasma oscillations is usually negligible, since it is determined by the parameter ω_c/ω_p (ω_c and ω_p are the cyclotron and plasma frequencies, respectively), which for metals assumes very small values (at $H=1$ T, $\omega_c/\omega_p \sim 10^{-5}$). However, for the free-carrier plasma in semiconductors this parameter can easily achieve values of the order of 1. Moreover, both in metals and semiconductors there exist low-frequency excitations with $\omega < \omega_c$ observed in the bulk[3,4] and in small samples.[5]

The study of magnetoplasma modes and their interaction with electromagnetic radiation in small samples is therefore interesting for semiconductors and, in the low-frequency regime, for metals as well. This implies that the magnetoplasma resonances of interest lie in the microwave or far-infrared spectral region. In this region the only competitive absorption mechanisms are the excitations of transverse-optical phonons (reststrahlen) and free-carrier processes due to the presence of crystal imperfections (e.g., impurities), which give a slowly varying absorption background. Thus, in high-mobility materials the magnetoplasma resonances can be observed in microwave or far-infrared—absorption experiments.[5–8]

From these resonances in semiconductors one can determine the carrier concentration and their effective mass, as well as the dielectric constant of the lattice. From an experimental point of view, such magnetoplasma resonances are particularly convenient because the resonance condition can be easily obtained by sweeping the static magnetic field. The use of microwave cavities[5] in these experiments allows excitation of the plasma either by electric or magnetic fields oriented arbitrarily with respect to the dc magnetic field. The magnetoplasma modes can be also excited by inelastically scattered electrons (electron-energy-loss spectroscopy).

The papers published so far have been concerned with the theory of magnetoplasma oscillations forced by the external electromagnetic radiation.[9–11] This approach has two important drawbacks. First, the physical picture of the excited magnetoplasma eigenmodes remains obscure. Second, the simplifying assumptions concerning the form of the exciting field exclude some of the possible eigenmodes, i.e., those not excited by a field of a simple form. In the present paper we avoid these deficiencies by first deriving a more complete set of the magnetoplasma eigenmodes (free oscillations) of a small (semi)conducting sphere, and only then calculating their interaction with the external electromagnetic field. This approach, used previously for the case of vanishing magnetic field,[12,13] also allows for the quantization of the magnetoplasma modes.

We make the following simplifying assumptions in our treatment. We assume the radius of the sphere to be small compared to the wavelength of incident radiation, but large compared to the Thomas-Fermi screening length. The first assumption allows neglect of the retardation effects,[14,15] while the second justifies the use of the jellium model for the plasma.[16] Furthermore, the energy band of the carriers is assumed to be spherical and parabolic. In our treatment we neglect the plasma—LO-phonon coupling[17] present in ionic semiconductors, i.e., we assume that the plasma and optical-phonon frequencies are well separated. Finally, the damping of the carrier motion is neglected, i.e., we assume that the free oscillations of the ideal plasma are not damped.

In Sec. II the basic equations for the electric potentials and the frequencies of the eigenmodes are derived. In Sec. III their solutions in the absence of the static magnetic field are briefly discussed. In the presence of the static magnetic field there occurs an important class of magnetoplasma modes, viz., modes with a vanishing space-charge density. These modes are discussed in Sec. IV. In Sec. V we discuss all high-frequency modes, including those with nonvanishing space-charge density, in the case of a weak static magnetic field. In Sec. VI we formulate the Hamiltonian of the interaction of the spherical magnetoplasma with external ac electromagnetic fields. The absorption of radiation by the magnetoplasma is calculated and discussed in Sec. VII. In Sec. VIII we compare our results with the results of the forced-oscillation approaches.

Finally, in order to determine the effect of a small departure from sphericity of the sample, in Appendix A we calculate the frequencies of all the above magnetoplasma modes for a small conducting ellipsoid.

II. BASIC EQUATIONS

Let us consider a sphere of radius r_0 filled with an ideal electron (or hole) plasma placed in a uniform compensating background having a dielectric constant ϵ and unit permeability. The free carriers of charge q and equilibrium concentration n are assumed to have an isotropic, energy-independent effective mass m^*. The system is placed in a static uniform magnetic field $\mathbf{B}_0$.

In the following we shall consider small collective oscillations of the spherical plasma, using the jellium model. The position of an oscillating carrier (i.e., the position of a volume element of the plasma) can then be written as the sum of the equilibrium position $\mathbf{r}$ ($r \lesssim r_0$) and a small displacement vector $\boldsymbol{\xi}(\mathbf{r},t)$ which is a continuous function of $\mathbf{r}$. In all equations we shall keep only terms linear in $\boldsymbol{\xi}$. The interaction between carriers within the plasma will be described in terms of a self-consistent electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$. These fields vanish for vanishing $\boldsymbol{\xi}$, and are therefore at least linear in $\boldsymbol{\xi}$.

The linearized equation of motion for a carrier in a lossless plasma is

$$m^*\ddot{\boldsymbol{\xi}}(\mathbf{r},t)=q\mathbf{E}^-(\mathbf{r},t)+\frac{q}{c}\dot{\boldsymbol{\xi}}\times\mathbf{B}_0 \,. \tag{1}$$

The electric field inside and outside the sphere, denoted by $\mathbf{E}^-$ and $\mathbf{E}^+$, respectively, satisfies the quasistatic Maxwell equations

$$\epsilon\nabla\cdot\mathbf{E}^-(\mathbf{r},t)=-4\pi qn\nabla\cdot\boldsymbol{\xi}(\mathbf{r},t)\ , \quad r<r_0 \tag{2}$$

$$\nabla\cdot\mathbf{E}^+(\mathbf{r},t)=0\ , \quad r>r_0 \tag{3}$$

$$\nabla\times\mathbf{E}^\pm(\mathbf{r},t)=0\ , \quad r<r_0 \text{ and } r>r_0\ . \tag{4}$$

The boundary conditions on the surface of the sphere, which complete our set of equations, are

$$\hat{\mathbf{r}}_0\times\mathbf{E}^+(\mathbf{r}_0,t)=\hat{\mathbf{r}}_0\times\mathbf{E}^-(\mathbf{r}_0,t)\ , \tag{5a}$$

$$\hat{\mathbf{r}}_0\cdot\mathbf{E}^+(\mathbf{r}_0,t)=\epsilon\hat{\mathbf{r}}_0\cdot\mathbf{E}^-(\mathbf{r}_0,t)+4\pi\sigma(\mathbf{r}_0,t)\ , \tag{5b}$$

where

$$\sigma(\mathbf{r}_0,t)=nq\hat{\mathbf{r}}_0\cdot\boldsymbol{\xi}(\mathbf{r}_0,t) \tag{6}$$

is the surface charge density and $\hat{\mathbf{r}}_0=\mathbf{r}_0/r_0$.

In the following we shall look for the eigenmodes of the system described by Eqs. (1)—(5). We shall thus assume the harmonic time dependence of the fields and displacements, e.g., $\boldsymbol{\xi}(\mathbf{r},t)=\boldsymbol{\xi}(\mathbf{r})e^{-i\omega t}$ and $\mathbf{E}(\mathbf{r},t)=\mathbf{E}(\mathbf{r})e^{-i\omega t}$, where $\boldsymbol{\xi}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ are complex amplitudes.

The quasistatic approximation [Eq. (4)] is valid if (for harmonic fields) we neglect terms proportional to $\epsilon(r_0/c)^2(\omega^2-\omega_p^2)$ inside the sphere and to $(\omega r/c)^2$ outside the sphere, where $\omega_p=(4\pi nq^2/m^*\epsilon)^{1/2}$ is the plasma frequency. As it turns out later, this is justified if the conditions

$$\epsilon(r_0\omega_p/c)^2 \ll 1\ , \tag{7a}$$

$$(\omega r/c)^2 \ll 1 \tag{7b}$$

are satisfied. These conditions allow us to decouple the equations for the electric and magnetic fields. Condition (7b) limits the range of validity of our equations outside the sphere to the so-called near zone.

Solving Eq. (1), we obtain

$$\boldsymbol{\xi}(\mathbf{r})=-\frac{q}{m^*(\omega^2-\omega_c^2)}\left\{\mathbf{E}^-(\mathbf{r})-i\frac{\omega_c}{\omega}\hat{\mathbf{z}}\times\mathbf{E}^-(\mathbf{r}) -\frac{\omega_c^2}{\omega^2}\hat{\mathbf{z}}[\hat{\mathbf{z}}\cdot\mathbf{E}^-(\mathbf{r})]\right\}, \tag{8}$$

where $\mathbf{B}_0$ was chosen parallel to the z axis, $\omega_c=(qB_0/m^*c)$ is the cyclotron frequency, and where we assumed that $\omega\neq|\omega_c|$. Equation (4) allows us to express the electric fields in terms of scalar potentials defined by $\mathbf{E}^\pm(\mathbf{r})=-\nabla\Phi^\pm(\mathbf{r})$. With the use of the solution (8), Eqs. (2), (3), and (5) can be written as

$$(\omega^2-\omega_p^2-\omega_c^2)\nabla^2\Phi^-(\mathbf{r})+(\omega_p\omega_c/\omega)^2\frac{\partial^2\Phi^-(\mathbf{r})}{\partial z^2}=0\ , \tag{9}$$

$$\nabla^2\Phi^+(\mathbf{r})=0\ , \tag{10}$$

$$\Phi^+(\mathbf{r}_0)=\Phi^-(\mathbf{r}_0)\ , \tag{11}$$

$$(\omega^2-\omega_c^2)\mathbf{r}_0\cdot\nabla\Phi^+(\mathbf{r}_0)=\epsilon(\omega^2-\omega_p^2-\omega_c^2)\mathbf{r}_0\cdot\nabla\Phi^-(\mathbf{r}_0) +\epsilon(\omega_p\omega_c/\omega)^2\left.\left\{z\frac{\partial\Phi^-(\mathbf{r})}{\partial z}\right\}\right|_{\mathbf{r}=\mathbf{r}_0} +i\epsilon(\omega_p^2\omega_c/\omega)\left.\left\{y\frac{\partial\Phi^-(\mathbf{r})}{\partial x}-x\frac{\partial\Phi^-(\mathbf{r})}{\partial y}\right\}\right|_{\mathbf{r}=\mathbf{r}_0}. \tag{12}$$

These equations form a complete set determining the potentials and eigenfrequencies for the spherical magnetoplasma. In the sections which follow we shall consider some special classes of solutions of these equations.

Although for our purposes it is sufficient to determine $\mathbf{E}(\mathbf{r})$, for the sake of completeness let us also write the set of equations for the magnetic field amplitude $\mathbf{B}(\mathbf{r})$ in the quasistatic approximation:

$$\nabla\times\mathbf{B}^-(\mathbf{r})=(4\pi/c)\mathbf{j}(\mathbf{r}) \ , \tag{13a}$$

$$\nabla\times\mathbf{B}^+(\mathbf{r})=0 \ , \tag{13b}$$

$$\nabla\cdot\mathbf{B}^\pm(\mathbf{r})=0 \ , \tag{13c}$$

where the current density $\mathbf{j}(\mathbf{r})$ is determined by the set of equations for the electric field, $\mathbf{j}(\mathbf{r})=-inq\omega\xi(\mathbf{r})$, with $\xi(\mathbf{r})$ given by Eq. (8). The boundary condition for $\mathbf{B}(\mathbf{r})$ is

$$\mathbf{B}^+(\mathbf{r}_0)=\mathbf{B}^-(\mathbf{r}_0) \ , \tag{13d}$$

where the surface current density was neglected, since it is of second order in ξ.

III. ZERO-MAGNETIC-FIELD MODES

Let us first consider the simplest case of the vanishing external magnetic field, i.e., $\omega_c=0$. In the case $\omega\neq\omega_p$, from Eqs. (9)—(12) we obtain the well-known normal-mode frequencies[13,18]

$$\omega=\omega_l=\left[\frac{4\pi nq^2}{m^*[\epsilon+(l+1)/l]}\right]^{1/2} , \tag{14}$$

and the corresponding potentials

$$\Phi_l^\pm(\mathbf{r})=(q/r_0)g_l^\pm(r/r_0)\sum_{m=-l}^{l}C_{lm}^\pm Y_{lm}(\theta,\phi) \ . \tag{15}$$

Here the spherical coordinates were introduced, $g_l^+(x)=x^l$ and $g_l^-(x)=x^{-l-1}$, Y_{lm} are orthonormalized spherical harmonics,[19] and $C_{lm}^\pm$ are arbitrary complex dimensionless coefficients. To every frequency ω_l there correspond $(2l+1)$ independent modes of motion, i.e., each frequency is $(2l+1)$-fold degenerate. It is worth noting that the space-charge density for these modes vanishes, so that the only source of the electric field in the excitations is the surface-charge density.

Let us now consider the case when $\omega=\omega_p$. Equation (10) together with Eq. (12) gives

$$\Phi^+(\mathbf{r})\equiv0 \ , \tag{16}$$

and then Eq. (11) leads to

$$\Phi^-(\mathbf{r}_0)=0 \ . \tag{17}$$

Thus, the oscillations with the frequency ω_p ($>\omega_l$) correspond to a vanishing potential (and field) outside the sphere, while the potential inside the sphere is *arbitrary* provided it vanishes on the surface of the sphere. Both space- and surface-charge densities associated with these modes are, in general, nonzero. The ω_p resonance was observed for metallic particles in electron-energy-loss measurements.[20]

An external magnetic field applied to the system can either modify the zero-field modes considered above or give rise to new heliconlike normal modes. In the low-magnetic-field range ($|\omega_c|\ll\omega_p$) we shall then expect two types of solutions for Eqs. (9)—(12): high-frequency solutions, corresponding to $\omega\approx\omega_l$ or $\omega\approx\omega_p$, and low-frequency solutions, corresponding to $\omega\sim|\omega_c|$.

IV. VANISHING SPACE-CHARGE-DENSITY MODES

In the presence of an arbitrary magnetic field it is difficult to find exact general solutions of Eqs. (9)—(12). However, in the preceding section we found that an important class of oscillations (the ω_l modes) was characterized by a vanishing space-charge density. Let us therefore make such an additional assumption in (9)—(12), i.e., let us demand that both terms in Eq. (9) vanish separately. The potentials inside and outside the sphere then satisfy the Laplace equation, whose solution can be written as

$$\Phi^\pm(\mathbf{r})=(q/r_0)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}C_{lm}^\pm g_l^\pm(r/r_0)Y_{lm}(\theta,\phi) \ . \tag{18}$$

In addition, the following equation must be satisfied:

$$0=\frac{\partial^2\Phi^-(\mathbf{r})}{\partial z^2}=\frac{q}{r_0^3}\sum_{l=2}^{\infty}\sum_{m=-(l-2)}^{l-2}C_{lm}^-\left[\frac{r}{r_0}\right]^{l-2}\left[\frac{2l+1}{2l-3}\right]^{1/2}(l^2-m^2)^{1/2}[(l-1)^2-m^2]^{1/2}Y_{l-2,m} \ . \tag{19}$$

This equation leads to vanishing of all C_{lm}^-, except for those with

$$m=\pm l,\pm(l-1) \ . \tag{20}$$

The continuity condition (11) gives $C_{lm}^+=C_{lm}^-$, and then the second boundary condition (12) leads to a set of equations

$$[(2l-1)(l+1)\omega^2(\omega^2-\omega_c^2)+l(2l-1)\epsilon\omega^2(\omega^2-\omega_p^2-\omega_c^2)$$
$$+m(2l-1)\epsilon\omega_c\omega_p^2\omega+(l^2-m^2)\epsilon\omega_c^2\omega_p^2]C_{lm}^-=0 \tag{21}$$

for every lm satisfying (20). For $l=0$, Eq. (21) leads to $C_{00}^-=0$ (since $\omega\neq|\omega_c|$). For $m=\pm l$, we obtain nonzero C_{lm}^- if the frequency is equal to[21]

$$\omega_{lm1}=\omega_l\{[1+(\omega_c/2\omega_l)^2]^{1/2}-(\omega_c/2\omega_l)\text{sgn}m\} \ ,$$
$$m=\pm l \ . \tag{22}$$

For $m=\pm(l-1)$, Eq. (21) can be divided by $\omega-\omega_c\text{sgn}m\neq0$ and reduced to

$$[\omega^3+\omega^2\omega_c\text{sgn}m-\omega\omega_l^2-(1/l)\omega_c\omega_l^2\text{sgn}m]C_{lm}^-=0 \ . \tag{23}$$

For $l=1$ we then obtain only one (positive and different from $|\omega_c|$) frequency,

$$\omega_{lm1}=\omega_1 \ , \quad l=1, \ m=0 \ . \tag{24}$$

For $l\geq2$ the exact expressions for the roots are rather cumbersome (they can be obtained from Cardan's formulas) and we shall write only their approximate forms in the case of a weak magnetic field, i.e., $|\omega_c| \ll \omega_p$. Up to second order in ω_c/ω_p, we obtain the high-frequency solutions in the form

$$\omega_{lm\,1}=\omega_l-\frac{m}{2l}\omega_c+\frac{(l-1)(l+3)}{8l^2}\frac{\omega_c^2}{\omega_l}\ , \quad l\geq 2\ ,m=\pm(l-1)\ , \tag{25}$$

and, for $m=-(l-1)\text{sgn}q$, the low-frequency solutions

$$\omega_{lm\,2}=\frac{|\omega_c|}{l}\left[1-\frac{l-1}{l^2}\left[\frac{\omega_c}{\omega_l}\right]^2\right]\ , \quad l\geq 2,\ m=-(l-1)\text{sgn}q\ . \tag{26}$$

The potentials corresponding to any of the frequencies given by Eqs. (22)—(26) are of the form

$$\Phi_{lm}^{\pm}(\mathbf{r})=(q/r_0)g_l^{\pm}(r/r_0)Y_{lm}(\theta,\phi)\ , \tag{27}$$

up to the multiplicative factor C_{lm}^{-}. The surface-charge density, which is the source of the "driving force" for the plasma oscillations, is given by

$$\sigma_{lm}(\mathbf{r}_0)=[(\epsilon l+l+1)/4\pi r_0]\Phi_{lm}^{-}(\mathbf{r}_0)\ . \tag{28}$$

This formula is obtained from the boundary condition (5b) and is generally valid for the harmonic potentials (27) [and also for (15)]. It is clear that every lm mode possesses an oscillating $(2l)$th electric multipole moment.

The above results show that, for the vanishing space-charge-density magnetoplasma modes, two classes of solutions exist: those with frequencies tending to ω_l when $\mathbf{B}_0\rightarrow\mathbf{0}$, i.e., (22), (24), and (25) (denoted by the index $\tau=1$), and those with frequencies tending to zero when $\mathbf{B}_0\rightarrow\mathbf{0}$, i.e., (26) (denoted by $\tau=2$). In order to gain some physical insight into the plasma oscillations corresponding to these solutions, we shall now examine the motion of the carriers for the simplest modes in both classes, i.e., for $\tau=l=1$ and for $\tau=l=2$.

For $l=1$ the exact frequencies are given by Eqs. (24) and (22) for $m=0$ and $m=\pm1$, respectively. The displacements ξ_{lm} corresponding to the Φ_{lm}^{-} potentials have the following forms for $l=1$:

$$\xi_{1,\pm1,1}(\mathbf{r})=\left[\frac{3}{8\pi}\right]^{1/2}\frac{q^2}{m^*r_0^2(\mp\omega_{1,\pm1,1}-\omega_c)\omega_{1,\pm1,1}}(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}}),$$
$$\xi_{1,0,1}(\mathbf{r})=\left[\frac{3}{4\pi}\right]^{1/2}\frac{q^2}{m^*r_0^2\omega_{1,0,1}^2}\hat{\mathbf{z}}\ , \tag{29}$$

where Eqs. (27) and (8) were used. The corresponding linearized current density is given by general formula

$$\mathbf{j}_{lm\tau}(\mathbf{r})=-inq\omega_{lm\tau}\xi_{lm\tau}(\mathbf{r})\ . \tag{30}$$

We thus see that all carriers move in the same way: circularly in the plane perpendicular to $\mathbf{B}_0$ for $m=\pm1$, and linearly along $\mathbf{B}_0$ for $m=0$. The resulting current is that of a rigid oscillation of uniformly charged sphere (on the compensating background) whose center is performing the $\xi_{1m\,1}(\mathbf{r},t)$ motion. The surface-charge density $\sigma_{1m}(\mathbf{r},t)$ obtained from Eq. (28) is consistent with this picture and represents an oscillating electric dipole moment.

For the simplest $(l=2)$ mode from the second class of solutions, Eqs. (8), (26), and (27) yield [up to terms $(\omega_c/\omega_p)^2$]

$$\xi_{2,-\text{sgn}q,2}(\mathbf{r})=\mathbf{K}\times\mathbf{r}+(i/2)\text{sgn}q(\omega_c/\omega_2)^2(\mathbf{K}\cdot\mathbf{r})\hat{\mathbf{z}}\ , \tag{31}$$

where

$$\mathbf{K}=i\frac{12|q|}{m^*\omega_c^2}(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\,\text{sgn}q)\ . \tag{32}$$

The current density $\mathbf{j}_{2,-\text{sgn}q,2}$ for $|\omega_c|/\omega_p \ll 1$ looks as if a uniformly charged sphere was rotating around $\mathbf{K}$, which, in turn, performed a rotation with a frequency $|\omega_c|/2$ in the cyclotron-resonance-active (CRA) sense in the plane perpendicular to $\mathbf{B}_0$. This mode then possesses a rotating magnetic dipole moment. It is worth noting that if the second term in (31) is neglected the surface-charge density (and thus also the electric quadrupole moment) of the mode vanishes.

V. HIGH-FREQUENCY MODES

Solving Eqs. (9)—(12) for an arbitrary dc magnetic field, we assumed a vanishing space-charge density, which allowed us to decouple Eq. (9). Another situation when Eq. (9) can be simplified arises in the case of a weak magnetic field ($|\omega_c| \ll \omega_p$) and $\omega\sim\omega_p$ or $\omega\sim\omega_l$, if we retain only terms up to first order in ω_c/ω_p.

Let us first consider the case

$$\omega=\omega_l+\omega'\ , \tag{33}$$

where ω' is a small correction of the order of ω_c. Equations (9) and (12) then become, respectively,

$$(\omega_l^2-\omega_p^2+2\omega_l\omega')\nabla^2\Phi^-(\mathbf{r})=0\ , \tag{34}$$

$$r_0\frac{\partial\Phi^+(\mathbf{r})}{\partial r}\bigg|_{\mathbf{r}=\mathbf{r}_0}=\epsilon\frac{\omega_l^2-\omega_p^2+2\omega_l\omega'}{\omega_l^2+2\omega_l\omega'}r_0\frac{\partial\Phi^-(\mathbf{r})}{\partial r}\bigg|_{\mathbf{r}=\mathbf{r}_0}-i\epsilon\frac{\omega_p^2\omega_c}{\omega_l^3}\frac{\partial\Phi^-(\mathbf{r})}{\partial\phi}\bigg|_{\mathbf{r}=\mathbf{r}_0}\ , \tag{35}$$

while Eqs. (11) and (10) remain unchanged. As $\omega_l^2-\omega_p^2$ is of zeroth order, the expression in parentheses in Eq. (34) cannot vanish. Thus Eq. (34) becomes

$$\nabla^2\Phi^-(\mathbf{r})=0\ , \tag{36}$$

and so $\Phi^-(\mathbf{r})$ and $\Phi^+(\mathbf{r})$ are given by expression (18). Inserting it into Eqs. (11) and (35), we obtain $C_{00}^{\pm}=0$, and for $l\geq1$ we have nonzero $C_{lm}^{\pm}$ only if

$$\omega_{lm\,1}=\omega_l-\frac{m}{2l}\omega_c\ ,\quad l\geq1\ ,\quad |m|\leq l \tag{37}$$

where again the index 1 denotes a high-frequency solution. The corresponding potentials are given by Eq. (27).

The Laplace equation (36) is valid up to first order in ω_c/ω_p, which means that the space-charge density for the modes considered is at least of second order. In some cases, namely for $m=\pm l\pm(l-1)$, it can vanish exactly.

One easily checks that the frequencies (22), (24), and (25) obtained in the preceding section fall within ω_c/ω_p accuracy into the appropriate frequencies (37). The surface-charge density is still given by (28).

Equation (37) shows that the degeneracy of the ω_l modes occurring in the $\mathbf{B}_0=0$ case has been lifted by the magnetic field, i.e., for a given l we obtain a ladder of $2l+1$ equidistant levels.

Let us now consider the second type of high-frequency modes, with

$$\omega=\omega_p+\omega'' , \tag{38}$$

where $\omega''\sim\omega_c$. In addition to Eqs. (10) and (11), we obtain, from (9) and (12) up to first order in ω_c/ω_p,

$$\omega''\nabla^2\Phi^-(\mathbf{r})=0 , \tag{39}$$

$$r_0\frac{\partial\Phi^+(\mathbf{r})}{\partial r}\bigg|_{\mathbf{r}=\mathbf{r}_0}=2\epsilon\frac{\omega''}{\omega_p}r_0\frac{\partial\Phi^-(\mathbf{r})}{\partial r}\bigg|_{\mathbf{r}=\mathbf{r}_0}-i\epsilon\frac{\omega_c}{\omega_p}\frac{\partial\Phi^-(\mathbf{r})}{\partial\phi}\bigg|_{\mathbf{r}=\mathbf{r}_0} . \tag{40}$$

For $\omega''\neq0$ we obtain $C^{\pm}_{lm}=0$ for every l,m. For $\omega''=0$, Eqs. (11) and (40) lead to conditions (16) and (17). Thus, to first order in ω_c/ω_p, both the frequencies and potentials remain unchanged in the presence of the magnetic field [although $\xi(\mathbf{r})$ is changed]. In particular, the potential outside the sphere vanishes. These modes and their interaction with electromagnetic field are discussed in detail in the following paper.[22]

Summarizing our analysis of normal modes of oscillations for a spherical magnetoplasma, we find that all possible modes with frequencies approaching the zero-magnetic-field solutions when $\mathbf{B}_0\to0$ have been examined. They were analyzed up to first order in ω_c/ω_p, although some were obtained for arbitrary $\mathbf{B}_0$. The modes with frequencies tending to zero when $\mathbf{B}_0\to0$ were investigated only in the case of zero-space-charge density, but for arbitrary magnetic field.

In an actual experimental situation the sample may not be ideally spherical. To take this possibility into account, in Appendix A we analyze the influence of small nonsphericity on all modes under consideration, assuming an ellipsoidal shape of the sample.

In the following sections we shall turn to the calculation of the absorption of electromagnetic radiation associated with the generation of the magnetoplasma $\omega_{lm\tau}$ modes considered above.

VI. THE HAMILTONIAN

In order to formulate the classical Hamiltonian describing our system, we shall confine ourselves to the case of small oscillations of the plasma with a vanishing space-charge density, i.e., either vanishing exactly or up to terms linear in ω_c/ω_p. The general complex displacement and the potential can be written as

$$\xi(\mathbf{r},t)=\sum_{\tau=1,2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}C_{lm\tau}(t)\xi_{lm\tau}(\mathbf{r}) , \tag{41}$$

$$\Phi^{\pm}(\mathbf{r},t)=\sum_{\tau=1,2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}C_{lm\tau}(t)\Phi^{\pm}_{lm}(\mathbf{r}) , \tag{42}$$

respectively, where $C_{lm\tau}(t)$ are complex dynamic variables and

$$\xi_{lm\tau}(\mathbf{r})=\frac{q^2}{m^*r_0^{l+1}(\omega_{lm\tau}^2-\omega_c^2)}\times\left[\nabla-(i\omega_c/\omega_{lm\tau})\hat{\mathbf{z}}\times\nabla-(\omega_c/\omega_{lm\tau})^2\hat{\mathbf{z}}\frac{\partial}{\partial z}\right]r^lY_{lm}(\theta,\phi) , \tag{43}$$

while $\Phi^{\pm}_{lm}$ are given by Eq. (27). We note that $\nabla\cdot\xi_{lm\tau}=0$ only up to first order in ω_c/ω_p for $\tau=1$, $m\neq\pm l,\pm(l-1)$, and exactly in all other cases. The potential energy of the plasma has the following form:

$$U=\tfrac{1}{2}nq\int_{r\le r_0}d^3r\,\Phi_R^-(\mathbf{r}+\xi_R(\mathbf{r},t),t)=\frac{nq}{2}\int_{r\le r_0}d^3r\,\xi_R(\mathbf{r},t)\cdot\nabla\Phi_R^-(\mathbf{r},t) , \tag{44}$$

where the subscript R denotes the real part. Here we have taken into account that the carrier with an equilibrium position $\mathbf{r}$ is situated at $\mathbf{r}+\xi_R$, and only terms up to second order in ξ_R were retained.

For a vanishing space-charge density, U can be transformed into a surface integral

$$(r_0^2/2)\int d\Omega\,\Phi_R^-(\mathbf{r}_0,t)\sigma_R(\mathbf{r}_0,t) .$$

In order to express U only in terms of ξ_R, the dependence of $C_{lm\tau}(t)$ on the displacements must be found. Insertion of Eqs. (6) and (42) into the boundary condition (5b) leads to

$$\sum_{\tau}C_{lm\tau}(t)=\frac{\omega_l^2m^*}{q^2lr_0^l}\int d^3r\,\xi(\mathbf{r},t)\cdot\nabla r^lY^*_{lm}(\theta,\phi) , \tag{45}$$

where we took into account that $\nabla\cdot\xi(\mathbf{r},t)=0$.

With the help of Eqs. (42) and (45) we can now obtain from (44) the required form of U, and the Hamiltonian of the system then becomes

$$H_0=(n/2m^*)\int d^3r\{\mathbf{p}_R(\mathbf{r},t)+(m^*\omega_c/2)[\mathbf{r}+\xi_R(\mathbf{r},t)]\times\hat{\mathbf{z}}\}^2+(nm^*/2)\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{\omega_l^2}{lr_0^{2l+1}}\left|\int d^3r\,\xi_R(\mathbf{r},t)\cdot\nabla r^lY_{lm}(\theta,\phi)\right|^2 , \tag{46}$$

where $\mathbf{p}_R$ is the canonical momentum conjugate to ξ_R. The Hamilton equations (with functional derivatives) for H_0, with the use of (45), lead to equations of motion consistent with (1).

The Hamiltonian H_0 can be expressed in terms of the dynamic variables $C_{lm\tau}(t)$. Using the transformation (41) and

$$\mathbf{p}(\mathbf{r},t)=(m^*\omega_c/2)\hat{\mathbf{z}}\times\mathbf{r}-m^*\sum_{\tau=1,2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}C_{lm\tau}(t)[i\omega_{lm\tau}\xi_{lm\tau}(\mathbf{r})+(\omega_c/2)\xi_{lm\tau}(\mathbf{r})\times\hat{\mathbf{z}}] , \tag{47}$$

we obtain, after some tedious calculation,

$$H_0=\sum_{l,m,\tau}\hbar\omega_{lm\tau}b^*_{lm\tau}(t)b_{lm\tau}(t) , \tag{48}$$

where

$$B_{lm\tau}(t)=C_{lm\tau}(t)/\gamma_{lm\tau}$$

and all nondiagonal terms turn out to cancel, as well as all terms proportional to $C_{lm\tau}(t)C_{l'm'\tau'}(t)$ or $C^*_{lm\tau}(t)C^*_{l'm'\tau'}(t)$. The Hamiltonian H_0 thus becomes a sum of independent oscillators with the $\omega_{lm\tau}$ the frequencies. The coefficients $\gamma_{lm\tau}$ are given by the following exact formulas:

$$\gamma_{lm1}=\left[\frac{4m^*\hbar\omega_{lm1}r_0}{q^4nl(\omega_{l,-m,1}^{-2}+\omega_l^{-2})}\right]^{1/2} , \tag{49}$$

for $m=\pm l$, and

$$\gamma_{lm\tau}=\left[\frac{4m^*\hbar\omega_{lm\tau}r_0}{q^4n}\right]^{1/2}\times\left[\frac{l\omega^2_{lm\tau}\pm2\omega_c\omega_{lm\tau}+\omega_c^2}{(\omega_{lm\tau}\pm\omega_c)^2\omega^2_{lm\tau}}+\frac{l}{\omega_l^2}\right]^{-1/2} , \tag{50}$$

for $\tau=1$, $m=\pm(l-1)$, or for $\tau=2$, $l\geq2$, $m=-(l-1)\mathrm{sgn}q$, with lower sign and ω_c replaced by $|\omega_c|$. For $\tau=1$ and arbitrary l,m, we have the approximate formula

$$\gamma_{lm1}=\left[\frac{2m^*\omega_l^3\hbar r_0}{q^4nl}\right]^{1/2} , \tag{51}$$

valid up to the first order in ω_c/ω_p. With that accuracy, Eqs. (49) and (50) reduce to (51) for $\tau=1$, $m=\pm l,\pm(l-1)$.

Let us now consider our system placed in a weak electromagnetic radiation field oscillating with a frequency ω. Assuming the vector potential inside the sphere $\mathbf{A}(\mathbf{r},t)=\mathbf{A}(\mathbf{r})e^{-i\omega t}$ and choosing the vanishing-scalar-potential gauge for the radiation field, we obtain the full Hamiltonian in the form

$$H=H_0+H_1+H_2 , \tag{52}$$

with

$$H_1=-(qn/m^*c)\int_{r\leq r_0}d^3r\{\mathbf{p}_R(\mathbf{r},t)+(m^*\omega_c/2)[\mathbf{r}+\xi_R(\mathbf{r},t)]\times\hat{\mathbf{z}}\}\cdot\mathrm{Re}[\mathbf{A}(\mathbf{r})e^{-i\omega t}] \tag{53}$$

and

$$H_2=-(qn/m^*c)\int_{r\leq r_0}d^3r\{\mathbf{p}_R(\mathbf{r},t)+(m^*\omega_c/2)[\mathbf{r}+\xi_R(\mathbf{r},t)]\times\hat{\mathbf{z}}\}\cdot[\xi_R(\mathbf{r},t)\cdot\nabla]\mathrm{Re}[\mathbf{A}(\mathbf{r})e^{-i\omega t}] . \tag{54}$$

We dropped here the terms quadratic in $\mathbf{A}$ since we are interested in (one-photon) absorption processes and not in photon scattering.[13] As was previously done, we used $\mathbf{r}+\xi_R$ for the actual position of the carrier and, expanding $\mathbf{A}(\mathbf{r}+\xi_R)$ up to first order in ξ_R, we obtained H_1 and H_2. Assuming that condition (7) is satisfied, and H_2 term is roughly of the order of $(\xi_R/r_0)H_1$.

One should realize that the carriers in the surface layer may not obey Eq. (41) since there may be surface reflections and other surface features which were not taken into account in our theory. The number of these carriers is approximately $4\pi r_0^2n\xi_R$, while in the entire sphere we have $\frac{4}{3}\pi r_0^3n$ carriers. Therefore our accuracy is of the order of ξ_R/r_0, and the term H_2 should thus be dropped. This does not mean that our surface-charge density $\sigma(\mathbf{r}_0,t)$ is incorrect; it arises from the movement of the carriers in the bulk and does not depend on what we assume about the surface.

Let us now consider the vector potential $\mathbf{A}(\mathbf{r})$ which enters the perturbing Hamiltonian H_1. It should be expressed by the vector potential $\mathbf{A}^r(\mathbf{r})$ (or the fields) in the absence of the sphere, as these quantities are measured in the experiment. We can write

$$\mathbf{A}(\mathbf{r})=\mathbf{A}^r(\mathbf{r})+\mathbf{A}^s(\mathbf{r}) , \tag{55}$$

where $\mathbf{A}^s(\mathbf{r})$ is the vector potential arising from the presence of the (nonconducting) sphere with the dielectric constant ϵ. Assuming that $k^{-1}\gg r_0$, where $k^{-1}\approx c/\omega$ is the spatial variation of $\mathbf{A}^r(\mathbf{r})$, we can expand $\mathbf{A}^r(\mathbf{r})$ around the center of the sphere:

$$A^r_\mu(\mathbf{r})=A^{r(0)}_\mu+\left[\frac{r_0\omega}{c}\right]\left[\frac{r_\nu}{r_0}\right]A^{r(1)}_{\mu\nu}+\left[\frac{r_0\omega}{c}\right]^2\left[\frac{r_\nu}{r_0}\right]\left[\frac{r_\kappa}{r_0}\right]A^{r(2)}_{\mu\nu\kappa}+\cdots , \tag{56}$$

where the summation convention is used, and where the coefficients $A^{r(0)}_\mu$, $A^{r(1)}_{\mu\nu}$, $A^{r(2)}_{\mu\nu\kappa}$, etc., are of the same order, and $\omega r_0/c$ is a dimensionless expansion parameter. We can also expand $\mathbf{A}^s(\mathbf{r})$ in this parameter:

$$\mathbf{A}^s(\mathbf{r})=\mathbf{A}^{s(0)}(\mathbf{r})+\left[\frac{r_0\omega}{c}\right]\mathbf{A}^{s(1)}(\mathbf{r})+\left[\frac{r_0\omega}{c}\right]^2\mathbf{A}^{s(2)}(\mathbf{r})+\cdots \; . \tag{57}$$

As our theory of the magnetoplasma modes is based on the quasistatic approximation [see Eq. (7)], clearly only the first two terms of the expansions (56) and (57) should be taken into account.[23] With the use of the Maxwell equations the functions $\mathbf{A}^{s(0)}(\mathbf{r})$ and $\mathbf{A}^{s(1)}(\mathbf{r})$ can be expressed by $A_\mu^{r(0)}, A_{\mu\nu}^{r(1)}$, and then by the radiation fields at $\mathbf{r}=\mathbf{0}$ (see Appendix B), so that the resulting vector potential inside the sphere is

$$A_\mu(\mathbf{r})=-\frac{3}{\epsilon+2}\frac{ic}{\omega}E_\mu^r(0)-\tfrac{1}{2}\varepsilon_{\mu\nu\kappa}r_\nu B_\kappa^r(0)-\frac{5}{2\epsilon+3}\frac{ic}{2\omega}[E_{\mu,\nu}^r(0)+E_{\nu,\mu}^r(0)]r_\nu \; , \tag{58}$$

where $\varepsilon_{\mu\nu\kappa}$ is the totally antisymmetric Levi-Cività tensor and the comma denotes differentiation.

Inserting Eqs. (41), (47), and (58) into Eq. (53), we obtain the perturbing Hamiltonian in terms of the oscillator amplitudes in the form

$$H_1=\frac{-iq^3nr_0}{2m^*\omega}\sum_{l,m,\tau}\gamma_{lm\tau}\omega_{lm\tau}\frac{1}{\omega_{lm\tau}^2-\omega_c^2}\times\left\{\frac{3}{\epsilon+2}\mathrm{Re}(iE_\mu^r e^{-i\omega t})D_\mu^{lm\tau}+\left[\frac{\omega r_0}{2c}\varepsilon_{\mu\nu\kappa}\mathrm{Re}(B_\kappa^r e^{-i\omega t})+\frac{5r_0}{2\epsilon+3}\mathrm{Re}\left[\frac{i}{2}(E_{\mu,\nu}^r+E_{\nu,\mu}^r)e^{-i\omega t}\right]\right]F_{\nu\mu}^{lm\tau}\right\}b_{lm\tau}(t)+\text{c.c.} \; , \tag{59}$$

where the argument $\mathbf{r}=\mathbf{0}$ of $\mathbf{E}^r$ and $\mathbf{B}^r$ was omitted, and

$$D_\mu^{lm\tau}=\left[\frac{2\pi}{3}\right]^{1/2}\delta_{\tau 1}\delta_{l1}\left[\left[1+\frac{\omega_c}{\omega_{1,-1,\tau}}\right]\delta_{m,-1}(\delta_{\mu 1}-i\delta_{\mu 2})-\left[1-\frac{\omega_c}{\omega_{11\tau}}\right]\delta_{m1}(\delta_{\mu 1}+i\delta_{\mu 2})+\sqrt{2}\left[1-\frac{\omega_c^2}{\omega_{10\tau}^2}\right]\delta_{m0}\delta_{\mu 3}\right] , \tag{60}$$

$$\begin{aligned}F_{\nu\mu}^{lm\tau}=&\left[\frac{2\pi}{15}\right]^{1/2}[1-\delta_{\tau 2}(1-\delta_{m,\mathrm{sgn}q})]\delta_{l2}\\&\times\left\{\left[\frac{2}{3}\right]^{1/2}\delta_{m0}\left[-\delta_{\nu 1}\delta_{\mu 1}+i\frac{\omega_c}{\omega_{20\tau}}(\delta_{\nu 1}\delta_{\mu 2}-\delta_{\mu 1}\delta_{\nu 2})-\delta_{\nu 2}\delta_{\mu 2}+2\left[1-\frac{\omega_c^2}{\omega_{20\tau}^2}\right]\delta_{\nu 3}\delta_{\mu 3}\right]\right.\\&-\delta_{m1}\left[\left[1-\frac{\omega_c}{\omega_{21\tau}}\right](\delta_{\mu 1}+i\delta_{\mu 2})\delta_{\nu 3}+\left[1-\frac{\omega_c^2}{\omega_{21\tau}^2}\right]\delta_{\mu 3}(\delta_{\nu 1}+i\delta_{\nu 2})\right]\\&+\delta_{m,-1}\left[\left[1+\frac{\omega_c}{\omega_{2,-1,\tau}}\right](\delta_{\mu 1}-i\delta_{\mu 2})\delta_{\nu 3}+\left[1-\frac{\omega_c^2}{\omega_{2,-1,\tau}^2}\right]\delta_{\mu 3}(\delta_{\nu 1}-i\delta_{\nu 2})\right]\\&\left.+\delta_{m2}\left[1-\frac{\omega_c}{\omega_{22\tau}}\right](\delta_{\mu 1}+i\delta_{\mu 2})(\delta_{\nu 1}+i\delta_{\nu 2})+\delta_{m,-2}\left[1+\frac{\omega_c}{\omega_{2,-2,\tau}}\right](\delta_{\mu 1}-i\delta_{\mu 2})(\delta_{\nu 1}-i\delta_{\nu 2})\right\} .\end{aligned} \tag{61}$$

Equations (60) and (61) show that (due to the condition $r_0 \ll k^{-1}$) only terms with $l=1,2$ appear in H_1, i.e., only ω_{1m1} and $\omega_{2m\tau}$ modes can be excited by external electromagnetic radiation.

At this point it is interesting to calculate the electric and magnetic multipole moments associated with various modes. Applying the quasistatic formula for the magnetic dipole moment

$$\mu^{lm\tau}(t)=\frac{1}{4c}C_{lm\tau}(t)\int_{r\le r_0}d^3r\,\mathbf{r}\times\mathbf{j}_{lm\tau}(\mathbf{r})+\text{c.c.} \; , \tag{62}$$

with $\mathbf{j}_{lm\tau}(\mathbf{r})$ given by (30), and for the "spherical" electric multipole moments

$$q_{l'm'}^{lm\tau}(t)=\frac{2l'+1}{8\pi}r^{l'+1}\int d\Omega[\Phi_{lm}^+(\mathbf{r})C_{lm\tau}(t)+\text{c.c.}]Y_{l'm'}^* \; , \tag{63}$$

with Φ_{lm}^+ given by (27), we obtain[24] [also using (22), (23), and (37)]

$$d_\nu^{lm\tau}(t)=\frac{1}{2}\left[\frac{3q^3nr_0}{m^*(\epsilon+2)}\frac{1}{\omega_{lm\tau}^2-\omega_c^2}D_\nu^{lm\tau}\gamma_{lm\tau}b_{lm\tau}(t)+\text{c.c.}\right] , \tag{64}$$

$$\mu_\nu^{lm\tau}(t)=\frac{1}{2}\left[\frac{iq^3nr_0^2}{2m^*c}\frac{\omega_{lm\tau}}{\omega_{lm\tau}^2-\omega_c^2}\varepsilon_{\nu\mu\kappa}F_{\mu\nu}^{lm\tau}\gamma_{lm\tau}b_{lm\tau}(t)+\text{c.c.}\right] , \tag{65}$$

$$Q_{\mu\nu}^{lm\tau}(t)=\frac{1}{2}\left[\frac{15q^3nr_0^2}{m^*(2\epsilon+3)}\frac{1}{\omega_{lm\tau}^2-\omega_c^2}(F_{\mu\nu}^{lm\tau}+F_{\nu\mu}^{lm\tau})\gamma_{lm\tau}b_{lm\tau}(t)+\text{c.c.}\right], \tag{66}$$

which represent the electric dipole, magnetic dipole, and electric quadrupole moments of the $lm\tau$ mode, respectively. With the help of Eqs. (64)–(66) the interaction Hamiltonian (59) can be rewritten in the following way:

$$H_1=\sum_{l,m,\tau}\{(i\omega_{lm\tau}/c)d_\nu^{lm\tau}(t)\text{Re}(A_\nu^r e^{-i\omega t})-\mu_\nu^{lm\tau}(t)\text{Re}(B_\nu^r e^{-i\omega t})+(i\omega_{lm\tau}/12c)Q_{\mu\nu}^{lm\tau}(t)\text{Re}[(A_{\mu,\nu}^r+A_{\nu,\mu}^r)e^{-i\omega t}]\}, \tag{67}$$

where all fields are taken at $\mathbf{r}=\mathbf{0}$. Using the Poisson brackets with the full Hamiltonian H_0+H_1, the time derivatives of the moments $d_\nu^{lm\tau}$ and $Q_{\mu\nu}^{lm\tau}$ can be determined to be equal to $-i\omega_{lm\tau}d_\nu^{lm\tau}$ and $-i\omega_{lm\tau}Q_{\mu\nu}^{lm\tau}$, respectively, plus corrections linear in $\mathbf{A}^r$. As the terms quadratic in $\mathbf{A}^r$ are dropped in H_1, we can substitute $\dot{d}_\nu^{lm\tau}$ and $\dot{Q}_{\mu\nu}^{lm\tau}$ for $-i\omega_{lm\tau}d_\nu^{lm\tau}$ and $-i\omega_{lm\tau}Q_{\mu\nu}^{lm\tau}$ in (67), respectively. Equation (67) then assumes the form of the standard multipole expansion of the interaction Hamiltonian in the adopted gauge with vanishing scalar potential. Thus, the interaction of the modes with external electromagnetic radiation is determined by their macroscopic moments.

VII. ABSORPTION

In order to calculate the optical absorption associated with the excitation of the modes, the full Hamiltonian $H=H_0+H_1$ can be quantized in a standard way, by replacing $b_{lm\tau}(t)$ and $b_{lm\tau}^*(t)$ amplitudes by annihilation and creation operators $b_{lm\tau}$ and $b_{lm\tau}^\dagger$, respectively. These operators satisfy the standard commutation rules for Bose operators.

The first-order perturbation calculation in H_1 shows that absorption occurs only if the radiation frequency ω is equal to one of the $\omega_{lm\tau}$ frequencies. The corresponding net (i.e., with stimulated emission subtracted) power $P_{lm\tau}$ absorbed by the sphere is nonzero only for $l=1$ and $l=2$ and is given by

$$P_{1m\tau}(\omega)=\delta(\omega-\omega_{1m\tau})\frac{3\omega_{1m1}\gamma_{1m1}^2q^2r_0^2\omega_1^4}{64\hbar(\omega_{1m1}^2-\omega_c^2)^2}\delta_{\tau 1}\left[2\left[1-\frac{\omega_c^2}{\omega_{1m1}^2}\right]^2|E_z^r|^2\delta_{m0}+\sum_{\alpha=\pm1}\left[1-\frac{\alpha\omega_c}{\omega_{1m1}}\right]^2|E_x^r-i\alpha E_y^r|^2\delta_{m\alpha}\right], \tag{68}$$

$$\begin{aligned}P_{2m\tau}(\omega)=&\delta(\omega-\omega_{2m\tau})[1-\delta_{\tau 2}(1-\delta_{m,-\text{sgn}q})]\frac{5\omega_{2m\tau}\gamma_{2m\tau}^2q^2r_0^4\omega_2^4}{3072\hbar(\omega_{2m\tau}^2-\omega_c^2)^2}\\&\times\left\{\frac{8}{3}\left|\left[3-\frac{2\omega_c^2}{\omega_{2m\tau}^2}\right]E_{z,z}^r+\frac{2\epsilon+3}{5}\frac{\omega_c}{c}B_z^r\right|^2\delta_{m0}\right.\\&+\sum_{\alpha=\pm1}\left[\left[1-\frac{\alpha\omega_c}{\omega_{2m\tau}}\right]^2\left|\left[2+\frac{\alpha\omega_c}{\omega_{2m\tau}}\right][E_{y,z}^r+E_{z,y}^r+i\alpha(E_{x,z}^r+E_{z,x}^r)]+\frac{2\epsilon+3}{5}\frac{\omega_c}{c}(B_y^r+i\alpha B_x^r)\right|^2\delta_{m\alpha}\right.\\&\left.\left.+4\left[1-\frac{\alpha\omega_c}{\omega_{2m\tau}}\right]^2|E_{x,y}^r+E_{y,x}^r+i\alpha(E_{x,x}^r-E_{y,y}^r)|^2\delta_{m2\alpha}\right]\right\}.\end{aligned} \tag{69}$$

The result is temperature independent due to the cancellation of the plasmon occupation functions for transitions with plasmon absorption and emission. The power does not depend on $\hbar$, indicating that the same result could be obtained by a classical calculation. This could be expected, as Planck's constant cancels out of the conservation equations relating photon and plasmon energies and the z components of the angular momenta. In case of the electron-plasmon interaction (characteristic energy-loss experiments) this is no longer true. Equations (68) and (69) are consistent with the adopted jellium model, as they depend only on charge and mass densities, and not on individual carrier parameters.

Let us point out several specific conclusions which follow from the expressions for the power absorbed.

The uniform electric field excites only the ω_{1m1} modes, i.e., the only modes possessing the electric dipole moment [see (64)]. The electric field parallel to $\mathbf{B}_0$ excites only the ω_{101} mode, while the CRA and cyclotron-resonance-inactive CRI circularly polarized fields excite the ω_{111} or $\omega_{1,-1,1}$ modes.

The uniform magnetic field excites only $\omega_{2m\tau}$ modes with $m=0,\pm1$, since according to (65) these are the only modes possessing the magnetic dipole moment (they also possess the electric quadrupole moments). The magnetic field parallel to $\mathbf{B}_0$ excites only the ω_{201} mode,[25] while the CRA-polarized magnetic field excites the $\omega_{2,-\text{sgn}q,2}$ and $\omega_{2,-\text{sgn}q,1}$ modes. The CRI-polarized field excites only the $\omega_{2,\text{sgn}q,1}$ mode.

A nonuniform electric field [i.e., $\mathbf{E}^r(0)=0$, $\nabla\mathbf{E}^r(0)\neq0$] excites only $\omega_{2m\tau}$ modes, i.e., the only modes possessing an electric quadrupole moment [see (66)]. Changing the orientation of a given nonuniform electric field in space, one can excite each of the ω_{2m1} modes with arbitrary m, as well as the $\omega_{2,-\text{sgn}q,2}$ mode.

One can see from (68) and (69) that $P_{1m\tau}$ and $P_{2m\tau}$ are proportional to r_0^3 and r_0^5, respectively. It follows, in particular, that the total absorption of a system of noninteracting spheres of a given total volume is independent of r_0 for $\omega=\omega_{1m\tau}$, but increases with r_0 for $\omega=\omega_{2m\tau}$.

It is interesting to estimate the relative intensities of the lines which should not be affected by the presence of weak damping of the carriers motion. Assuming all components of the fields to be of the same order, $\epsilon=10$, $\epsilon(\omega r_0/c)^2=\frac{1}{4}$, and $|\omega_c|/\omega_p=\frac{1}{4}$, we obtain

$$\int d\omega\, P_{2m1} \Big/ \int d\omega\, P_{1m1} \approx 4\times 10^{-3}$$

and

$$\int d\omega\, P_{2m2} \Big/ \int d\omega\, P_{1m1} \approx 10^{-3} .$$

Thus for $|\omega_c| \ll \omega_p$ the absorption for the frequencies with $l=2$ is much weaker than for those with $l=1$.

VIII. CONCLUSIONS

An analysis of free magnetoplasma oscillations of a small conducting sphere yielded an infinite number of $lm\tau$ eigenmodes, all with a vanishing space-charge density (exactly, or up to first order in ω_c/ω_p). The frequencies of these modes fall into two classes: those tending to zero for $\mathbf{B}_0 \to \mathbf{0}$ ($\tau=2$) and those tending to ω_l ($\tau=1$). The first class corresponds to helicon waves in an infinite medium, and the second to surface magnetoplasmons on a plane boundary. In addition, there also exist eigenmodes with nonvanishing space-charge density (bulk magnetoplasmons) which are discussed in detail in the following paper.[22] The external electromagnetic radiation, acting as a perturbation, leads to transitions between the levels of the quantized magnetoplasma. The absorption, however, turns out to be classical, i.e., it does not depend on $\hbar$. In the quasistatic approximation adopted in the present paper only nine of the $lm\tau$ modes interact with the external electromagnetic radiation. However, the remaining modes could be excited in some other way, e.g., they might be observable in characteristic-energy-loss experiments.

Some of our results can be compared with the theory of Ford, Furdyna, and Werner[10] (FFW), and the later results of Ford and Werner[11] (FW). FFW and FW use a classical approach, based on the Maxwell equations, where the external fields are included from the very beginning, i.e., they consider the forced motion of the plasma. Their theory also includes phenomenological damping. For the external fields FFW use only uniform magnetic or electric fields. Thus, they do not obtain the electric quadrupole resonances (our $\tau=1, l=2$ case) which can be excited by a linearly varying electric field. In the case of the magnetic excitation FFW neglect the electric field outside the sphere, and, likewise, for the electric excitation the magnetic field outside the sphere is neglected. In the case of electric excitation this approximation yields an error of second order in $\omega r_0/c$. For the magnetic excitation the approximation affects the boundary condition for the total current density at the surface of the sphere, leading to an error of zeroth order in $(\omega r_0/c)^2$ (note that for this excitation there exists a nonvanishing surface-charge density). One can observe directly that the lowest-order solution of FFW does not fulfill the boundary condition for the tangent component of the electric field at the surface of the sphere. In our treatment both the electric and the magnetic fields outside the sphere are included—we neglect only contributions of order $(\omega r_0/c)^2$ and higher in our quasistatic approximation. Our results should therefore coincide with the zero order in $(\omega r_0/c)^2$, no-damping results of FFW in the case of the electric excitation, but not in the case of the magnetic excitation.

The above remarks also suggest that the corrections of order of $\epsilon(\omega_p/\omega)^2(\omega r_0/c)^2$ calculated by FFW are comparable to the errors due to their approximation concerning fields outside the sphere.

In the electric excitations of FFW one can recognize our $\tau=1$, $l=1$, $m=0,\pm 1$ modes. The resonant frequencies of FFW which can be inferred from their expressions for the power absorbed agree with ours given by Eqs. (22) and (24). Similarly, our Eq. (68) for the power absorbed coincides with the corresponding formula of FFW (note here that FFW use the convention $\omega_c > 0$ for electrons).

The magnetic excitation of FFW corresponds to our $\tau=2$, $l=2$, $m=-\text{sgn}q$ mode. The resonant frequencies obtained from the expressions of FFW for the power absorbed do not agree with ours inferred from Eq. (23) for $l=2$. In particular, for $|\omega_c| \ll \omega_p$ our formula (26) does not agree with the resonant frequency of FFW, in that the factor $(l-1)/l^2=\frac{1}{4}$ is replaced by $-\frac{1}{2}$. It should be also noted that the approximation used by FFW precludes the excitation of the high-frequency modes $\tau=1$, $l=2$, $m=0,\pm 1$ by the uniform magnetic field. For these modes the electric field outside the sphere is particularly essential.

In their later paper[11] Ford and Werner give the formal, rigorous solution for the problem of a gyrotropic sphere of arbitrary size and for arbitrary dc magnetic field. However, they were able to obtain analytical results only in the Rayleigh limit and for $|q/k| \gg 1$, where q and k are the wave numbers of radiation inside and outside the sphere, respectively. In that case FW reproduce the formulas of FFW. The condition $|q/k| \gg 1$ is not fulfilled in many important cases. For instance, in the limit of lossless plasma and for $\mathbf{B}_0=\mathbf{0}$ it reduces to $\epsilon(1-\omega_p^2/\omega^2) \gg 1$ and does not hold for $\omega=\omega_p$ or $\omega=\omega_l$. Only in the case when the condition $|q/k| \gg 1$ may be relaxed (zeroth-order approximation for the electric excitation of FW and FFW) are the expressions of FW accurate. Still their results in the general case can be useful for a numerical treatment.[6]

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APPENDIX A: MAGNETOPLASMA OSCILLATIONS IN AN ELLIPSOID

Retaining all assumptions made for the sphere, we obtain the same set of equations (1)–(4) determining carrier displacements and fields inside and outside the ellipsoid. Equations (5) and (6) should now be taken at the surface of the ellipsoid with the unit vector normal to the surface $\hat{\mathbf{r}}_\perp$ instead of $\hat{\mathbf{r}}_0$.

In the zero-magnetic-field case we choose the system of coordinates along the principal axes of the ellipsoid and solve the Laplace equations in ellipsoidal coordinates. Apart from the $\omega=\omega_p$ mode we obtain frequencies of the vanishing space-charge-density modes in a fairly complicated form which will not be presented here.

In the presence of an arbitrarily oriented external magnetic field $\mathbf{B}_0$ we choose the z axis along $\mathbf{B}_0$, as shown in Fig. 1, so that Eqs. (9) and (10) remain unchanged, while the boundary conditions (11) and (12) assume the form

$$\Phi^+(\mathbf{r})=\Phi^-(\mathbf{r}) , \tag{A1}$$

$$\begin{aligned}(\omega^2-\omega_c^2)\hat{\mathbf{r}}_\perp\cdot\nabla\Phi^+(\mathbf{r})&=\epsilon(\omega^2-\omega_p^2-\omega_c^2)\hat{\mathbf{r}}_\perp\cdot\nabla\Phi^-(\mathbf{r})\\&\quad+\epsilon(\omega_p\omega_c/\omega)^2(\hat{\mathbf{r}}_\perp\cdot\hat{\mathbf{z}})\frac{\partial\Phi^-(\mathbf{r})}{\partial z}\\&\quad+i\epsilon(\omega_p^2\omega_c/\omega)\hat{\mathbf{r}}_\perp\cdot[\hat{\mathbf{z}}\times\nabla\Phi^-(\mathbf{r})] ,\end{aligned} \tag{A2}$$

where $\mathbf{r}$ runs on the surface of the ellipsoid. Introducing spherical coordinates, we can express r as a function of θ and ϕ from the equation of an ellipsoid, so that (A1) and (A2) depend only on θ and ϕ.

We shall now introduce two nonsphericity parameters:

$$\eta_1=1-(r_0/a)^2 , \tag{A3}$$

$$\eta_2=1-(r_0/b)^2 , \tag{A4}$$

where a, r_0, and b are the semiaxes, and we choose $b\le r_0\le a$. In the following we assume η_1, $|\eta_2|\ll 1$ and confine ourselves to the first-order perturbation calculus in η_1 and η_2, i.e., we linearize (A1) and (A2) in these parameters.

1. Vanishing space-charge-density modes

Demanding that both components of (9) vanish separately, we obtain

$$\begin{aligned}(r_0/q)\Phi^-(\mathbf{r})&=C_{lm}(r/r_0)^lY_{lm}(\theta,\phi)\\&\quad+\sum_{l'=1}^{\infty}\sum_{\substack{m'=\pm l',\\ \pm(l'-1)}}C'_{l'm'}(r/r_0)^{l'}Y_{l'm'}(\theta,\phi) ,\end{aligned} \tag{A5}$$

$$\begin{aligned}(r_0/q)\Phi^+(\mathbf{r})&=C_{lm}(r/r_0)^{-(l+1)}Y_{lm}(\theta,\phi)\\&\quad+\sum_{l''=1}^{\infty}\sum_{m''=-l''}^{l''}C''_{l''m''}(r/r_0)^{-(l''+1)}\\&\quad\quad\times Y_{l''m''}(\theta,\phi) ,\end{aligned} \tag{A6}$$

where $l\ge 1$, $m=\pm l,\pm(l-1)$. Here the C_{lm} are the zero-order amplitudes, while $C'_{l'm'}$ and $C''_{l''m''}$ are linear in η_1 and η_2. We also have

$$\omega=\omega_{lm\tau}+\omega'_{lm\tau} , \tag{A7}$$

where $\omega_{lm\tau}$ is given by (22)—(26), while $\omega'_{lm\tau}$ is linear in η_1 and η_2. Inserting (A5)—(A7) into (A1) and (A2), and keeping only terms linear in η_1 and η_2, we obtain two equations in θ and ϕ. Integrating these equations with spherical harmonics $Y_{\tilde{l}\tilde{m}}(\theta,\phi)$ leads[26] to a system of linear equations for $(C'_{l'm'}/C_{lm})$ and $(C''_{l''m''}/C_{lm})$. The value of $\omega'_{lm\tau}$ can be obtained using only the equation with $(\tilde{l},\tilde{m})=(l,m)$. For $m=\pm l$ and $\tau=1$ we obtain

$$\omega'_{lm1}=\frac{(\omega_{lm1}^2-\omega_c^2)(2l+1)\omega_{lm1}^2}{2(2l+3)[\epsilon m\omega_c\omega_p^2-2\omega_{lm1}^3(l\epsilon+l+1)]}W(\theta_0,\phi_0) , \tag{A8}$$

and for $m=\pm(l-1)$ (and arbitrary τ) we obtain

$$\omega'_{lm\tau}=\frac{\omega_{lm\tau}(\omega_{lm\tau}^2-\omega_c^2)[4l/\epsilon(2l+3)+1-(\omega_p/\omega_{lm\tau})^2]}{2\{2(\omega_c\omega_p/\omega_{lm\tau})^2+m\omega_p^2(\omega_c/\omega_{lm\tau})-2\omega_{lm\tau}^2[l+(l+1)/\epsilon]\}}W(\theta_0,\phi_0) , \tag{A9}$$

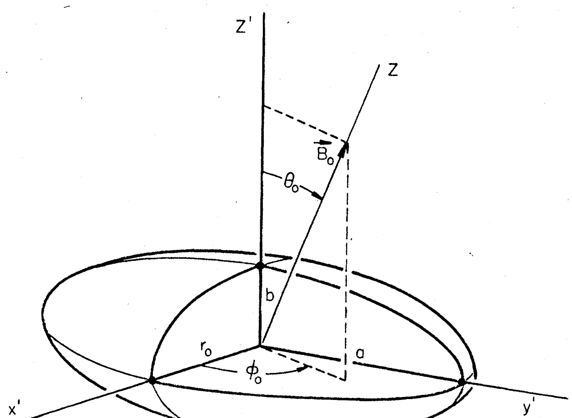

FIG. 1. Coordinate systems showing the relative orientation of the ellipsoid and the magnetic field. The z direction is taken along the static magnetic field $\mathbf{B}_0$. The surface of the ellipsoid is determined in primed coordinates by $(x'/r_0)^2+(y'/a)^2+(z'/b)^2=1$, where r_0, a, and b are the principal semiaxes. The direction of $\mathbf{B}$ relative to the principal axes of the ellipsoid is determined by angles θ_0 and ϕ_0.

where

$$W(\theta_0,\phi_0)=\eta_1(1-3\sin^2\theta_0\sin^2\phi_0)+\eta_2(1-3\cos^2\theta_0) . \tag{A10}$$

The factor depending on θ_0 and ϕ_0 achieves the largest absolute values when $\mathbf{B}_0$ is along the shortest or the longest semiaxis of the ellipsoid, and it vanishes for $\mathbf{B}_0$ in $(\pm1,\pm1,\pm1)$ directions in the principal-axes system of the ellipsoid.

Equations (A8) and (A9) can be used to obtain the eigenmode frequencies in an ellipsoid without the magnetic field. In that case the system of coordinates should be chosen along the principal axes of the ellipsoid, i.e., we set $\theta_0=0$ or $\theta_0=\phi_0=\pi/2$ (see Fig. 1). Expressions for the eigenfrequencies in spheroids ($\eta_1=0$ or $\eta_2=0$) may be found in Ref. 27. The general formula for ω_{lm} is correct in this paper, but there are some errors in the expansion in terms of the eccentricity of the ellipse generating the spheroid. If this expansion is performed properly, the results coincide with ours.

2. High-frequency modes

Assuming $\omega \sim \omega_p \gg |\omega_c|$ and keeping only the first-order terms in ω_c/ω_p, η_1, and η_2 (i.e., neglecting all their products), we obtain the $\omega=\omega_p$ mode and $\omega=\tilde{\omega}_{lm}$ frequencies given by the zeros of a $(2l+1)\times(2l+1)$ determinant. In order to obtain analytical formulas for $\tilde{\omega}_{lm}$ one must assume that the perturbation given by the magnetic field $\mathbf{B}_0$ is stronger than that arising from nonsphericity, i.e., that the magnetic field first splits the $(2l+1)$-degenerate mode $\omega=\omega_{pl}$ into ω_{lm1} modes, and then the nonsphericity slightly modifies these frequencies,

$$\omega=\omega_{lm1}+\omega'_{lm1} , \tag{A11}$$

so that we deal with a nondegenerate perturbation calculus. The potentials inside and outside the ellipsoid are then given by (A5) and (A6), respectively, with the restriction on m abandoned and the summation over $m'=-l',\dots,l'$. A procedure similar to the one used in subsection 1 above leads to

$$\omega'_{lm1}=-\frac{(2l+1)(3m^2-l^2-l)\omega_{pl}}{4l(\epsilon l+l+1)(2l-1)(2l+3)}W(\theta_0,\phi_0) . \tag{A12}$$

The factor depending on the orientation of $\mathbf{B}_0$ is the same as in (A8) and (A9). For $|\omega_c|/\omega_p \ll 1$, (A8) and (A9) reduce to (A12) for $m=\pm l$ and $m=\pm(l-1)$, respectively. The validity of (A12) is restricted by the condition that the shift arising from nonsphericity should be much smaller than the splitting caused by $\mathbf{B}_0$. This leads to the inequality

$$|\omega_c|/\omega_{pl} \gg |\eta_i|/2\epsilon , \quad i=1,2 . \tag{A13}$$

For large values of ϵ this will be satisfied for $|\eta_i| \approx |\omega_c|/\omega_p$ or even higher. It follows from (A8), (A9), and (A12) that $|\omega'_{lm1}|/\omega_{lm1} \sim |\eta_i|/\epsilon$ and $|\omega'_{lm2}|/\omega_{lm2} \sim |\eta_i|$. Thus, for $\epsilon \gg 1$ the effect of nonsphericity on high-frequency modes is very small.

APPENDIX B: VECTOR POTENTIAL INSIDE THE SPHERE

We must express the expansion coefficients $\mathbf{A}^{s(\alpha)}$ in Eq. (57) ($\alpha=0,1$) in terms of $\mathbf{A}^{r(\alpha)}$ of Eq. (56). The wave equation, the gauge, and the boundary conditions (at the surface of the sphere and at infinity) can be separated in orders of $r_0\omega/c$, and if the terms quadratic in $r_0\omega/c$ are neglected, they become, respectively,

$$A^{s(\alpha)}_{\mu,\nu\nu}(\mathbf{r})=0 , \tag{B1}$$

$$A^{s(\alpha)}_{\mu,\mu}(\mathbf{r})=0 , \quad A^{r(1)}_{\mu,\mu}=0 , \tag{B2}$$

$$\varepsilon_{\kappa\nu\mu}[A^{s(\alpha)}_{\mu,\nu}(\mathbf{r}_{0+})-A^{s(\alpha)}_{\mu,\nu}(\mathbf{r}_{0-})]=0 , \tag{B3}$$

$$\varepsilon_{\kappa\nu\mu}r_{0\nu}[A^{s(\alpha)}_{\mu}(\mathbf{r}_{0+})-A^{s(\alpha)}_{\mu}(\mathbf{r}_{0-})]=0 , \tag{B4}$$

$$r_{0\mu}[A^{s(0)}_{\mu}(\mathbf{r}_{0+})-\epsilon A^{s(0)}_{\mu}(\mathbf{r}_{0-})]=(\epsilon-1)r_{0\mu}A^{r(0)}_{\mu} , \tag{B5a}$$

$$r_{0\mu}[A^{s(1)}_{\mu}(\mathbf{r}_{0+})-\epsilon A^{s(1)}_{\mu}(\mathbf{r}_{0-})]=(\epsilon-1)r_{0\mu}r_{0\nu}A^{r(1)}_{\mu\nu} , \tag{B5b}$$

$$A^{s(\alpha)}_{\mu}(\mathbf{r})\to 0 \quad \text{for } r\to\infty . \tag{B6}$$

Here, $\mathbf{r}_{0+}$ and $\mathbf{r}_{0-}$ denote $\mathbf{r}$ tending to $\mathbf{r}_0$ from outside and inside the sphere, respectively.

The solutions of Eq. (B1) for $\alpha=0$ and $\alpha=1$, satisfying (B2), (B3), and (B6), can be sought in the form

$$A^{s(0)}_{\mu}(\mathbf{r})=a_\mu\theta(r_0-r)+r_0^3(m_\mu r^{-3}-3m_\nu r_\nu r_\mu r^{-5})\theta(r-r_0), \tag{B7}$$

$$A^{s(1)}_{\mu}(\mathbf{r})=r_0^{-1}a_{\mu\nu}r_\nu\theta(r_0-r)+\tfrac{1}{6}r_0^4 d_{\nu\kappa}\left[\frac{1}{r}\right]_{,\nu\kappa\mu}\theta(r-r_0) , \tag{B8}$$

where $\theta(x)$ is the Heaviside step function and $d_{\nu\kappa}=d_{\kappa\nu}$, $d_{\nu\nu}=0$. Outside the sphere, $\mathbf{A}^{s(0)}$ and $\mathbf{A}^{s(1)}$ represent the dipole and the quadrupole fields, respectively. The constant coefficients in (B7) and (B8) can be determined with the use of Eqs. (B4) and (B5) and the result is

$$a_\mu=m_\mu=-[(\epsilon-1)/(\epsilon+2)]A^{r(0)}_{\mu} , \tag{B9}$$

$$a_{\mu\nu}=d_{\mu\nu}=-[(\epsilon-1)/(2\epsilon+3)][A^{r(1)}_{\mu\nu}+A^{r(1)}_{\nu\mu}] . \tag{B10}$$

The total vector potential inside the sphere can now be written as

$$A_\mu(\mathbf{r})=\frac{3}{\epsilon+2}A^{r(0)}_{\mu}+\frac{\omega}{2c}(A^{r(1)}_{\mu\nu}-A^{r(1)}_{\nu\mu})r_\nu + \frac{\omega}{2c}\frac{5}{2\epsilon+3}(A^{r(1)}_{\mu\nu}+A^{r(1)}_{\nu\mu})r_\nu , \tag{B11}$$

where (55)—(57) and (B7)—(B10) were used. With the use of (56) the coefficients $A^{r(0)}_{\mu}$ and $A^{r(1)}_{\mu\nu}$ can be easily expressed by the fields in the center of the sphere, which leads to (58).

[1]G. Mie, Ann. Phys. (Leipzig) **25**, 377 (1908).
[2]*Electromagnetic Surface Modes,* edited by A. D. Boardman (Wiley, Chichester, 1982).
[3]R. Bowers, C. Legendy, and F. Rose, Phys. Rev. Lett. **7**, 339 (1961); R. Bowers, *Plasma Effects in Solids* (Dunod, Paris, 1965).
[4]J. D. Wiley, P. S. Peercy, and R. N. Dexter, Phys. Rev. **181**, 1173 (1969); **181**, 1181 (1969).
[5]T. A. Evans and J. K. Furdyna, Phys. Rev. B **8**, 1461 (1973); F. E. Rose, M. T. Taylor, and R. Bowers, Phys. Rev. **127**, 1122 (1962); S. A. Werner, T. K. Hunt, and G. W. Ford, Solid State Commun. **14**, 1217 (1974).
[6]J. R. Dixon, Jr. and J. K. Furdyna, Phys. Rev. B **18**, 6770 (1978).
[7]J. R. Dixon, Jr. and J. K. Furdyna, J. Appl. Phys. **51**, 3762 (1980).
[8]J. R. Dixon, Jr. and J. K. Furdyna, Solid State Commun. **35**, 195 (1980).
[9]G. W. Ford and S. A. Werner, Phys. Rev. B **8**, 3702 (1973).
[10]G. W. Ford, J. K. Furdyna, and S. A. Werner, Phys. Rev. B **12**, 1452 (1975).
[11]G. W. Ford and S. A. Werner, Phys. Rev. B **18**, 6752 (1978).
[12]C. J. Crowell and R. H. Ritchie, Phys. Rev. **172**, 436 (1968).
[13]J. C. Ashley, T. L. Ferrel, and R. H. Ritchie, Phys. Rev. B **10**, 554 (1974).
[14]R. Ruppin and R. Englman, J. Phys. C **1**, 630 (1968).
[15]R. Fuchs and K. L. Kliewer, J. Opt. Soc. Am. **58**, 319 (1968).
[16]A. A. Lushnikov, V. V. Maksimenko, and A. J. Simonov, in

Electromagnetic Surface Modes, Ref. 2, Chap. 8.
[17]I. Yokota, J. Phys. Soc. Jpn. **16**, 2075 (1961).
[18]U. Kreibig and P. Zacharias, Z. Phys. **231**, 128 (1970).
[19]J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, New York, 1973).
[20]P. E. Batson, Solid State Commun. **34**, 477 (1980).
[21]G. Dresselhaus, A. F. Kip, and C. Kittel, Phys. Rev. **98**, 869 (1955).
[22]S. Goettig and W. Trzeciakowski, following paper, Phys. Rev. B **31**, 7726 (1985).
[23]We shall be interested in $\omega=\omega_{lm\tau}$.
[24]Electric moments $d_\nu^{lm\tau}$ and $Q_{\mu\nu}^{lm\tau}$ are expressed by $q_{l'm'}^{lm\tau}$ in the standard manner. See, e.g., Ref. 19.
[25]The magnetic field of radiation parallel to $\mathbf{B}_0$ also excites the "bulk mode," described in detail in Ref. 22.
[26]Many formulas useful for this calculation can be found in L. Robin, *Fonctions Spheriques de Légendre et Fonctions Sphéroidales* (Gauthier-Villars, Paris, 1957—1959).
[27]A. Moussiaux, A. Ronveaux, and A. Lucas, Can. J. Phys. **55**, 1423 (1977).