Dynamic renormalization-group theory of interfaces: Model A

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The dynamics of the interface of an Ising-like system driven by relaxational dynamics is studied within the framework of a first-order ϵ expansion near four dimensions. The starting point uses the static and dynamic equations of the bulk system (model A) and an interface is produced by imposing suitable boundary conditions. The interfacial dispersion relation is of the form $\omega_q = -i\Gamma q^z \Omega(q\xi)$, with $z=2+O(\epsilon^2)$ and $\Omega(x)$ universal, and satisfies the Goldstone theorem for the spontaneously broken Euclidean symmetry.

I. INTRODUCTION

There has been considerable interest, recently, in surface and interfacial properties of systems near the critical point.[1] Static properties in such inhomogeneous systems have been investigated by a variety of methods, including the use of renormalization-group (RG) techniques.

Dynamical phenomena have also received a great deal of attention, and some progress has been made in investigations of critical dynamics involving rigid bounding surfaces.[2] The case of the dynamics of interfaces separating coexisting phases is perhaps richer, and little progress has been achieved in this area of critical phenomena using RG methods. Studies of this kind would allow for a better understanding of capillary waves and capillary-wave-like excitations. Critical dynamics of such interfaces has only begun to be explored in any systematic way,[3,4] and much remains to be done within mean-field theory as well as within the RG theory of fluctuations.

It has also been appreciated that the motion of interfacial structures may play an important role in the understanding of various nonequilibrium phenomena,[5] such as the evolution of order and spinodal decomposition. Furthermore, interface instabilities themselves have been receiving a good deal of attention from fundamental perspectives as well as from practical ones in fields of materials research.[6]

From a fundamental point of view, one would like to begin at the bulk level, that is, from a (reduced) Hamiltonian which correctly describes homogeneous bulk properties. Then, by arranging the thermodynamic parameters appropriately and by imposing suitable boundary conditions, one may produce an interface in the system. Interface properties then should follow from the knowledge of thermodynamic, correlation, and response functions of the full inhomogeneous system. For real fluids, in which case the velocity fields are coupled to a (generally) conserved order parameter, this is a difficult program. Felderhof[7] and Langer and Turski[8] have made some progress in this regard by treating the statistical mechanics of the fluid at mean-field level and neglecting viscosity. These calculations essentially amount to the hydrodynamics of a diffuse interface, but even then the calculations are not without their difficulties. In the long-wavelength limit, one recovers the interface capillary-wave dispersion relation (containing the properly identifiable surface tension at mean-field level), in agreement with that coming from purely macroscopic hydrodynamics.

The program suggested above holds out the hope of further progress with implications on general predictions of scaling. For if the interface dispersion relation can be found (say, from the pole of an appropriate response function), scaling suggests that it takes the form

$$\omega_q \sim q^z \Omega(q\xi) , \tag{1.1}$$

where q is the $(d-1)$-dimensional wave vector in the plane of the interface and ξ is the bulk correlation length. The exponent z is a dynamic exponent which may be identified with the dynamic scaling exponent of the bulk homogeneous system. As $q\xi \to \infty$, i.e., as the critical regime is approached, the interface disappears and at critically there should be one characteristic frequency $\omega_q \sim q^z$.[9] On the other hand, in hydrodynamic regime, $q\xi \to 0$, various invariance properties determine the spectrum, and one expects $\omega_q \sim q^\sigma$ at fixed $T<T_c$, with σ typically an integer or simple fraction. The scaling function $\Omega(x)$ must contain a singularity as $x\to 0$ to go over to this new behavior. The correlation and response functions of the inhomogeneous system should also exhibit interesting crossover behavior between the two regimes.

This scaling picture has been used by Bausch *et al.*[3] to derive the bulk dynamic exponent z beginning from a phenomenological drumhead model of an interface. The above argument suggests that if the drumhead system can be driven to criticality (indicating *bulk* criticality), the characteristic frequency $\omega_q \sim q^z$ allows identification of the bulk dynamic exponent z. Bausch *et al.* were able to perform such a calculation in bulk $d=1+\epsilon$ dimensions and thereby extract an estimate of z to second order in ϵ. The starting drumhead model for the interface has only been shown to follow from the Landau-Ginzburg-Wilson ϕ^4 model in the limit $T \ll T_c$.[10] Furthermore, one might imagine mechanisms which could make the identification of the bulk exponent from the surface dispersion relation questionable (e.g., the presence of possible surface renormalization counterterms).

Further investigation of some of these issues would seem desirable. As a first step we demonstrate here the computation of the dispersion relation ω_q for an interface of an Ising-like system described by the usual ϕ^4 reduced Hamiltonian and model-A relaxational dynamics.[9] A brief summary of this work has appeared elsewhere.[4]

The remainder of this paper is divided as follows. In Sec. II we define our model for the dynamic interface and adapt the standard field-theoretic dynamic RG formalism[11,12] to the case of an inhomogeneous system in which a diffuse interface is present. A perturbation scheme is constructed and used to compute the inhomogeneous system's linear response function to one-loop order. In Sec. III, a renormalization scheme is set up to make perturbation theory finite, i.e., cutoff-independent, in the critical regime. A quantity closely related to the renormalized response function is calculated explicitly to first order in $\epsilon=4-d$, and from this the interface dispersion relation ω_q follows. This is shown to be of the form $\omega_q \sim q^z \Omega(q\xi)$, with $z=2+O(\epsilon^2)$, and the function $\Omega(x)$ is calculated explicitly. A discussion of this result follows in Sec. IV, together with our conclusions. In Appendix A the dispersion relation for the corresponding one-phase bulk ordered system is derived, to $O(\epsilon)$, for comparison with the interface spectrum. In Appendix B the result $\lim_{q\to 0}\omega_q=0$ for the interface dispersion relation is shown to follow nonperturbatively from a general Ward identity.

II. THEORETICAL MODEL AND PERTURBATION THEORY

In this paper we will be concerned with the interfacial dynamics of a system modeled by the kinetic Ising model,[13] or by its continuum version, model A.[14] Purely relaxing systems of this kind include Ising antiferromagnets and order-disorder alloys. In the continuum version of the model, $\phi(x,t)$ is the nonconserved local order-parameter field (such as the staggered magnetization density, or the sublattice concentration), and the equation of motion is given by

$$\frac{\partial\phi(x,t)}{\partial t}=-\Gamma_0\frac{\delta\mathscr{H}}{\delta\phi(x,t)}+\eta(x,t)\ . \tag{2.1}$$

Here, $\mathscr{H}$ is the usual Landau-Ginzburg-Wilson effective Hamiltonian,

$$\mathscr{H}=\int d^dx\left[\tfrac{1}{2}r_0\phi^2+\tfrac{1}{2}(\nabla\phi)^2+\frac{\lambda_0}{4!}\phi^4-h\phi\right]\ , \tag{2.2}$$

which is believed to describe appropriately the static properties of our system in the limit of long wavelength and large correlation length. Γ_0 is the (bare) kinetic coefficient, $r_0 \propto T-T_c^{\mathrm{MF}}$ (with T_c^{MF} the mean-field critical temperature), and λ_0 is the (bare) coupling constant. $\eta(x,t)$ is a Gaussian random force obeying

$$\langle\eta(x,t)\eta(x',t')\rangle=2\Gamma_0\delta(x-x')\delta(t-t')\ ,$$

so that the fluctuation-dissipation theorem is satisfied.[15] We introduce an interface in analogy with equilibrium calculations[16,17] by requiring that $\langle\phi(x,t)\rangle=M(z)$, where $M(z)$ is the equilibrium order-parameter profile satisfying the boundary condition

$$\lim_{z\to\pm\infty} M(z)=\pm M_B\ . \tag{2.3}$$

Here, M_B is the bulk order parameter, and we are dealing with a flat interface perpendicular to the spatial z direction. For the $h=0$ interface and the Hamiltonian of Eq. (2.2), the interfacial profile is known to one-loop order.[16,17] Using minimal subtraction methods, we have

$$M(z)=M_B\tanh\left[\frac{\kappa_R z}{2}\right]\times\left[1-\frac{\pi\sqrt{3}}{12}u\,\mathrm{sech}^2\left[\frac{\kappa_R z}{2}\right]+O(u^2)\right]\ , \tag{2.4}$$

where

$$M_B^2=\frac{3\kappa_0^2}{u}[1-\tfrac{1}{2}u\ln\kappa_0^2+O(u^2)]\ , \tag{2.5}$$

$$\kappa_R=\kappa_0[1+\tfrac{1}{8}u(\ln\kappa_0^2+\pi\sqrt{3}-3)+O(u^2)]\ . \tag{2.6}$$

Here and in the following, $\kappa_0^2=|2\tau|$, $\tau=(T-T_c)/T_c$, and $u\to u^*=2\epsilon/3+O(\epsilon^2)$ is the dimensionless renormalized coupling constant with its fixed-point value u^*. M_B and κ_R exponentiate to $\sim\kappa_0^\beta$ and $\sim\kappa_0^\nu$, respectively.

The quantity of interest in the description of a dynamical interface is the linear response function $R(x,t;x',t')$, defined in the usual way be

$$\langle\phi(x,t)\rangle-M(z)=\int dx'\,dt'\,R(x,t;x',t')h(x',t')+O(h^2)\ , \tag{2.7}$$

where $h(x,t)$ is the external field coupling to $\phi(x,t)$. As usual, the response function is related to the correlation function,

$$C(x,t;x',t')=\langle\phi(x,t)\phi(x',t')\rangle-M(z)M(z')\ , \tag{2.8}$$

through the fluctuation-dissipation theorem[15]

$$\tfrac{1}{2}\omega C(x,x';\omega)=\mathrm{Im}R(x,x';\omega)\ , \tag{2.9}$$

where ω is the frequency of the Fourier component of C or R appearing in Eq. (2.9).

Thus far, averages, denoted by $\langle\ \rangle$, have been taken with respect to the random force distribution

$$P(\eta)\sim\exp\left[-(4\Gamma_0)^{-1}\int dx\,dt\,\eta^2\right]\ .$$

However, it is convenient to replace the random force $\eta(x,t)$ by an auxiliary field $\hat{\phi}(x,t)$,[11,12] defined in such a way that, for a given operator $\mathscr{O}(\phi)$,

$$\langle\mathscr{O}\rangle_\eta=\frac{\int D\eta\,\mathscr{O}(\phi(\eta))P(\eta)}{\int D\eta\,P(\eta)}=\frac{\int D\phi\,D\hat{\phi}\,\mathscr{O}(\phi)W(\phi,\hat{\phi})}{\int D\phi\,D\hat{\phi}\,W(\phi,\hat{\phi})}=\langle\mathscr{O}\rangle_{\phi,\hat{\phi}}\ . \tag{2.10}$$

This procedure generates expectation values taken with respect to the statistical weight

$$W(\phi,\hat\phi)=\exp\left[-\int dx\,dt\,\mathscr{L}(\phi,\hat\phi)\right],$$

where

$$\mathscr{L}(\phi,\hat\phi)=\Gamma_0\hat\phi^2+i\hat\phi\left[\frac{\partial\phi}{\partial t}+\Gamma_0\frac{\delta\mathscr{H}}{\delta\phi}\right]$$
$$=\Gamma_0\hat\phi^2+i\hat\phi\left[\frac{\partial}{\partial t}+\Gamma_0(-\nabla^2+r_0)\right]\phi+i\frac{\lambda_0\Gamma_0}{3!}\hat\phi\phi^3 \tag{2.11}$$

is the Lagrangian of the resulting field theory. As is common to all field-theoretic RG treatments,[11,12] a Jacobian contribution,

$$j(\phi)=-\tfrac{1}{2}\Gamma_0\frac{\partial}{\partial\phi}\left[\frac{\delta\mathscr{H}}{\delta\phi}\right], \tag{2.12}$$

to $\mathscr{L}$ is generated, but may be omitted because of the requirements of causality. The field $\hat\phi(x,t)$ has the physical meaning of a response field, since it can be shown that (for model A)

$$R(x,t;x',t')=i\Gamma_0\langle\phi(x,t)\hat\phi(x',t')\rangle\,. \tag{2.13}$$

In the presence of one or more ordered phases, we introduce the shift $\phi(x,t)\to M(z)+\phi(x,t)$, where $M(z)$ is the interfacial profile and where now $\langle\phi(x,t)\rangle=0$. The new Lagrangian is then

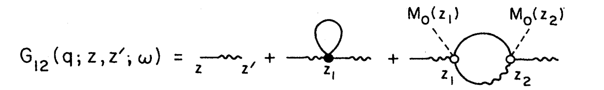

FIG. 1. Diagrammatic representation of the response function to one-loop order for $T<T_c$. The circles denote the third- and fourth-order interaction couplings in Eq. (2.14).

$$\mathscr{L}=\Gamma_0\hat\phi^2+i\hat\phi\left[\frac{\partial}{\partial t}-\Gamma_0\nabla^2+\Gamma_0r_0+\frac{\lambda_0\Gamma_0}{2}M^2(z)\right]\phi$$
$$+i\frac{\lambda_0\Gamma_0}{2}M(z)\hat\phi\phi^2+i\frac{\lambda_0\Gamma_0}{3!}\hat\phi\phi^3\,, \tag{2.14}$$

with the usual omission of terms linear in ϕ and $\hat\phi$, consistent with $\langle\phi\rangle=\langle\hat\phi\rangle=0$. At this point, a perturbation expansion in the terms $M\hat\phi\phi^2$ and $\hat\phi\phi^3$ can be generated in order to extract the static and dynamic interfacial properties. To start with, the free propagators

$$G^{(0)}_{\alpha\beta}(1,2)=\langle\phi_\alpha(x_1,t_1)\phi_\beta(x_2,t_2)\rangle_0\,,$$

where $\phi_1=\phi$ and $\phi_2=\hat\phi$, must be evaluated. We proceed in the usual fashion[18] by introducing external fields $l_1(x,t)$ and $l_2(x,t)$, coupling to ϕ_1 and ϕ_2, and by calculating the free theory's generating functional,

$$Z_0[l_1,l_2]=\frac{\int D\phi_1D\phi_2\exp\left[-\int dx\,dt\,\mathscr{L}_0+\int dx\,dt\,l_\alpha\phi_\alpha\right]}{\int D\phi_1D\phi_2\exp\left[-\int dx\,dt\,\mathscr{L}_0\right]}$$
$$=\exp\left[\tfrac{1}{2}\int dx\,dt\,dx'\,dt'\,l_\alpha(x,t)G^{(0)}_{\alpha\beta}(x,t;x',t')l_\beta(x',t')\right]. \tag{2.15}$$

If $\mathscr{L}_0=\frac{1}{2}\phi_\alpha\hat A_{\alpha\beta}\phi_\beta$ is the (symmetrized) free part of the Lagrangian, Eq. (2.14), we find that the $G^{(0)}_{\alpha\beta}$ satisfy the matrix differential equation

$$\hat A_{\alpha\beta}(x,t)G^{(0)}_{\beta\gamma}(x,t;x',t')=\delta_{\alpha\gamma}\delta(x-x')\delta(t-t')\,. \tag{2.16}$$

Introducing the operators

$$\hat D(x,t)=\frac{\partial}{\partial t}+\Gamma_0\hat\Delta(x),\quad \hat\Delta(x)=-\nabla_x^2+r_0+\tfrac{1}{2}\lambda_0M^2(z)\,, \tag{2.17}$$

one finds that Eq. (2.16) can be explicitly written as

$$\begin{bmatrix}0 & i\hat D(x_1,-t_1)\\ i\hat D(x_1,t_1) & 2\Gamma_0\end{bmatrix}\begin{bmatrix}G^{(0)}_{11}(1,2) & G^{(0)}_{12}(1,2)\\ G^{(0)}_{21}(1,2) & G^{(0)}_{22}(1,2)\end{bmatrix}=\delta(1-2)\,. \tag{2.18}$$

If we denote by $e^{-i\omega t+iq\rho}\zeta^{(\mu)}(z)$ the eigenfunctions of $\hat D$, where ρ is the position vector in the plane of the interface and the $\zeta^{(\mu)}(z)$ are the eigenfunctions of the kink operator,[16,17]

$$[-\partial_z^2+r_0+\tfrac{1}{2}\lambda_0M_0^2(z)]\zeta^{(\mu)}(z)=E^{(\mu)}\zeta^{(\mu)}(z)\,, \tag{2.19}$$

then Eq. (2.18) can be solved by making use of the spectral decomposition

$$G^{(0)}_{\alpha\beta}(1,2)=\int d\omega\, dq \sum_{\mu} e^{-i\omega(t_1-t_2)+iq(\rho_1-\rho_2)}\zeta^{(\mu)}(z_1)[\zeta^{(\mu)}(z_2)]^* G^{(0)}_{\alpha\beta}(q,\omega;\mu) . \tag{2.20}$$

The spectral components $G^{(0)}_{\alpha\beta}(q,\omega;\mu)$ satisfy

$$\begin{bmatrix} 0 & i(i\omega+\Gamma_0 q^2+\Gamma_0 E^{(\mu)}) \\ i(-i\omega+\Gamma_0 q^2+\Gamma_0 E^{(\mu)}) & 2\Gamma_0 \end{bmatrix} G^{(0)}(q,\omega;\mu)=1 , \tag{2.21}$$

and thus, the spectral representations of the free propagators, in the mixed form appropriate for the geometry of the flat interface, are given by

$$\begin{aligned} G^{(0)}_{11}(q;z,z';\omega)&=\sum_{\mu}\zeta^{(\mu)}(z)[\zeta^{(\mu)}(z')]^*\frac{2\Gamma_0}{|\omega+i\Gamma_0(q^2+E^{(\mu)})|^2} ,\\ G^{(0)}_{12}(q;z,z';\omega)&=G^{(0)}_{21}(-q;z,z';-\omega)=\sum_{\mu}\zeta^{(\mu)}(z)[\zeta^{(\mu)}(z')]^*\frac{1}{\omega+i\Gamma_0(q^2+E^{(\mu)})} ,\\ G^{(0)}_{22}(q;z,z';\omega)&=0 . \end{aligned} \tag{2.22}$$

For completeness, we reproduce here the spectrum of the kink operator, Eq. (2.19), which consists of two bound states, $\mu=0,1$, and of a continuum of phase-shifted plane-wave states, $\mu=k$,[16]

$$\begin{aligned} &E^{(0)}=0,\ \ \zeta^{(0)}(z)=\left[\frac{3\kappa_0}{8}\right]^{1/2}\operatorname{sech}^2\left[\frac{\kappa_0 z}{2}\right] ,\\ &E^{(1)}=\tfrac{3}{4}\kappa_0^2,\ \ \zeta^{(1)}(z)=\left[\frac{3\kappa_0}{4}\right]^{1/2}\operatorname{sech}\left[\frac{\kappa_0 z}{2}\right]\tanh\left[\frac{\kappa_0 z}{2}\right] ,\\ &E^{(k)}=k^2+\kappa_0^2,\ \ \zeta^{(k)}(z)=\frac{1}{\omega_k^{1/2}}e^{ikz}\left[2k^2+\tfrac{1}{2}\kappa_0^2-\tfrac{3}{2}\kappa_0^2\tanh^2\left[\frac{\kappa_0 z}{2}\right]+3i\kappa_0 k\tanh\left[\frac{\kappa_0 z}{2}\right]\right] , \end{aligned} \tag{2.23}$$

with

$$\omega_k=4(k^2+\kappa_0^2)(k^2+\tfrac{1}{4}\kappa_0^2) .$$

The full response function,

$$R(q;z,z';\omega)=i\Gamma_0 G_{12}(q;z,z';\omega),\ \ G_{12}(q;z,z';\omega)=\int d\rho\, dt\, e^{i\omega t-iq\rho}\langle\phi(\rho,z;t)\hat{\phi}(0,z';0)\rangle , \tag{2.24}$$

can be calculated perturbatively by making use of standard mixed-representation techniques. To one-loop order, $G_{12}(q;z,z';\omega)$ is given in terms of Feynman diagrams as shown in Fig. 1. Here the propagators $G^{(0)}_{11}$ and $G^{(0)}_{12}$ are represented by a continuous and a continuous wavy line, respectively, and the two different interaction vertices of Eq. (2.14) are represented by open and solid circles.

The essential physics of the dynamic interface is already captured by the zero-order theory. Indeed, the unperturbed (Van Hove) theory's response function is, from Eq. (2.22),

$$R^{(0)}(q;z=z'=0;\omega)=\frac{q^2+\tfrac{3}{4}\kappa_0^2-i(\omega/\Gamma_0)}{2[q^2-i(\omega/\Gamma_0)][q^2+\kappa_0^2-i(\omega/\Gamma_0)]^{1/2}} . \tag{2.25}$$

This result indicates that the inhomogeneous system's response function contains, in principle, two characteristic frequencies: $\omega_B(q)=-i\Gamma_0(q^2+\kappa_0^2)$, the equivalent of the bulk characteristic frequency, and $\omega_q=-i\Gamma_0 q^2$, corresponding to the capillary-wave dispersion relation. It is important to recognize that the amplitude of the simple-pole surface singularity in $R^{(0)}(q;0,0;\omega)$ vanishes as the critical point is approached and the interface disappears. What survives is then a branch cut singularity at the bulk frequency $\omega_B(q)$. As we shall explain in Sec. III, we have been unable to confirm that these features are also present in the response function to one-loop order. However, it is reasonable to expect that the surface-pole singularity has a vanishing amplitude (as $T\to T_c^-$) to all orders in perturbation theory. If this is the case, then one may imagine the possibility of the interfacial dispersion relation having an exponent different, in principle, from that of the bulk characteristic frequency, z. The full response function would still retain the proper bulk information as $T\to T_c$ and would reproduce the bulk critical dynamics.

From the diagrammatic expansion of Fig. 1, we now have the bare response propagator G_{12} to one-loop order,

$$\begin{aligned} G_{12}(q;z,z';\omega)=G^{(0)}_{12}(q;z,z';\omega)&-\tfrac{1}{2}i\lambda_0\Gamma_0\int dz_1\, G^{(0)}_{12}(q;z,z_1;\omega)\int dp\, d\nu\, G^{(0)}_{11}(p;z_1,z_1;\nu)G^{(0)}_{12}(q;z_1,z';\omega)\\ &-\lambda_0^2\Gamma_0^2\int dz_1\, dz_2\, G^{(0)}_{12}(q;z,z_1;\omega)\int dp\, d\nu\, G^{(0)}_{11}(p;z_1,z_2;\nu)\\ &\times G^{(0)}_{12}(p-q;z_2,z_1;\omega-\nu)G^{(0)}_{12}(q;z_2,z';\omega)M_0(z_1)M_0(z_2) , \end{aligned} \tag{2.26}$$

where $M_0(z)$ is the mean-field, or zero-order, interfacial profile.

III. RENORMALIZATION AND INTERFACIAL DISPERSION RELATION

An inspection of the one-loop contributions to Eq. (2.26) indicates that for dimensions $d \leq 4$ the perturbation expansion is divergent and, therefore, a renormalization scheme must be used. As in the case of bulk dynamic RG calculations,[11,12] one adds counterterms to the Lagrangian of Eq. (2.14) and makes use of the bulk renormalization constants Z_ϕ, Z_{ϕ^2}, Z_u, and Z_Γ. No surface counterterm is introduced in the case of an interface and we now proceed to verify, explicitly to one-loop order, that this renormalization scheme yields a finite perturbation theory. This will demonstrate that the inhomogeneous system's response function obeys the same RG equation as for the bulk response and, hence, that there can be, at least to $O(\epsilon)$, only one dynamic critical exponent, the bulk one, z.

Technically, the only novel feature in the present renormalization scheme is associated with the renormalization of the bare propagators. Indeed, the renormalization constants now also enter through the renormalized eigenfunctions $\zeta_R^{(\mu)}(z)$ and eigenvalues $E_R^{(\mu)}$ appropriate to the renormalized operator,

$$\hat{D}_R = \frac{\partial}{\partial t} + Z_\Gamma \Gamma \hat{\Delta}_R, \quad \hat{\Delta}_R = -\nabla^2 + Z_{\phi^2}\tau + \tfrac{1}{2} Z_u u M^2(z) . \tag{3.1}$$

Here, $\Gamma = Z_\Gamma^{-1}\Gamma_0$, $\tau = Z_{\phi^2}^{-1}(r_0 - r_{0c}) \propto T - T_c$, $u\mu_0^\epsilon = Z_u^{-1}\lambda_0$, and $M(z)$ is the full order-parameter profile, Eq. (2.4). μ_0^{-1} sets the length scale of the RG and, as usual, r_{0c} is the critical value. At this point the response propagator G_{12} could be calculated within the framework of renormalized perturbation theory. Unfortunately, owing to the complicated spectral representations for the bare propagators, Eq. (2.22), the calculation of the full response function appears to be impractical even to one-loop order. Nonetheless, the surface-wave dispersion relation can be extracted from renormalized perturbation theory by making the reasonable ansatz that ω_q is the only pole of the interface response $\tilde{G}_R(q,\omega)$ found by taking the full renormalized response propagator and projecting onto the Goldstone-mode eigenfunctions ($\mu = 0$), which generate localized interface distortions, i.e.,

$$\tilde{G}_R(q,\omega) = \int dz\, dz' [\zeta_R^{(0)}(z)]^* G_{12R}(q;z,z';\omega) \zeta_R^{(0)}(z') . \tag{3.2}$$

To one-loop order, Eq. (3.2) and the renormalized version of Eq. (2.26) yield

$$\begin{aligned}\tilde{G}_R(q,\omega) = {} & \frac{1}{\omega + i\Gamma(q^2 + E_R^{(0)})} - \frac{1}{2} u \frac{i\Gamma}{(\omega + i\Gamma q^2)^2} \int_{-\infty}^{+\infty} dz [\zeta^{(0)}(z)]^* G_0(z,z) \zeta^{(0)}(z) \\ & + u^2 \frac{i\Gamma}{(\omega + i\Gamma q^2)^2} \sum_{\mu,\mu'} |I(\mu,\mu')|^2 \int dp \frac{1}{(p^2 + E^{(\mu)})[p^2 + (p-q)^2 + E^{(\mu)} + E^{(\mu')} + i(\omega/\Gamma)]} .\end{aligned} \tag{3.3}$$

In the above equation we have defined

$$\begin{aligned}G_0(z,z) &= \int dp\, d\nu\, G_{11}^{(0)}(p;z,z;\nu) = \sum_\mu \int dp \frac{|\zeta^{(\mu)}(z)|^2}{p^2 + E^{(\mu)}} \\ &= \frac{\kappa_0^2}{\epsilon}\left[-1 + \tfrac{3}{2}\operatorname{sech}^2\left[\frac{\kappa_0 z}{2}\right]\right] + \kappa_0^2\left[\tfrac{1}{2}\ln\kappa_0^2 - \tfrac{3}{4}(\ln\kappa_0^2 - 1)\operatorname{sech}^2\left[\frac{\kappa_0 z}{2}\right]\right. \\ &\qquad \left. - \frac{\pi\sqrt{3}}{4}\operatorname{sech}^2\left[\frac{\kappa_0 z}{2}\right]\tanh^2\left[\frac{\kappa_0 z}{2}\right]\right] + O(\epsilon)\end{aligned} \tag{3.4}$$

and

$$I(\mu,\mu') = \int_{-\infty}^{+\infty} dz\, M_0(z) \zeta^{(0)}(z) \zeta^{(\mu)}(z) [\zeta^{(\mu')}(z)]^* . \tag{3.5}$$

These coefficients satisfy $I(\mu,\mu') = [I(\mu',\mu)]^*$ and $I(\mu,\mu) = 0$, and are given explicitly as follows:

$$\begin{aligned}I(0,1) &= \frac{9\pi}{128}\left[\frac{\kappa_0^3}{u}\right]^{1/2} , \\ I(0,k) &= -i\left[\frac{3\kappa_0^2}{u\omega_k}\right]^{1/2} \frac{\pi k^2(k^2 + \kappa_0^2)^2}{2\kappa_0^4 \sinh[\pi(k/\kappa_0)]} , \\ I(1,k) &= -\left[\frac{3\kappa_0^2}{2u\omega_k}\right]^{1/2} \frac{\pi(k^2 + \frac{3}{4}\kappa_0^2)(k^2 + \frac{1}{4}\kappa_0^2)^2}{\kappa_0^4 \cosh[\pi(k/\kappa_0)]} , \\ I(k,k') &= i\left[\frac{\kappa_0}{2u\omega_k\omega_{k'}}\right]^{1/2} \frac{6\pi(k-k')^2}{\kappa_0^2 \sinh\{\pi[(k-k')/\kappa_0]\}} \{(2k^2 - \kappa_0^2)(2k'^2 - \kappa_0^2) + kk'[3(k-k')^2 + 6\kappa_0^2] \\ &\qquad + \tfrac{1}{2}[(k-k')^4 + 5\kappa_0^2(k-k')^2 - 2\kappa_0^4]\} .\end{aligned} \tag{3.6}$$

Also, from first-order perturbation theory on the Schrödinger-like equation (2.19), appropriate to the renormalized operator $\widehat{\Delta}_R$ of Eq. (3.1), we have

$$E_R^{(0)}=E^{(0)}+u\int_{-\infty}^{+\infty}dz\,\zeta^{(0)}(z)\widehat{\Delta}^{(1)}(z)[\zeta^{(0)}(z)]^*+O(u^2)=\left[\frac{\kappa_0^2}{5\epsilon}+\frac{3\kappa_0^2}{20}\left[-\frac{2}{3}\ln\kappa_0^2+\frac{\pi\sqrt{3}}{7}-1\right]\right]u+O(u^2)\,, \tag{3.7}$$

where $\widehat{\Delta}_R(x)=-\nabla_\rho^2+\widehat{\Delta}^{(0)}(z)+u\Delta^{(1)}(z)+O(u^2)$. As for the third contribution to Eq. (3.3), it should be noticed that for $(\mu,\mu')\neq(k,k')$ the p integral is finite and, since the extra single k integral converges for large momentum, it need only be calculated to $O(1)$, i.e., in $d=4$, that is

$$\int dp\frac{1}{(p^2+E^{(\mu)})[p^2+(p-q)^2+E^{(\mu)}+E^{(\mu')}+i(\omega/\Gamma)]}$$
$$=\pi\int_0^1 dx(x(1+x)^2q^2+(1+x)^3\{E^{(\mu)}+x[E^{(\mu')}+i(\omega/\Gamma)]\})^{-1/2}+O(\epsilon)\,. \tag{3.8}$$

However, for $(\mu,\mu')=(k,k')$ and extra divergence is introduced through one of the k integrals in $\sum_{k,k'}$ associated with $I(k,k')$. The divergent contribution is separated out and expanded in $\epsilon=4-d$; this leads us to the familiar bulk integral

$$\int dk\int dp\frac{1}{(k^2+p^2+\kappa_0^2)[k^2+p^2+(k-\bar{k})^2+(p-q)^2+2\kappa_0^2+i(\omega/\Gamma)]}$$
$$=\frac{1}{2\epsilon}-\frac{1}{4}-\frac{1}{2}\int_0^1 dx(1+x)^{-2}\ln\left[\frac{x}{(1+x)^2}(q^2+\bar{k}^2)+\kappa_0^2+i\frac{x}{1+x}\frac{\omega}{\Gamma}\right]+O(\epsilon)\,, \tag{3.9}$$

where we have set $\bar{k}=k-k'$. At this point, one can verify that the ϵ poles contained in the three contributions to $\widetilde{G}_R(q,\omega)$ in Eq. (3.3) cancel exactly. This represents a demonstration that, to $O(\epsilon)$, the bulk renormalization constants suffice to renormalize the interfacial response. We obtain

$$\widetilde{G}_R(q,\omega)=\frac{1}{\omega+i\Gamma q^2}-u\frac{i\Gamma\kappa_0^2}{(\omega+i\Gamma q^2)^2}\Phi\left[\frac{q^2}{\kappa_0^2},i\frac{\omega}{\Gamma\kappa_0^2}\right], \tag{3.10}$$

where $\Phi(x,y)$ is a universal function given in terms of lengthy multiple integrals. The dispersion relation is the solution of $\widetilde{G}_R(q,\omega_q)^{-1}=0$, and from Eq. (3.10) we find, setting $u=u^*=\frac{2}{3}\epsilon$ and $\kappa=\xi^{-1}$,

$$\omega_q=-i\Gamma\left[q^2+\frac{2}{3}\epsilon\kappa^2\Phi\left[\frac{q^2}{\kappa^2},\frac{q^2}{\kappa^2}\right]+O(\epsilon^2)\right]=-i\Gamma q^z\Omega(q\xi)\,. \tag{3.11}$$

The function $\Omega(x)$ has the following parametric representation:

$$\Omega(x)=1+\frac{\epsilon}{x^2}\left\{\frac{1}{5}+\frac{\pi\sqrt{3}}{210}-\frac{2}{3}\pi\left[|\hat{I}(0,1)|^2[Q(0,1|x)+Q(1,0|x)]\right.\right.$$
$$+\int_{-\infty}^{+\infty}\frac{dz}{2\pi}|\hat{I}(0,z)|^2[Q(0,z|x)+Q(z,0|x)]+\int_{-\infty}^{+\infty}\frac{dz}{2\pi}|\hat{I}(1,z)|^2[Q(1,z|x)+Q(z,1|x)]$$
$$+\int_{-\infty}^{+\infty}\frac{dz}{2\pi}\frac{dz'}{2\pi}\left[|\hat{I}(z,z')|^2-\frac{288\pi^2(z-z')^4(1+z^2)(1+z'^2)}{(1+4z^2)(1+4z'^2)\sinh^2[\pi(z-z')]}+\frac{54\pi^2(z-z')^4(5+4z^2+4z'^2)}{(1+4z^2)(1+4z'^2)\sinh^2[\pi(z-z')]}\right]Q(z,z'|x)$$
$$\left.\left.-9\int_0^\infty dz\frac{z^4}{\sinh^2(\pi z)}\int_0^{1/2}dy\,\ln[1+y(1-y)z^2+y(2-y)x^2]\right]\right\}+O(\epsilon^2)\,, \tag{3.12}$$

where we have introduced the notation

$$Q(\mu,\mu'|q\xi)=\Delta\int_0^1 dx[(1+x)^3(E^{(\mu)}+xE^{(\mu')})\xi^2+x(2+x)(1+x)^2q^2\xi^2]^{-1/2}\,, \tag{3.13}$$

as well as

$$\hat{I}(\mu,\mu')=(u/\kappa_0^3)^{1/2}I(\mu,\mu')\,,$$

and we have used the dimensionless integration variable $z=k/\kappa_0$. The limit forms of this function are as follows

$$\Omega(x)\simeq 1+\epsilon(C+O(x^2)),\quad x\ll 1$$
$$\Omega(x)\simeq 1+\epsilon((C_1/x^2)\ln x+(C_2/x^2)+O(x^{-3})),\quad x\gg 1 \tag{3.14}$$

with

$$C\simeq 0.229,\quad C_1=\frac{\pi\sqrt{3}}{210}+\frac{3}{10}\ln\left[\frac{4}{3}\right],\quad C_2=\frac{1}{5}\,. \tag{3.15}$$

IV. DISCUSSION AND CONCLUSIONS

Several aspects of our result, Eqs. (3.11)–(3.13), ought to be considered at this point.

(i) Our dispersion relation ω_q is gapless in the limit $q \to 0$. This is consistent with the nonperturbative result $\lim_{q\to 0}\omega_q = 0$, which follows from a dynamical version of the Goldstone theorem for the spontaneously broken Euclidean symmetry for all $T < T_c$ (see Appendix B). This is to be compared with the gap in the bulk homogeneous phase characteristic frequency, $\omega_B(k=0) \neq 0$, which is consistent with the nonconserved order-parameter relaxation of model A (see Appendix A).

(ii) The exponent z in ω_q, Eq. (3.11), has been identified with the bulk dynamic exponent $z = 2 + O(\epsilon^2)$, in agreement with scaling. However, it was suggested above that it is reasonable to expect that the amplitude of the surface singularity in $R(q;z=z'=0;\omega)$ vanishes for $T = T_c$ to all orders in perturbation theory. Then if a breakdown of simple dynamical scaling occurs with ω_q having an exponent different from the bulk one, such "breakdown" would occur with a vanishing amplitude in the response function. A calculation to $O(\epsilon^2)$ would be most welcome in order to investigate this issue.

(iii) A higher-order calculation should also reveal a singularity in $\Omega(x)$ as $x \to 0$, as discussed in the Introduction. In the present $O(\epsilon)$ calculation, this singularity does not appear, owing to the fact that the value of z has no $O(\epsilon)$ correction and also to the fact that the $O(\epsilon)$ value of z coincides with the hydrodynamic-limit exponent for ω_q. In the case of model B for a conserved order parameter, $\Omega(x)$ has a singularity at mean-field level of the form $\Omega(x) \sim x^{-1}$, as $x \to 0$. In fact, $z = 4$ in this approximation, whereas the interface relaxes as $\omega_q \sim q^3$ in the hydrodynamic limit, $q\xi \ll 1$. In general, the existence of this singularity in $\Omega(x)$ is a consequence of the presence of long-wavelength Goldstone-mode-like excitations in the system with an interface. An analogous situation is presented by the isotropic ferromagnet for all $T < T_c$, in which case the singularity of the longitudinal correlation function, $G_{\parallel}(k) \sim k^{-\epsilon}$, is determined, in the limit $k\xi \to 0$, by spin-wave theory.[19]

In conclusion, we have shown in this work that it is possible, despite the formidable computational difficulties, to derive within the renormalization-group ϵ expansion an interfacial dispersion relation starting from the statistical mechanics of the bulk system. This approach has been developed here for an Ising-like system with purely relaxational dynamics; extension of this work to more realistic dynamical models is possible, although not without severe computational problems. Several issues within the interfacial dynamics of model A still remain to be investigated.

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APPENDIX A: BULK HOMOGENEOUS PHASE DISPERSION RELATION

In the case of a homogeneous ordered phase, $T < T_c$, $\langle \phi(x) \rangle = M_B$, the dispersion relation is the solution of $\Gamma_{12}(k,\omega) = 0$ since $\Gamma_{12}(k,\omega) G_{12}(k,\omega) = 1$. The bare two-point vertex response function is given, to one-loop order, by

$$\Gamma_{12}(k,\omega) = \omega + i\Gamma_0(k^2 + r_0 + \tfrac{1}{2}\lambda_0 M_B^2) + i\frac{\lambda_0\Gamma_0}{2}\int d^d p \frac{1}{p^2 + r_0 + \frac{1}{2}\lambda_0 M_B^2}$$
$$+ \lambda_0^2\Gamma_0^2 M_B^2 \int d^d p \frac{1}{(p^2 + r_0 + \frac{1}{2}\lambda_0 M_B^2)\{-\omega + i\Gamma_0[p^2 + (p-k)^2 + 2r_0 + \lambda_0 M_B^2]\}} . \tag{A1}$$

We now make use of the ordinary bulk renormalization constants $Z_\phi = 1 + O(u^2)$, $Z_{\phi^2} = 1 + u/(2\epsilon) + O(u^2)$, $Z_\Gamma = 1 + O(u^2)$, and $Z_u = 1 + 3u/(2\epsilon) + O(u^2)$, and use Eq. (2.5) for M_B, in order to obtain the renormalized version of Eq. (A1). The result is, to $O(\epsilon)$,

$$\Gamma_{12R}(k,\omega) = \omega + i\Gamma\left\{k^2 + \kappa_0^2 + \epsilon\kappa_0^2\left[\tfrac{1}{6}(\ln\kappa_0^2 + 3) + \int_0^{1/2} dx \ln\left[x(1-x)\frac{k^2}{\kappa_0^2} + ix\frac{\omega}{\Gamma\kappa_0^2} + 1\right]\right]\right\} . \tag{A2}$$

The divergent d-dimensional integrals contributing to Eq. (A1) have been calculated in $d = 4 - \epsilon$ dimensions using, in particular, familiar expansions like Eq. (3.9). By solving $\Gamma_{12R}(k, \omega_B(k)) = 0$, we arrive at the result

$$\omega_B(k) = -i\Gamma k^z\left[1 + \frac{1}{k^2\xi^2} + \frac{\epsilon}{k^2\xi^2}\left[1 - \frac{\pi\sqrt{3}}{6} + \int_0^{1/2} dx \ln[1 + x + x(2-x)k^2\xi^2]\right]\right] , \tag{A3}$$

in which $\xi = \xi_0 |\tau|^{-\nu}$ is the $T < T_c$ bulk correlation length, with $\xi = \kappa_R^{-1} + O(\epsilon^2)$ and κ_R given by Eq. (2.6). A notable consequence of Eq. (A3) is that $\omega_B(k=0) \neq 0$, which signifies that the $k=0$ mode, that is, the nonconserved order parameter $\int dx \langle \phi(x) \rangle$, also relaxes, as expected, under model-A relaxational dynamics. To our knowledge, the scaling function in Eq. (A3), which interpolates between bulk hydrodynamic and critical behavior, has not appeared in the literature.

APPENDIX B: WARD IDENTITY FOR THE DYNAMIC INTERFACIAL RESPONSE FUNCTION

The infinite system with an interface is described by a Lagrangian, given by Eq. (2.11), which is invariant under spatial translations of the field ϕ,

$$\phi(x,t)\rightarrow\phi'(x,t)=\phi(x,t)+\epsilon^i\frac{\partial\phi(x,t)}{\partial x_i} . \tag{B1}$$

The presence of an interface, however, means that this Euclidean symmetry is spontaneously broken, and this fact can be used to demonstrate some general properties of the response functions valid to all orders in perturbation theory. If $W[l,\hat{l}]=\ln Z[l,\hat{l}]$ is the connected-part generating functional of the theory (see Sec. II), then invariance of the Lagrangian, Eq. (2.11), under the transformation (B1) means

$$W[l,\hat{l}]=W\left[l-\epsilon^i\frac{\partial l}{\partial x_i},\hat{l}\right] , \tag{B2}$$

and hence

$$\epsilon^i\int dx\,dt\frac{\partial l(x,t)}{\partial x_i}\frac{\delta W[l,\hat{l}]}{\delta l(x,t)}=0 . \tag{B3}$$

This relationship can be used to generate a number of Ward identities concerning various correlation functions of the theory. By taking a further derivative $\delta/\delta l(x',t')$, we obtain

$$\epsilon^i\int dx'\,dt'\frac{\partial l(x',t')}{\partial x_i'}\frac{\delta^2 W[l,\hat{l}]}{\delta l(x',t')\delta l(x,t)}=\epsilon^i\frac{\partial}{\partial x_i}\frac{\delta W[l,\hat{l}]}{\delta l(x,t)} , \tag{B4}$$

that is, for an inhomogeneous field $l(x,t)=l(z)$,

$$\int dz'\frac{\partial l(z')}{\partial z'}\int d\rho'\,C(\rho',z';\rho,z;\omega=0)=\frac{\partial}{\partial z}M(z) . \tag{B5}$$

In the presence of an interface, as the magnitude of $l(z)$ vanishes, the profile $M(z)$ remains and, therefore, the correlation function diverges. By virtue of Eq. (2.9), this also means

$$\left[\int d\rho\,R(\rho;z,z';\omega=0)\right]^{-1}=R(q=0;z,z';\omega=0)^{-1}=0 , \tag{B6}$$

which implies that, barring anomalous behavior,

$$\lim_{q\rightarrow 0}\omega_q=0 , \tag{B7}$$

to all orders in perturbation theory.

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