Random fields, random anisotropies, nonlinear σ models, and dimensional reduction

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The effects of weak random fields and random higher-rank anisotropies are investigated for n-component magnets with $n \geq 2$ for low temperatures near four dimensions. All of the random fields and anisotropies are marginal at the lower critical dimension, $d_c = 4$. By use of supersymmetry, it is shown that at zero temperature the formal perturbation expansion in powers of the strengths, $\{\Delta_\mu\}$, of the random fields and all the anisotropies only depend on a simple combination, $\widetilde{\Delta}$, of the $\{\Delta_\mu\}$. Furthermore, this expansion in powers of $\widetilde{\Delta}$ is equivalent to the expansion of the pure n-component system in powers of T in two dimensions less. It would be natural to conclude that this implies a generalized dimensional reduction for exponents; however, we present renormalization-group calculations which indicate that this is not valid. Functional renormalization-group recursion relations are derived which couple together all the random fields and anisotropies. It is demonstrated that, in contrast to earlier claims, there is no perturbative fixed point of the renormalization group in $4+\epsilon$ dimensions. The flows go into regimes where nonperturbative effects are important and it is argued that dimensional reduction is likely to break down. This is the first example of which we are aware of a problem for which an infinite number of marginal operators play an important role. Various suggestions concerning the behavior of random-anisotropy magnets are discussed in the light of the present results.

I. INTRODUCTION

The equivalence in a dimensionality expansion about six dimensions, of the critical behavior of n-component magnets in random magnetic fields and pure systems in two dimensions less has been known for some time.[1–6] However, the equivalence or lack of it in lower dimensions is still controversial. Most of the controversy has focused on the Ising model for which the questions are most experimentally relevant.[7,8] Naive extrapolation of the random field $6-\epsilon$ expansion and the pure $4-\epsilon$ expansion suggests that since the lower critical dimension of the pure Ising model is 1, the lower critical dimension of the random-field Ising model will be 3.

Simple though rather compelling domain-wall arguments, originally due to Imry and Ma[9] and recently made more precise by others,[10–13] suggest, however, that the lower critical dimension of the random-field Ising model is 2. If this answer is correct, which this author believes it to be, then something must go wrong with the dimensional reduction somewhere between six dimensions and three.

For magnets with continuous symmetry, i.e., $n \geq 2$, both dimensional reduction[1–4] and Imry-Ma arguments yield 4 as the lower critical dimension. It is thus natural to ask whether, for this case, dimensional reduction holds all the way from six dimensions down to four. The primary purpose of this paper is to investigate in detail the behavior of random-field magnets with continuous symmetry near four dimensions in order to attempt to answer this question.

It has been claimed in the literature[3,14] that for $n > 2$, to leading order in $\epsilon = d-4$, the critical behavior of the random-field model is the same as the pure system in $2+\epsilon$ dimensions.[15–17] It will be argued that these calculations are incorrect. We construct renormalization-group recursion relations near four dimensions and show that an infinite number of marginal operators corresponding to all possible higher-rank random anisotropies are generated from the random field. The feedback of these random anisotropies is such as to destroy the fixed point found in $4+\epsilon$ by Young.[3] The renormalization-group flows carry the Hamiltonian into a regime when nonperturbative effects are likely to be important and probably destroy dimensional reduction at least near four dimensions. Unfortunately, the full renormalization group is found to have no fixed point of order ϵ in $4+\epsilon$ dimensions and thus the critical behavior in $4+\epsilon$ is not obtained.

Much of this paper is somewhat didactic. We investigate the problem by conventional perturbative methods and show that these may lead to misleading conclusions. The functional renormalization-group calculations derived yield a fuller picture with which the perturbative results can be reconciled. Nonperturbative effects are shown to be important by considering a simple case of the model. At the end of the paper various proposals for phase diagrams of random-field and random-anisotropy models, several of which rely on the perturbative results shown here to be misleading, are discussed in light of the present work.

A. Model and problems

For the random-field problem the fluctuations tend to be dominated by the quenched disorder rather than the thermal fluctuations. For the ϕ^4 theory in a random field near six dimensions, this is manifested as the most divergent diagrams being those which occur at zero temperature. The conventional way[1,2] to perform the $6-\epsilon$ expansion has been to rescale in such a way that temperature is held fixed while various of the other terms in the effective replicated Hamiltonian become infinite, with certain ratios of them remaining finite. However, it is better to consider the strength of the random field to be fixed near to the critical fixed point and allow the temperature to be renormalized (as has also been done in Ref. 6). The domination of the quenched disorder over the thermal disorder then causes the fixed point which controls the critical behavior in the $6-\epsilon$ expansion to occur at zero temperature. A schematic flow diagram is shown in Fig. 1. The renormalization-group eigenvalue for the temperature T at the critical fixed point is

$$\lambda_T=-2 \tag{1.1}$$

to all orders in ϵ;[6] however, T is dangerously irrelevant because, for example, the free energy is proportional to T times the logarithm of the partition function.[2] This is the simplest way of understanding the breakdown of the usual hyperscaling relation for the specific-heat exponent which occurs:[2] In the $6-\epsilon$ expansion, the primary effect of temperature is to modify the hyperscaling relation to be

$$(d+\lambda_T)\nu=2-\alpha\ . \tag{1.2}$$

Other than this and related effects,[18] the temperature *appears* to play an unimportant role and it is convenient to think of dimensional reduction as relating the transition *at* zero temperature as a function of the strength of the random field to the transition of a pure system in two dimensions less as a function of temperature.

In this paper we study in detail the behavior of magnets in random fields and also random higher-rank anisotropies[14,19] for weak disorder near four dimensions. From the above discussion, it is clear that we should focus our attention on low temperatures with the expectation that the effects of the disorder will dominate. In a way analogous to that for pure systems at low temperatures near two dimensions,[15–17] we will perform a spin-wave expansion about a ferromagnetically ordered state of n-component spins S^i with a fixed-length constraint $S\cdot S=1$. (Here and henceforth all dot products will be in spin space.) The general Ginzburg-Landau-Wilson Hamiltonian we consider is (with *no* factor of temperature included)

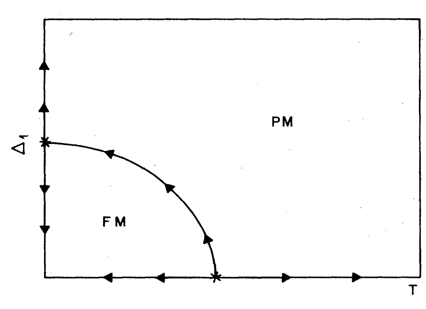


FIG. 1. Schematic renormalization-group flow diagram for a random-field magnet near six dimensions as a function of temperature T and mean-square random-field strength Δ_1. Ferromagnetic (FM) and paramagnetic (PM) phases are shown.

$$\mathcal{H}=\int d^dr\Big[\tfrac{1}{2}\nabla S(\mathbf{r})\cdot\nabla S(\mathbf{r})-h_1^i(\mathbf{r})S^i(\mathbf{r}) -h_2^{ij}S^iS^j-h_3^{ijk}S^iS^jS^k -\cdots-h_\mu^{i_1\cdots i_\mu}S^{i_1}\cdots S^{i_\mu}-\cdots\Big]\ , \tag{1.3}$$

with summation on repeated spin indices implied.

In addition to a random field, h_1^i, we have included for future use random second-rank anisotropy h_2^{ij} and general random μth-rank tensor anisotropies each with zero-expectation and Gaussian correlations

$$[h_\mu^{i_1\cdots i_\mu}(\mathbf{r})h_\nu^{j_1\cdots i_\nu}(\mathbf{r}')]_{\rm av}=\Delta_\mu\delta_{\mu\nu}\delta^{i_1j_1}\cdots\delta^{i_\mu j_\mu}\delta(\mathbf{r}-\mathbf{r}')\ . \tag{1.4}$$

$[\ ]_{\rm av}$ denotes averages over the disorder. We will consider the expansion of correlation and response functions about the ordered state with all the $\mathbf{S}(\mathbf{r})$ aligned in the nth direction, in powers of the Δ_μ and the temperature T. As near six dimensions, the temperature is formally irrelevant and we are thus led to consider expansions in powers of the Δ_μ at $T=0$. By power counting it can be seen that *all* of the Δ_μ are marginal in $d=4$ because of the fixed-length constraint. This is in contrast to the behavior near six dimensions where all the random anisotropies with $\mu>1$ are irrelevant since they couple to higher powers of S than the random field. We will show that formally the expansions of all correlation functions of the fixed-length spin model in powers of the Δ_μ at $T=0$ depend *only* on the quantity

$$\widetilde{\Delta}=\sum_{\mu=1}^{\infty}\mu\Delta_\mu \tag{1.5}$$

and that this expansion in powers of $\widetilde{\Delta}$ is exactly equivalent to the expansion in powers of T of the pure system in two dimensions less. However, we will argue that there are almost certainly nonperturbative corrections to the expansion for the random problem which do not exist for the pure system. Nevertheless, one might expect that the equivalence of the expansion in powers of $\widetilde{\Delta}$ with the expansion in power of T of the pure system in two dimensions less would imply that a $4+\epsilon$ renormalization-group expansion for the random system would be essentially equivalent to the pure $2+\epsilon$ expansion.[15–17] It will be shown, however, that the situation here is rather more complicated and that the other operators play an important role in the renormalization-group flows.

We will construct isotropic momentum-space renormalization-group recursion relations for the random model up to second order (the first nontrivial order) in the Δ_μ which will turn out to couple together all of the Δ_μ. This renormalization-group expansion *must*, therefore, contain the whole infinite set of marginal operators Δ_μ since they will all be generated by renormalization of a Hamiltonian containing initially only Δ_1, i.e., a random field.

A slice through the flow diagram for d just above 4 is shown schematically in Fig. 2, which is discussed at the end of Sec. V. Note that there is an invariant surface $\widetilde{\Delta}=\widetilde{\Delta}^*$ which could be mistaken for a fixed point if the other variables were ignored.

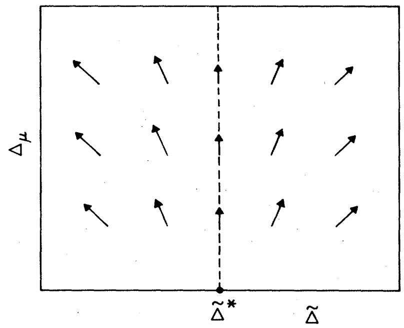


FIG. 2. Schematic slice of the renormalization-group flows in $d=4+\epsilon$. The invariant surface $\tilde{\Delta}=\tilde{\Delta}^*=O(\epsilon)$ is cut by the dotted line shown.

However, it will be shown that the *full* renormalization-group equations truncated at second order do not have a singly unstable fixed point in $4+\epsilon$ dimensions, at which the Δ_μ are uniformly of order ϵ. In fact, the renormalization-group flows tend to take the effective Hamiltonian into regimes where nonperturbative effects are likely to be important. Reconciliation of this apparent discrepancy with the straightforward perturbative results will be discussed.

In the light of these results near four dimensions, it is natural to ask whether similar effects might destroy the formal dimensional reduction near six dimensions as well. However, there are two important differences between the behavior in $6-\epsilon$ dimensions and that in $4+\epsilon$ dimensions. Firstly, as mentioned previously, the higher-rank anisotropies are all irrelevant near six dimensions so that the renormalization-group analysis is much simpler. Secondly, and crucially, it can be shown (see, e.g., Ref. 2) that there exists a perturbative fixed point in $6-\epsilon$ dimensions in contrast to the case in $4+\epsilon$ dimensions discussed here. Once a perturbative fixed point exists, it is reasonable to believe that the formal perturbative calculations should correctly yield the asymptotic expansion for the critical exponents order by order in ϵ. In Sec. VI we briefly discuss the possible problems with dimensional reduction near six dimensions.

B. Outline

The organization of the paper is as follows. In Sec. II the perturbation theory and its possible failure via nonperturbative effects are discussed for the simple X-Y ($n=2$) case, while in the following section renormalization-group flow equations are derived and analyzed for this case.

For the general n case, the $T=0$ perturbation theory is shown in Sec. IV to be equivalent to the pure model in two dimensions less by the method of Parisi and Sourlas[4] and of Cardy[5] using supersymmetry. Renormalization-group equations for the general case are derived and analyzed in Sec. V and reconciliation with the perturbative results illustrated. Finally, Sec. VI contains discussion of the results and some tentative conclusions, and Sec. VII presents speculation concerning possible phases and relation to other work. Details of some of the calculations and proofs are relegated to the Appendices, along with a discussion of problems which appear in the general n case using conventional diagrammatic methods.

II. X-Y MODEL PERTURBATION THEORY

To gain some insight into the more general case, we first analyze in detail the X-Y ($n=2$) case, which can be greatly simplified by the introduction of angle variables through the transformation

$$\mathbf{S}(\mathbf{r})=(\cos\theta(\mathbf{r}),\sin\theta(\mathbf{r}))\ . \tag{2.1}$$

If we ignore vortices, which we may do in perturbation theory, then θ can be considered to be a single-valued variable on $(-\infty,\infty)$ and the Hamiltonian, Eq. (1.3), becomes

$$\mathscr{H}=\int d^dr\left\{\tfrac{1}{2}[\nabla\theta(\mathbf{r})]^2 - \sum_{m=1}^{\infty}\{g_m^x\cos[m\theta(\mathbf{r})]+g_m^y\sin[m\theta(\mathbf{r})]\}\right\}, \tag{2.2}$$

where we have dropped random terms which do not depend on θ. The g_m^i are Gaussian m-fold anisotropies distributed with correlations

$$[g_m^i(\mathbf{r})g_l^j(\mathbf{r})]_{\text{av}}=\Gamma_m\delta_{ml}\delta^{ij}\delta(\mathbf{r}-\mathbf{r}')\ , \tag{2.3}$$

with the Γ_m related to the Δ_μ by

$$\Gamma_m=\sum_{k=0}^{\infty}\frac{1}{2^{m+2k-1}}\begin{bmatrix} m+2k \\ k\end{bmatrix}\Delta_{m+2k}\ . \tag{2.4}$$

In angle variables with no vortices it is immediately apparent that by the transformation $\theta'=m\theta$ any model with only one Γ_m nonzero is essentially equivalent (after redefining T and Γ_m) to any other.[20,21]

A. Zero-temperature perturbation theory

We first consider perturbation theory for the ground state, i.e., at zero temperature. In the context of charge-density waves, Efetov and Larkin[22] showed some time ago that at zero temperature the perturbation expansion of the two-point angle correlation function for the case with only Γ_1 nonzero is trivial to *all* orders in Γ_1. Their result, which is straightforward to extend to the general case with all the Γ_m nonzero, can be readily proved graphically.

It is found that the two-point function depends *only* on the combination of the Γ_m's, $\tilde{\Delta}=\sum_m m^2\Gamma_m$. It is given by

$$[\theta(\mathbf{q})\theta(\mathbf{q}')]_{\text{av}}=\delta(\mathbf{q}+\mathbf{q}')(\tilde{\Delta}/q^4)\ , \tag{2.5}$$

with *no* corrections proportional to any power of the Γ_m. This result can be generalized to all angle correlation functions; the conclusion is that all correlation functions are, to all orders in perturbation theory, simply Gaussian with a propagator given by Eq. (2.5).

The simplest (though formal) derivation of these results is via the method of Parisi and Sourlas.[4] Since we are interested in zero temperature, we can obtain a generating function for correlation functions by summing over all fields with a constraint that the field configurations extremize the Hamiltonian (ignoring, for the time being, the possibility of more than one extremal solution for a given field configuration). We define

$$Z_g[e(\mathbf{r})]=\int \left[\prod_{\mathbf{r}} D\theta(\mathbf{r})\right]\delta\left[\frac{\delta\mathscr{H}}{d\theta(\mathbf{r})}\right] \times\exp\left[\int \theta(\mathbf{r})e(\mathbf{r})\right]J_g\{\theta(\mathbf{r})\}\ , \qquad (2.6)$$

where the Jacobian

$$J_g\{\theta(\mathbf{r})\}=\exp\left[\mathrm{Tr}\ln\frac{\delta^2\mathscr{H}}{\delta\theta(\mathbf{r})\delta\theta(\mathbf{r}')}\right] \qquad (2.7)$$

is needed to give each configuration equal weight so that

$$Z_g\{0\}=1\ . \qquad (2.8)$$

Introducing an auxiliary field $\hat{\theta}$ to eliminate the δ functions and anticommuting fields ψ and $\bar{\psi}$ to cancel the Jacobian, we can write[4]

$$Z_g\{e\}=\int D\theta D\hat{\theta}D\bar{\psi}D\psi\exp\left[\int \mathscr{L}_h+e\theta\right]\ , \qquad (2.9)$$

with

$$\mathscr{L}_h=-i\hat{\theta}\nabla^2\theta-\bar{\psi}(-\nabla^2)\psi + \sum_m\{-i\hat{\theta}g_m^y m\cos(m\theta)+i\hat{\theta}g_m^x m\sin(m\theta) -\bar{\psi}[m^2g_m^y\sin(m\theta)+m^2g_m^x\cos(m\theta)]\psi\}\ , \qquad (2.10)$$

Because each $Z_g\{0\}$ is 1, averaged correlation functions can be obtained from the average Z:

$$Z\{e\}\equiv[Z_g\{e\}]_{\mathrm{av}}=\int D\theta D\hat{\theta}D\psi D\hat{\psi} \times\exp\left[\int\left[-i\hat{\theta}\nabla^2\theta-\frac{\tilde{\Delta}}{2}\hat{\theta}^2 -\bar{\psi}(-\nabla^2)\psi+e\theta\right]\right]\ , \qquad (2.11)$$

where we have dropped the four fermion terms which vanish because of the anticommutation relation

$$\{\psi(\mathbf{r}),\bar{\psi}(\mathbf{r}')\}=0\ . \qquad (2.12)$$

If we integrate out $\hat{\theta}$, Eq. (2.11) just becomes the product of a free-fermion partition function and that of a free field with Lagrangian density

$$\mathscr{L}=-(1/2\tilde{\Delta})(\nabla^2\theta)^2\ ; \qquad (2.13)$$

derivatives with respect to e give the correlations quoted above. While this derivation is very formal and has several potential problems which will be discussed later, it does correctly reproduce all perturbative results. It will also work on a lattice (with lattice Laplacians and integrals replaced by sums), provided the randomness has no correlations between sites.

B. Finite-temperature perturbation theory

In order to calculate finite-temperature correlation functions, another formalism is clearly necessary, and the most convenient is replicas. We will, in particular, be interested in *connected* correlation functions which vanish at zero temperature. The replicated effective Hamiltonian is simply

$$\overline{\mathscr{H}}=\int\frac{1}{2T}\sum_\alpha(\nabla\theta_\alpha)^2-\frac{1}{2T^2}\sum_{\alpha,\beta}Q(\theta_\alpha-\theta_\beta)\ , \qquad (2.14)$$

where sums over the replica indices α and β run from 1 to p, with p to be taken to zero at the end of the calculation.[23] The 2π-periodic function $Q(\theta)$ is given by

$$Q(\theta)=\sum_{m=1}^{\infty}\Gamma_m\cos(m\theta)\ . \qquad (2.15)$$

An expansion of the second term in Eq. (2.14) in powers of θ yields all possible interactions with two replica indices; many of them are related by symmetry or periodicity. We represent the vertex $\theta_\alpha^n\theta_\beta^m$ as shown in Fig. 3(a), with the dotted line carrying any momentum and *not* conserving the replica index. It carries a factor $1/T^2$ times

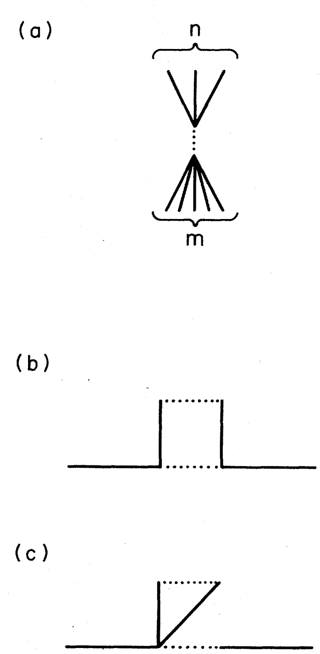


FIG. 3. Vertex $\theta_\alpha^n\theta_\beta^m$ is represented by the graph in (a) where the dotted line carries a factor $1/T^2$ times the $(n+m)$th derivative of $Q(\theta)$ at $\theta=0$ and a combinatoric factor. In (b) and (c) are shown two graphs contributing to the zero-temperature disconnected correlation functions $[\theta^2]_{\mathrm{av}}$. They are related by moving a solid line from one end of a dotted line to the other and cancel each other.

derivatives of Q at $\theta=0$. Solid lines represent $(T/q^2)\delta_{\alpha\beta}$ and so conserve the replica index. The zero-temperature expansion will just be given by those graphs with l solid lines, k dotted lines, and no solid closed loops with $l=2k$, i.e., trees connected by dotted lines; for example, Figs. 3(b) and 3(c).

The cancellations which yield the trivial perturbation expansion at $T=0$ arise from the equivalence up to combinatoric factors, of graphs like Figs. 3(b) and 3(c), which differ only by moving one end of a solid line from one end of a dotted line to the other. (This is why it is crucial that the dotted line have no momentum dependence, i.e., that the randomness is uncorrelated.) It can readily be shown that the tree graphs (zero temperature) can be grouped so that the equivalent graphs cancel. However, this does *not* occur at finite temperature. The simplest example is the contribution of order Q^2T to the *disconnected* part, $\langle\theta_\alpha\theta_\beta\rangle$ with $\alpha\neq\beta$, of the propagator which represents $[\langle\theta\rangle^2]_{\rm av}$, where the angular brackets, $\langle\ \rangle$, denote thermal averaging. This is represented by the graphs in Figs. 4(a)—4(c), which, as can explicitly be seen, do *not* cancel. In contrast, the *connected* part $[\langle(\theta-\langle\theta\rangle)^2\rangle]_{\rm av}$ of the propagator which is proportional to $T\delta_{\alpha\beta}$ can be shown to have *no* nontrivial corrections at any order in T and Δ. This follows from the result proved in Appendix A. Other connected correlation functions, for example, $[\langle(\theta-\langle\theta\rangle)^4\rangle]_{\rm av}$, will vanish at $T=0$, but have nontrivial corrections of order Δ to their leading low-T behavior.

The correlation functions of $e^{i\theta}$ can be formally obtained at $T=0$ by summing the perturbation expansion or directly from the Parisi-Sourlas formalism above. In four dimensions, the resulting $[e^{i\theta(\mathbf{r})}e^{-i\theta(\mathbf{r})}]_{\rm av}$ fall off as a power law with a continuously variable exponent proportional to $\tilde{\Delta}$. Since the thermal contributions to the correlation functions are nonzero but less divergent, it is tempting to conclude that the 4D X-Y model has a fixed line at $T=0$ (with T irrelevant) for a range of $\tilde{\Delta}$ about zero, which would be analogous to the 2D pure X-Y model.[24]

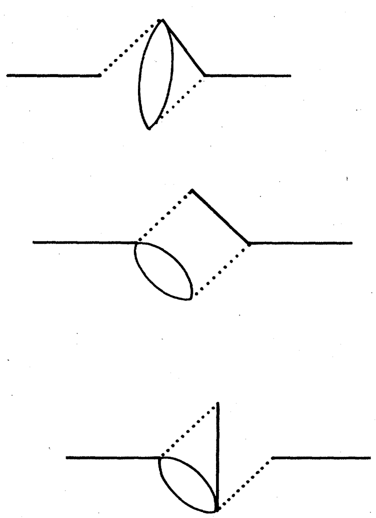
(a)
(b)
(c)


FIG. 4. Graphs which contribute to the leading low-temperature correction to the disconnected correlation function. They do not cancel.

In the next section we construct renormalization-group recursion relations and show that the situation is somewhat more complex.

C. Nonperturbative effects

We have shown that to all orders in perturbation theory the correlation functions at zero temperature for the X-Y model ignoring vortices are simply Gaussian. We now consider possible nonperturbative effects.

From the derivation of the perturbative results via the formal Parisi-Sourlas tricks, it is clear that a possible source of failure in the trivial perturbative result is the existence of many extrema of the Hamiltonian equation (2.2). This has been suggested[25] as a possible mechanism for the breakdown of dimensional reduction for a random-field ϕ^4 theory near six dimensions; it corresponds to supersymmetry breaking. However, there are several advantages of the model under consideration here as far as showing explicitly that the perturbative results break down. The first is that the perturbative results here yield an exact expression for all correlation functions rather than relating them to unknown correlation functions of a pure model. Second, it is much easier to produce an argument for many extrema here because of the periodicity of the random-field terms. Finally, the perturbative result works for a simple lattice model.

For the lattice model with independent Gaussian random fields (for simplicity, we consider the case with *only* random fields) it is straightforward to see that many extrema of the Hamiltonian arise. Consider a group G of sites completely surrounding a particular central site S. There will be some probability,

$$p\sim e^{-CZ^2/\Delta_1} \tag{2.16}$$

(with C a constant), that the random-field magnitudes $g=[(g_1^x)^2+(g_1^y)^2]^{1/2}$ in G and at S will each be larger than a value Z which we will take to be much larger than 1 (in units in which the nearest-neighbor interaction strength is 1). Then in the ground state of the system the phase at each site in G and at S will lie close, 2π, to the preferred value of the phase at that site. If Z is very large, then the phase at site S can be changed by close to 2π and a new metastable state found by changing the phases in G by amounts of order $1/Z$. The effect outside G of the change of the phase at site S can then be arbitrarily well screened out for Z large by the stiffness of the phases in G. Thus there will be several extrema of $\mathscr{H}$ differing only locally.

The formal Parisi-Sourlas[4] calculation will weight both the ground state and the metastable states equally and the singly unstable saddle points in between with *negative* weights. It is thus clear that, unless there is some miraculous cancellation of these effects, the simple perturbative result will be incorrect in any dimension for *any* Δ_1 by amounts of order at least e^{-1/Δ_1}. (We note that similar nonperturbative effects can be shown to occur for a *bounded* random-field distribution.) Since the pure X-Y model is exactly Gaussian in spin-wave theory, a precise statement of the failure of the dimensional reduction would be that for the lattice model the truncated four-point correlation function of the θ's is nonzero in the limit of vanishing temperature. This should be amenable to rigorous proof.

The breakdown of the trivial perturbative result, for, e.g., $[\theta(\mathbf{q})\theta(\mathbf{q}')]_{\rm av}$, can be explicitly demonstrated for a system consisting of only a small finite number of spins; in particular, two. The formal Parisi-Sourlas[4] derivation of the perturbative result can be carried out for a finite system; however, one might anticipate problems due to the infinite degeneracy in the absence of the disorder. These can be remedied by (for example) constraining the average phase to be zero, which has a negligible effect in the thermodynamic limit but a large effect in a small system.

For two coupled spins with a constraint on the average phase $\bar{\theta}=(\theta_1+\theta_2)/2$, the errors of the perturbative result are exponentially small; they arise from large values of the random fields which cause the constrained Hamiltonian to have several extrema. However, if there is no constraint on $\bar{\theta}$, then the Hamiltonian *always* has many extrema since it is periodic in $\bar{\theta}$ (with $\theta_1-\theta_2$ fixed). It can be shown, in this case, that the asymptotic expansion of $[(\theta_1-\theta_2)^2]_{\rm av}$ for small Δ_1 has a leading term proportional to Δ_1, in agreement with the formal perturbative result, but then a subdominant term proportional to $\Delta_1^{3/2}$! (This drastic breakdown of the perturbative result for this simple case has been independently found by Villain and Séméria.[26])

An important question is whether for *infinite* systems in various dimensions the formal perturbative result fails by exponentially small amounts or at some power of the disorder. We will return to this question in the next section. In either case, it is apparent that the perturbative results should not be taken too literally—especially as far as summing them up to yield spin-spin correlation functions as has been done by Dotsenko and Feigelman[27] in $d=3$. Note that the perturbative result yields a spin-spin correlation function falling off *more* rapidly than an exponential in $d<3$, a result which is almost certainly incorrect.

III. *X-Y*—MODEL RENORMALIZATION GROUP

In this section we construct a momentum-shell spin-wave renormalization group which should, in principle, be valid for small T and Δ. [For notational simplicity we will use $O(\Delta)^n$ (without a subscript) to generically denote combinations of n Δ_μ's.] Since we would like to be able to include the effects of temperature, we cannot use the Parisi-Sourlas[4] formalism as a starting point, so for convenience we use replicas (although use of dynamics in the manner of DeDominicis[28] works equally well). (One might worry that replica symmetry breaking analogous to the Almeida-Thouless[29] instability for spin glasses could occur in the presence of random fields or anisotropies. However, it is straightforward to show that, in the perturbative regime near the ferromagnetic fixed point about which we will expand, the system is at least locally stable to replica symmetry breaking.[30] We thus need not concern ourselves with this complication here.) Starting with the effective Hamiltonian

$$\overline{\mathscr{H}}=\int \frac{1}{2T}\sum_\alpha(\nabla\theta_\alpha)^2-\frac{1}{2T^2}\sum_{\alpha,\beta}Q(\theta_\alpha-\theta_\beta) \tag{2.14}$$

[with Q given by Eq. (2.15)], it is easy to power count. Since θ is periodic, it should not be rescaled. To leading order in T and Δ we obtain the differential recursion relations

$$\frac{dT}{dl}=(2-d)T \tag{3.1}$$

and

$$\frac{\partial Q}{\partial l}=(4-d)Q\ . \tag{3.2}$$

Thus the whole function Q is marginal in four dimensons. The other operators which are allowed by symmetry either include higher gradients (and are, hence, more irrelevant) or are of the form

$$\frac{1}{T^p}\sum_{\alpha_1,\ldots,\alpha_p}Q_p(\theta_{\alpha_1},\ldots,\theta_{\alpha_p})\ , \tag{3.3}$$

with Q_p unchanged under $\theta_\beta\to\theta_\beta+C$ for all β and 2π periodic in each of the θ_β separately. These kinds of terms arise from non-Gaussian correlations in the randomness and will be generated under renormalization. However, by power counting we have

$$\frac{\partial Q_p}{\partial l}=[d+p(2-d)]Q_p \tag{3.4}$$

and, hence, the couplings Q_p for $p\geq 3$ are irrelevant near four dimensions. In addition, it can readily be shown that because p must be taken to zero, contributions to one and two replica index terms from the Q_p will contain enough powers of T so that the $1/T^p$ in Eq. (3.3) will not make Q_p dangerously irrelevant.

We thus proceed by keeping only the operators in Eq. (2.14); the first is clearly needed to make the problem well defined and we are, at any rate, interested in the effects of small but finite T. Coupled recursion relations for temperature and all the coefficients of the Taylor-series expansion of $Q(\theta)$ about zero can be derived by usual graphical expansion methods. By examining the form of the nonvanishing terms as $p\to 0$, we see that dT/dl will contain terms of the form T^2 and $T^{n+1}\Delta^m$, but *no* terms of the form Δ^m. All perturbative fixed points (if any) will thus be at $T=0$; as we recall, this was also the case near the upper critical dimension.[6]

The leading term in the recursion relation for Q is just of order Δ^2 with corrections of order $T\Delta^2$, etc. To calculate the order Δ^2 term [which may include derivatives of $Q(\theta)$ with respect to θ], we can ignore the momentum dependence of external lines in graphs. Using this simplification, we can derive a functional recursion relation for Q in a few lines. This functional recursion relation—when expanded in θ—generates all the one-loop renormalizations of $\theta_\alpha^n\theta_\beta^m$ vertices or order Δ^2. We divide θ_α into high-momentum, $\theta_\alpha^>$, and low-momentum, $\theta_\alpha^<$, parts, and expand the effective Hamiltonian to second order in the $\theta_\alpha^>$ which are to be integrated out. The desired limit of the external momenta small can be obtained by setting the momentum of each $\theta_\alpha^<$ to zero, so that the linear terms in $\theta_\alpha^>$ vanish by momentum conservation. The part of the Hamiltonian quadratic in $\theta_\alpha^>$ has the form

$$\mathscr{H}^> = -\frac{1}{2}\int_{\mathbf{q}}^{>} \sum_{\alpha,\beta} \left[\frac{q^2\delta_{\alpha\beta}}{T} + \frac{Q''(\theta_\alpha^< - \theta_\beta^<)}{T^2} - \sum_\gamma \frac{Q''(\theta_\alpha^< - \theta_\gamma^<)\delta_{\alpha\beta}}{T^2} \right] \times \theta_\alpha^>(\mathbf{q})\theta_\beta^>(-\mathbf{q}) , \quad (3.5)$$

where primes on Q denote derivatives with respect to its argument; here, $\theta_\alpha^< - \theta_\beta^<$. Integrating out the $\theta_\alpha^>$ yields, to leading order in Q, a term of the form $\sum_{\alpha,\beta} Q''(\theta_\alpha^< - \theta_\beta^<)/T$, which we ignore since it renormalizes Q at order T and a second-order term of the desired form,

$$\left[\frac{1}{4T^2} \sum_{\alpha,\beta} [Q''(\theta_\alpha^< - \theta_\beta^<)]^2 - \frac{1}{2T^2} \sum_{\alpha,\beta} Q''(0) Q''(\theta_\alpha^< - \theta_\beta^<) \right] \int_{\mathbf{q}}^{>} \frac{1}{q^4} , \quad (3.6)$$

as well as irrelevant terms with three replica sums.

In Appendix A we prove that there are *no* renormalizations of the temperature. From this and expression equation (3.6), we find differential recursion relations near four dimensions,

$$\frac{dT}{dl} = (2-d)T , \quad (3.1)$$

and

$$\frac{\partial Q(\theta)}{\partial l} = (4-d)Q(\theta) + C_4\{\tfrac{1}{2}[Q''(\theta)]^2 - Q''(\theta)Q''(0)\} + O(TQ, Q^3, \dots) , \quad (3.7)$$

where $C_d = (2\pi)^{-d}$ times the surface area of the unit d-dimensional sphere, and we have set $d=4$ in the second term anticipating an expansion in $\epsilon = d-4$.

Surprisingly, in contrast to the simple behavior of the perturbation expansion, the recursion relations even at zero temperature are nontrivial. There is, however, no inconsistency. The only parameter on which the $T=0$ perturbation expansion depends is, from Sec. II,

$$\widetilde{\Delta} = -Q''(0) . \quad (3.8)$$

By differentiating Eq. (3.7) twice with respect to θ, we find that

$$\frac{dQ''(0)}{dl} = (4-d)Q''(0) + O(Q^3) . \quad (3.9)$$

Hence $\widetilde{\Delta}$ is conserved by the zero-temperature renormalization transformation, at least at this order. It is convenient to define

$$U(\theta) \equiv Q''(\theta) - Q''(0) . \quad (3.10)$$

The recursion relation (3.7) truncated at second order then becomes

$$\frac{\partial U}{\partial l} = (4-d)U + \tfrac{1}{2}C_4(U^2)'' . \quad (3.11)$$

Thus $U(\theta)$, although it does not enter the zero-temperature perturbation expansion at all, renormalizes nontrivially and will affect the small but finite-temperature renormalizations and correlation functions.

A. Renormalization-group flows

We now investigate the renormalization-group flows arising from the truncated Eq. (3.11). The initial condition on the even function U is that all its Fourier coefficients are *negative* (corresponding to positive Γ_m) and $U(0)=0$. From this it is straightforward to show that U remains positive. By expanding U in a Fourier series it can be seen that these conditions are preserved by the flow.

We are primarily interested in possible fixed points of Eq. (3.11). Nontrivial fixed points cannot exist for nonzero ϵ since

$$\frac{d}{dl}\int_{-\pi}^{\pi} U(\theta)d\theta = -\epsilon \int_{-\pi}^{\pi} U(\theta)d\theta . \quad (3.12)$$

We thus restrict our consideration to $\epsilon=0$, where naively, by Eqs. (3.9) and (3.12) and dimensional reduction, one might expect a fixed line similar to the two-dimensional pure X-Y model.

B. Flows in $d=4$

By expanding U about the origin, we can obtain an autonomous equation for $U''(0)$:

$$\frac{dU''(0)}{dl} = -\epsilon U''(0) + 3C_4[U''(0)]^2 , \quad (3.13)$$

from which it follows that $U''(0)$ diverges for $\epsilon=0$ at a length scale e^{l^*} with

$$l^* = \frac{1}{3C_4 U_0''(0)} \quad (3.14)$$

(where the subscript 0 denotes the unrenormalized value). At this scale the function $U(\theta)$ develops a $|\theta|^{3/2}$ cusp at the origin and we must consider the effects of higher-order terms in U in the recursion relation which may prevent this singularity.

The form of the higher-order terms can be found by a detailed analysis of the two-loop graphs. The terms contributing to the renormalization of Q at order Δ^N will all involve a total of $4(N-1)$ derivatives with respect to θ. Since there must be at least two derivatives on each Q, we can always consider the simpler recursion relations of $\widetilde{\Delta} = Q''(0)$, and $U(\theta)$. At order Q^3 it can be explicitly verified that, for purely combinatoric reasons, there is no renormalization of $\widetilde{\Delta}$. From the arguments in the preceding section we expect this to be true to all orders. Because of combinatoric cancellations, it can also be shown that the only surviving term in the U equation has the form

$$[U'(\theta)^2 U(\theta)]'' . \quad (3.15)$$

Because of the larger number of derivatives in the higher-order terms such as Eq. (3.15), the divergence of $U''(0)$ at length scale l^* will either be lessened or possibly replaced by a fixed point in $U''(0)$ which could lead to a fixed-point function $U^*(\theta)$. In either case, the behavior will be controlled by the recursion relation for U near $\theta=0$ when the derivatives of U at zero are of order 1. We thus cannot expect to determine the existence or properties of a fixed-point function from perturbation theory in U. Furthermore, since there would be no small parameter at the fixed point, the operators which were irrelevant at the ferromagnetic fixed point might play a role. In particular, since the length scale at which the solution to the truncated flow equation (3.13) diverges scales as $e^{\text{const}/U_0''(0)}$, the nonperturbative effects discussed in Sec. II might enter at the same scale.

It is worth noting here, however, that if the higher-order terms in the expansion of the recursion relation in powers of U have the same form as the first two, in that they are total derivatives, then the full renormalization group might conceivably contain a fixed line. The existence (or lack thereof) of a fixed-point function or a fixed line may be worth exploring further by an approximate nonlinear renormalization group.

The simplest and, *a priori*, the most likely behavior, is that the system has neither a fixed point nor a fixed line, but simply scales to a disordered paramagnet. The correlation length, roughly e^{l^*}, would then be exponentially large for small Δ analogous to the pure two-dimensional Heisenberg model rather than the pure 2D X-Y model. However, it is possible that the system could scale to some kind of spin-glass fixed point—this possibility will be discussed briefly in Sec. VI.

C. Flow in $d=4+\epsilon$

Formally, in more than four dimensions, all the parts of U (e.g., all its derivatives at the origin) are irrelevant. However, the infinite number of operators with the same negative eigenvalue, $-\epsilon$, can cause problems.

Let us consider initially just a random field so that

$$U_0(\theta)=\Delta_1(1-\cos\theta) . \tag{3.16}$$

We expand U about the origin:

$$U(\theta)=\sum_{n=1}^{\infty} u_{2n}(-1)^{n+1}\theta^{2n} , \tag{3.17}$$

where we have defined the u_{2n} so that initially they are all positive. They are just the $2n$ point vertices in the θ's. The truncated recursion relation (3.13) leads to recursion relations for the u_{2n},

$$\frac{du_{2n}}{dl}=-\epsilon u_{2n}+\tfrac{1}{2}C_4(2n+1)(2n+2)\sum_{k=1}^{n} u_{2n+2-2k}u_{2k} , \tag{3.18}$$

all the terms in the sum in Eq. (3.18) are positive if the u_{2n} are positive, and, hence, all u_{2n} will remain positive. The solutions to Eq. (3.18) will therefore be bounded below by the solutions to

$$\frac{dy_n}{dl}=[-\epsilon+C_4(2n+1)(2n+2)\bar{u}_2]y_n , \tag{3.19}$$

with

$$y_n(l=0)=u_{2n}(l=0)=\frac{1}{(2n)!} \tag{3.20}$$

and

$$\bar{u}_2=u_2^{(0)}e^{-\epsilon l}<u_2(l) , \tag{3.21}$$

as long as the right-hand side of Eq. (3.19) is positive. For any *fixed* ϵ, no matter how small Δ_1 is, the right-hand side of Eq. (3.19) will always be positive for sufficiently large n. The maximum value of u_{2n} will then be greater than the maximum value of y_n from Eq. (3.19), which yields, for n large enough so that

$$D_n=\frac{C_4(2n+1)(2n+2)}{\epsilon u_2^{(0)}}>1 , \tag{3.22}$$

$$u_{2n}^{\max}>y_n^{\max}=\frac{u_{2n}^{(0)}}{D_n}e^{D_n-1} . \tag{3.23}$$

Since $D_n\sim n^2$ and $u_{2n}^{(0)}\sim 1/(2n)!$, $u_{2n}^{\max}$ will become arbitrarily large for n sufficiently large. This suggests that no matter how small the initial random-field strength Δ_1, the higher-order terms in the flow equations for U will become important at sufficiently long length scales.

Thus it is, in principle, possible that, although the disorder is formally irrelevant in more than four dimensions, the ferromagnetic fixed point has a vanishing domain of attraction in the space of physical initial Hamiltonians. This is due to the nonuniformity of the truncation of the recursion relation;[31] however, it may also occur within the full nonlinear renormalization group. If it does occur, then one should perhaps question whether ferromagnetism exists even for weak disorder in more than four dimensions. However, it is most likely that the disorder will only cause nontrivial long-distance behavior of correlation functions but not destroy the ferromagnetic order entirely.

D. Flows and nonperturbative effects below four dimensions

In less than four dimensions the disorder will renormalize to be of order 1 on a length scale ξ_c, which is the scale for crossover from weak to strong disorder. From the form of the recursion relations, Eq. (3.2), we have that for small Δ

$$\xi_c\sim\Delta^{-1/(4-d)} . \tag{3.24}$$

This is the length scale at which nonperturbative effects, in particular, many extrema of the Hamiltonian, will occur. The behavior on length scales longer than ξ_c is unclear, and for the actual X-Y model vortices may cause significant deviations from the spin-wave model considered here. One can draw some conclusions, however, concerning the failure of the perturbative results. The nonperturbative effects will cause many extrema of the Hamiltonian with a spatial density

$$\rho \sim \xi_c^{-d} \sim \Delta^{d/(4-d)} ,$$

which is a *power law* in the strength of the disorder. Thus in less than four dimensions the perturbative results for the correlation functions probably fail by a power of the disorder rather than an exponentially small amount, in particular, the perturbative result for the spin-spin correlation function: that it decays as $e^{-C\tilde{\Delta}r^{4-d}}$ for $d<4$ is almost certainly incorrect at least in less than three dimensions. On length scales longer than ξ_c the spin-spin correlation function is most unlikely to fall off faster than exponentially, even in one dimension. Note that at a distance ξ_c the perturbative correlation function is still of order 1.

We may ask, in general, how the system behaves on scales larger than ξ_c. The simplest possibility is that it is just a paramagnet in any $d<4$ with correlation length ξ_c; we postpone discussion of more novel possibilities until the last section.

IV. GENERAL n: ZERO TEMPERATURE AND SUPERSYMMETRY

We now return to the general n case, which in the fixed-length limit is just the n-component nonlinear σ model in a random magnetic field and with random μth-rank anisotropies. Analysis of the zero-temperature perturbation expansion is tedious since it is necessary to use σ and π fields, and the resulting expressions which arise from the random anisotropies are rather complicated. However, to generate the $T=0$ perturbation expansion we can again use Parisi-Sourlas[4]—type tricks.[32]

We would like to minimize the Hamiltonian equation (1.3) subject to the constraint that $S^2=1$. We thus introduce Lagrange multipliers $\lambda(\mathbf{r})$ and define

$$\tilde{\mathscr{H}}=\mathscr{H}-\int \frac{\lambda(\mathbf{r})}{2}[S(\mathbf{r})^2-1]d^dr . \tag{4.1}$$

Analogously to the X-Y case, we define a generating function

$$Z_h\{\mathbf{f}(\mathbf{r})\}=\int \prod_{\mathbf{r}} DS(\mathbf{r})D\lambda(\mathbf{r})\exp\left[\int f\cdot S\right]\delta\left[\frac{\delta\tilde{\mathscr{H}}}{\delta S(\mathbf{r})}\right]\delta\left[\frac{\delta\tilde{\mathscr{H}}}{\delta\lambda(\mathbf{r})}\right]J_h\{S(\mathbf{r}),\lambda(\mathbf{r})\} , \tag{4.2}$$

with the Jacobian given by

$$J_h=\det\begin{bmatrix}\dfrac{\delta^2\tilde{\mathscr{H}}}{\delta S_i(\mathbf{r})\delta S_j(\mathbf{r}')} & \dfrac{\delta^2\tilde{\mathscr{H}}}{\delta S_i(\mathbf{r})\delta\lambda(\mathbf{r}')}\\ \dfrac{\delta^2\tilde{\mathscr{H}}}{\delta\lambda(\mathbf{r})\delta S_j(\mathbf{r}')} & \dfrac{\delta^2\tilde{\mathscr{H}}}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')}\end{bmatrix} . \tag{4.3}$$

We then introduce auxiliary n-component fields $\hat{S}_i$ to eliminate the first set of δ functions in Eq. (4.2) and complex Fermi fields to cancel the Jacobian: an n-component one, ψ_i, with its conjugate, $\bar{\psi}_i$, as well as an auxiliary pair, χ and $\bar{\chi}$. The generating function then becomes

$$Z_h\{\mathbf{f}\}=\int DS\,D\hat{S}\,D\psi\,D\bar{\psi}\,D\chi\,D\bar{\chi}\,D\lambda\,\delta(1-S^2)\exp\int\left[\mathscr{L}_0+\sum_{\mu=1}^{\infty}\mathscr{L}_h^{\mu}+f\cdot S\right] , \tag{4.4}$$

with the bare "Lagrangian"

$$\mathscr{L}_0=\bar{\psi}\cdot(\nabla^2+\lambda)\psi+\bar{\psi}\cdot S\chi+\bar{\chi}S\cdot\psi+i\hat{S}\cdot(-\nabla^2-\lambda)S , \tag{4.5}$$

and the μth-rank random anisotropy part given by

$$\mathscr{L}_h^{\mu}=\left[-i[\hat{S}_{i_1}S_{i_2}\cdots S_{i_\mu}+S_{i_1}\hat{S}_{i_2}S_{i_3}\cdots S_{i_\mu}+(\mu-2)\text{ permutations}]+\sum_{1\le j\ne k\le\mu}\frac{\bar{\psi}_{i_j}\psi_{i_k}}{S_{i_j}S_{i_k}}\prod_{l=1}^{\mu}S_{i_l}\right]h_\mu^{i_1\cdots i_\mu} . \tag{4.6}$$

Since all the h_μ are independent (correlations will be irrelevant for weak disorder), we can average over each of the h_μ separately. However, before doing this it is convenient to formally integrate out the auxiliary fields λ, χ, and $\bar{\chi}$. This will result, formally, in three extra δ-function constraints: $\delta(-i\hat{S}\cdot S+\bar{\psi}\cdot\psi)$, $\delta(\bar{\psi}\cdot S)$, and $\delta(S\cdot\psi)$. These can be used to dramatically simplify the expressions obtained after averaging over the h_μ. After liberal use of these constraints, the fixed-length constraint, and the Fermi commutation rules, we find that for *each* μ

$$\ln[\exp\mathscr{L}_h^{\mu}]_{\rm av}=\frac{\mu\Delta_\mu}{2}\hat{S}\cdot\hat{S} . \tag{4.7}$$

Combining Eq. (4.7) for each μ with $\mathscr{L}_0$ we finally arrive at the averaged generating function

$$Z\{\mathbf{f}\}=\int DS\,D\hat{S}\,D\psi\,D\bar{\psi}\,\delta(S^2-1)\delta(\bar{\psi}\cdot S)\delta(S\cdot\psi)\delta(-i\hat{S}\cdot S+\bar{\psi}\cdot\psi)\exp\left[\int\mathscr{L}\right]+f\cdot S , \tag{4.8}$$

with

$$\mathscr{L}=\bar{\psi}\cdot\nabla^2\psi+(-i\hat{S})\cdot\nabla^2 S+(\tilde{\Delta}/2)(-i\hat{S})^2\ , \tag{4.9}$$

which, as claimed in the Introduction, *only* depends on the combination of the anisotropy strengths $\tilde{\Delta}$.

We can now define an n-component superfield[6,32] with components given by

$$\Phi_i\equiv\frac{S_i}{\tilde{\Delta}^{1/2}}+\bar{\theta}\psi_i+\bar{\psi}_i\theta+\theta\bar{\theta}(-i\hat{S}_i\tilde{\Delta}^{1/2})\ , \tag{4.10}$$

with θ and $\bar{\theta}$ anticommuting c numbers. We then find that Eq. (4.8) (at $f=0$) is formally equivalent to the supersymmetric nonlinear σ model,

$$Z=\int D\Phi\,\delta\left[\Phi\cdot\Phi-\frac{1}{\tilde{\Delta}}\right]\exp\tfrac{1}{2}\int d\bar{\theta}\,d\theta\,d^d r\,\Phi\cdot\nabla_{ss}^2\Phi \tag{4.11}$$

with

$$\nabla_{ss}^2=\nabla^2+\frac{\partial^2}{\partial\bar{\theta}\,\partial\theta} \tag{4.12}$$

the supersymmetric Laplacian.

From the standard dimensionality reduction[4] for supersymmetric models, it follows that the perturbation expansion in powers of $\tilde{\Delta}$ in d dimensions will be the same as the expansion in powers of T of the pure nonlinear σ model in $d-2$ dimensions. However, now the result is much more general than for the ϕ^4 theory in a random field—here the perturbative dimensional reduction works in the presence of *any* combination of independent μth-rank anisotropies. It is presumably possible to derive this result directly from a perturbation expansion—however, this will certainly be far more difficult than for the simple X-Y case or the ϕ^4 theory.

As for the X-Y case discussed in Sec. II, the equivalence of the random model at $T=0$ near $d=4$ to the pure model in two dimensions less is likely to hold *only* in perturbation theory, since there will again be many solutions of the extremal conditions even for arbitrarily small $\tilde{\Delta}$. We now turn to construction of renormalization-group recursion relations.

V. GENERAL n: REPLICAS AND RENORMALIZATION GROUP

In order to construct a renormalization group which, in principle, allows calculation of low but nonzero temperature properties, we again use replicas.

The replicated effective Hamiltonian is simply

$$\bar{\mathscr{H}}=\int\frac{1}{2T}\sum_\alpha(\nabla S_\alpha\cdot\nabla S_\alpha)-\frac{1}{2T^2}\sum_{\alpha,\beta}R(S_\alpha\cdot S_\beta)\ , \tag{5.1}$$

where the function $R(X)$ represents all the anisotropies and is given by

$$R(X)=\sum_{\mu=1}^{\infty}\Delta_\mu X^\mu\ . \tag{5.2}$$

We impose the fixed-length constraint

$$S_\alpha\cdot S_\alpha=1 \tag{5.3}$$

for each α from 1 to p. The derivatives of R with respect to X will be denoted by primes. By analogous arguments to those in Sec. III, all other operators allowed by symmetry in the Hamiltonian are irrelevant and, at least to lowest order, we need not include them.

To expand about a putative ordered ground state with all the replicas ordered along the same direction, we write, as usual,[16,17]

$$S_\alpha=(\pi_\alpha,\sigma_\alpha)\ , \tag{5.4}$$

where π is an $n-1$ component field and

$$\sigma_\alpha=(1-\pi_\alpha^2)^{1/2}\ . \tag{5.5}$$

We can then expand $R(S_\alpha\cdot S_\beta)$ about the aligned state with $X_{\alpha\beta}\equiv S_\alpha\cdot S_\beta=1$ for all α,β. To change variables from S to π we need to introduce a Jacobian; however, since it will enter the effective Hamiltonian *without* a $1/T$ in front, it will not affect the recursion relations to leading order in T and we may thus ignore it to the order desired here.

We perform a momentum-shell renormalization group by breaking down π_α into a slowly varying low-momentum $\pi_\alpha^<$ and a rapidly varying high-momentum $\pi_\alpha^>$ and then integrate out the $\pi_\alpha^>$ field. We must allow for a rescaling of the π field; the renormalized field will then be given by (with b the spatial rescaling factor)

$$\pi_{R\alpha}(\mathbf{r})=\zeta\pi_\alpha^<(\mathbf{r}b)\ , \tag{5.6}$$

with ζ to be determined by requiring that, with $\sigma_{R\alpha}=(1-\pi_{R\alpha}^2)^{1/2}$, the theory is rotationally invariant in spin space.

The conventional method for integrating out the high momenta is to expand the effective Hamiltonian in powers of the π_α and then to use graphical methods with the internal lines representing the $\pi_\alpha^>$ which are to be integrated out and the external lines representing the $\pi_\alpha^<$. In order to obtain the rotationally invariant Hamiltonian in terms of the spin operators, the expansion in powers of the $\pi_{R\alpha}$ must be resummed.

It is not clear or *a priori* that this resumming will result in a Hamiltonian of the same form as the original problem which included only a random field. In Appendix B we show explicitly that terms will be generated under renormalization which *must* come from higher-rank anisotropies in the renormalized Hamiltonian.

Here we will use a different method which is a generalization of that introduced in Sec. III for the X-Y case. While we will expand in the $\pi_\alpha^>$, we will *not* expand in the $\pi_\alpha^<$, and thus the resumming does not need to be done. We first compute the temperature renormalization, which we note could have been obtained by more conventional methods.

The renormalization of the temperature to order RT can be computed straightforwardly from the renormalization of the $(\nabla\pi_\alpha)^2$ term in the effective Hamiltonian. It only depends on the coefficient $(1/T)$ of the $(\pi_\alpha\cdot\nabla\pi_\alpha)^2$ and on the expansion of $R(S_\alpha\cdot S_\beta)$ to second order in the π fields:

$$\frac{1}{T_R}=\frac{1}{T}\zeta^2 b^{d-2}[1+R'(1)C_4\ln b+O(T,RT,R^2,\dots)] , \tag{5.7}$$

where we have set $d=4$ in the second term. The field rescaling ζ can be determined either by requiring that the renormalized coefficient of the $(\pi_\alpha\cdot\nabla\pi_\alpha)^2$ term also be $1/T_R$, by requiring that the renormalization of a uniform field in the σ and π directions be the same, or by requiring (see below) that the disorder terms only depend on $S_\alpha\cdot S_\beta$; all of these just use the rotational invariance in spin space and, hence, are clearly equivalent. We obtain

$$\zeta=1-\tfrac{1}{2}R'(1)(n-1)C_4\ln b+O(T,RT,R^2) , \tag{5.8}$$

whence the differential recursion relation

$$\frac{dT}{dl}=(2-d)T+C_4(n-2)TR'(1)+O(T^2,TR^2,\dots) , \tag{5.9}$$

which reduces—upon changing variables to angle variables—to Eq. (3.1) for the X-Y case, $n=2$.

Obtaining the functional recursion relation for $R(X)$ is much more complicated since each diagram with external $\pi^<$ legs and internal $\pi^>\pi^>$ propagators can arise from many different terms in the Taylor-series expansion of $R(X)$ about $X=1$. We want to treat *all* one-loop diagrams with arbitrary numbers of external legs. This we can do by keeping functions of the slowly varying $\pi^<$ which, to the desired order, we can take to be spatially uniform, i.e., at $q=0$. We thus expand $\overline{\mathscr{H}}$ to second order in the $\pi^>$ and set the momentum of the $\pi^<$ to zero, except for the terms with no $\pi^>$. The gradient term just becomes

$$\frac{1}{2T}\sum_\alpha(\nabla\pi_\alpha)^2+\frac{(\pi_\alpha\cdot\nabla\pi_\alpha)^2}{1-\pi_\alpha^2}=\frac{1}{2T}\sum_\alpha\left[(\nabla\pi_\alpha^<)^2+\frac{(\pi_\alpha^<\cdot\nabla\pi_\alpha^<)^2}{1-(\pi_\alpha^<)^2}\right]$$
$$+\frac{1}{2T}\sum_\alpha\left[(\nabla\pi_\alpha^>)^2+\frac{(\pi_\alpha^<\cdot\nabla\pi_\alpha^>)^2}{1-(\pi_\alpha^<)^2}\right]+O(\pi^>\nabla\pi^>\nabla\pi^<,\pi^{>2}(\nabla\pi^<)^2) . \tag{5.10}$$

The second term in Eq. (5.10) can be written in the form

$$\frac{1}{2T}\sum_{\alpha,\beta,a,b}\nabla\pi_{\alpha a}^>\nabla\pi_{\beta b}^>G_{\alpha\beta ab}^{-1} , \tag{5.11}$$

where a and b run over spin indices from 1 to $n-1$, and the matrix G over α and a is given by

$$G_{\alpha\beta ab}=\delta_{\alpha\beta}(\delta_{ab}-\pi_{\alpha a}^<\pi_{\alpha b}^<) . \tag{5.12}$$

Expanding R to second order in the $\pi^>$ we obtain (dropping the linear term which vanishes by the high-low momentum separation)

$$\sum_{\alpha,\beta}R(S^\alpha\cdot S^\beta)=\sum_{\alpha,\beta}R\{\pi_\alpha^<\cdot\pi_\beta^<+[1-(\pi_\alpha^<)^2]^{1/2}[1-(\pi_\beta^<)^2]^{1/2}\}+\sum_{\alpha,\beta}\pi_{\alpha a}^>\pi_{\beta b}^>\Gamma_{\alpha\beta ab} , \tag{5.13}$$

where Γ is of the form

$$\Gamma_{\alpha\beta ab}=f_{ab}(\pi_\alpha^<,\pi_\beta^<)+\delta_{\alpha\beta}\sum_\gamma g_{ab}(\pi_\alpha^<,\pi_\gamma^<) . \tag{5.14}$$

In this form, it is straightforward to integrate out the $\pi^>$ perturbatively, yielding

$$\sum_{\alpha,\beta}\frac{R_R}{T_R^2}=\frac{b^d}{T^2}\sum_{\alpha,\beta}R\{\pi_\alpha^<\cdot\pi_\beta^<+[1-(\pi_\alpha^<)^2]^{1/2}[1-(\pi_\beta^<)^2]^{1/2}\}+\frac{1}{2T^2}\int_q^>\frac{1}{q^4}[\mathrm{Tr}(G\Gamma G\Gamma)]+O\left[\frac{R}{T},\frac{R^3}{T^2},\dots\right] , \tag{5.15}$$

where G and Γ are treated as matrices over α and a. The order Γ term which could have been included in (5.15) renormalizes R at order RT and we hence ignore it. Equation (5.15) is not manifestly rotationally invariant. We must expand the first term to leading order in $\zeta-1$ (for b near 1) after substituting $\zeta\pi_{R\alpha}$ for $\pi_\alpha^<$, and the resulting expression will then cancel the nonrotationally invariant parts of the second term.

The details of the calculation of $\mathrm{Tr}(G\Gamma G\Gamma)$ which are rather tedious but straightforward will not be reproduced here. Note that only the ff and fg terms will contribute to the renormalization of R; the gg terms will involve three replica sums and hence will renormalize (or generate) irrelevant non-Gaussian correlations in the disorder, as discussed in Sec. III. The differential functional recursion relation for R is given by

$$\frac{\partial R(X)}{\partial l}=(4-d)R+C_4\{2(n-2)R'(1)R(X)-R'(1)R'(X)(n-1)X+R'(1)R''(X)(1-X^2)$$
$$+\tfrac{1}{2}[R'(X)]^2(n-2+X^2)+R'(X)R''(X)(X^3-X)+\tfrac{1}{2}[R''(X)]^2(1-X^2)^2\} , \tag{5.16}$$

where the first term in the curly brackets comes from the temperature renormalization equation (5.9) and we have neglected terms of order RT, R^3, etc. This rather complicated expression simplifies drastically for the X-Y case. If we substitute $Q(\theta)=R(\cos\theta)$, then Eq. (5.16) reduces to the simple recursion relation for Q, Eq. (3.7).

If the Hamiltonian initially includes a random magnetic field, $R_0(\mathcal{X})=\Delta_1\mathcal{X}$, all the higher-rank anisotropies will be generated by the renormalization-group flows. If, on the other hand, $\mathscr{H}$ initially contains only second-rank or other even-rank anisotropy but no odd-rank anisotropies, then this property will be preserved by the flow as should be expected from the symmetry $S\rightarrow -S$ of the random Hamiltonian.

We note that $R'(1)$ plays a special role in the renormalization-group equations: In particular, ζ and the thermal recursion relation only depend on $R'(1)$. From the definition of R in terms of the Δ_μ, Eq. (5.2), it is clear that

$$R'(1)=\widetilde{\Delta}\ . \tag{5.17}$$

The special role of $R'(1)$ should thus have been anticipated.

Furthermore, since $T=0$ correlation functions only depend on $\widetilde{\Delta}$ in perturbation theory, we should expect that $R'(1)$ will renormalize simply at zero temperature. Differentiation of Eq. (5.16) at $\mathcal{X}=1$ yields

$$\frac{dR'(1)}{dl}=(4-d)R'(1)+C_4(n-2)[R'(1)]^2+O(R^3)\ , \tag{5.18}$$

which is just like the recursion relation for temperature in two dimensions less.

At this point it is instructive to examine what has gone wrong with the calculations of Young[3] for the $4+\epsilon$ random-field case and Pelcovits[14] for the random-anisotropy case. Instead of performing a functional renormalization-group transformation which allows all possible terms in the Hamiltonian that are allowed by symmetry to be generated, these authors[14] expand the nonlinear σ model in the π fields as done in Appendix B and use only the renormalization of the $(\nabla\pi_\alpha)^2$ and $\pi_\alpha\pi_\beta$ terms and a uniform magnetic field to yield the renormalized Hamiltonian.[14] This works for the pure nonlinear σ model in $2+\epsilon$ dimensions[16,17] since for that case there is only *one* term of the correct dimension allowed by symmetry. However, for the random $4+\epsilon$ problem the renormalized coefficient of the $\pi_\alpha\pi_\beta$ term need not arise solely from the expansion in π's of a random field (or second-rank anisotropy), but can come from the expansion of higher-rank anisotropies, e.g., the $(\sigma_\alpha\sigma_\beta)^2\pi_\alpha\cdot\pi_\beta$ term from a random third-rank anisotropy. This difficulty is illustrated in Appendix B. It is seen that at least some higher-rank anisotropies are generated. In principle, it is possible to derive Eq. (5.16) by evaluating *all* one-loop graphs with an arbitrary number of external legs (the X-Y case in angle variables was originally done this way); however, the combinatorics involved would be extremely complicated. We note that similar difficulties arising from an infinite number of marginal operators occur in other random problems; some of these might be handled by methods similar to those used here.

A. Fixed points in $d=4+\epsilon$

One might hope that in $d=4+\epsilon$ the functional equation for R would exhibit a singly unstable fixed point with R and its derivatives uniformly of order ϵ which could be found as a fixed point of the truncated equation (5.16). The uniformity condition on R is important for the same reasons as for the X-Y case in Sec. III since neglected terms of order R^3 will generally contain more derivatives with respect to $\mathcal{X}$ and thus can become important if the derivatives of R are not all small. In Appendix C we prove that the truncated equation has no singly unstable fixed points which have a Taylor-series expansion about the ordered point $\mathcal{X}=1$. The flows from Eq. (5.16) will generally go into regimes where nonperturbative effects cannot be neglected.

How can the absence of a perturbative fixed point of the full renormalization group be reconciled with the apparent $4+\epsilon$ fixed points found previously?[3,14] What was essentially found in the earlier calculations was a fixed point for the special combination $\widetilde{\Delta}$ of the anisotropies which was interpreted as a real fixed point by ignoring all but one of the Δ_μ. In fact, the *hyperplane* $\widetilde{\Delta}=\widetilde{\Delta}^*=C_4\epsilon/(n-2)$, $T=0$ is an invariant hyperplane of the full renormalization group, at least at quadratic order in the Δ_μ and probably to all orders. However, at any point in the physical region of this invariant hyperplane, there will always be flows within the hyperplane and thus no fixed points of the *full* renormalization group. Thus the putative $4+\epsilon$ fixed point found previously does not even exist in the full renormalization group. The flow projected onto a plane of $\widetilde{\Delta}$ and one of the Δ_μ's is schematically illustrated in Fig. 2.

VI. DISCUSSION

We have argued in the bulk of this paper that the behavior of random-field systems with continuous symmetry near four dimensions is rather more complicated than would be expected from straightforward dimensional reduction.

By use of supersymmetry we have shown that for fixed-length spins the formal perturbation expansions of all zero-temperature correlation functions in powers of the strengths of the random field and all possible random anisotropies only depend on a special linear combination of them, $\widetilde{\Delta}$. This expansion in powers of $\widetilde{\Delta}$ is identical to the expansion of the pure nonlinear σ model in powers of temperature in two dimensions less. Renormalization-group recursion relations were constructed which generate at leading nontrivial order all of the random anisotropies from a random field, and all of the even-rank anisotropies from a second-rank anisotropy. The functional renormalization group which includes all of the anisotropies does not have a singly unstable fixed point in $4+\epsilon$ dimensions, although it does exhibit a zero-temperature invariant hyperplane with $\widetilde{\Delta}=\widetilde{\Delta}^*=O(\epsilon)$. The temperature is irrelevant perturbatively; however, it is argued that the renormalization-group flows will take the system into a regime where nonperturbative effects, particularly the existence of many extrema of the Hamiltonian, will play an important role and break the perturbative supersymmetry. It is likely that in this nonperturbative regime, temperature will become more important.

We conclude that the nonperturbative effects will probably change the critical behavior in $4+\epsilon$ dimensions, although there is still likely to be a phase transition from a ferromagnet to a disordered phase. Unfortunately, due to the absence of a perturbative fixed point of the full renormalization group in $4+\epsilon$ dimensions, we cannot make any predictions about the critical behavior (indeed, the transition could be first order). Thus it is still possible that the exponents are the same as those of the pure systems in $2+\epsilon$ dimensions,[15–17] although there is certainly little reason to believe this. It might appear that a renormalization-group argument could still be made to support dimensional reduction for the zero-temperature critical behavior by constructing an anisotropic renormalization group which preserves the supersymmetry[6] but not the full spatial rotational symmetry. This would, at least naively, need to contain only the parameter $\widetilde{\Delta}$ and would yield a fixed point in $4+\epsilon$ dimensions with exponents obeying dimensional reduction. However, in order to achieve this it would be necessary to discard all nonsupersymmetric couplings. Since the perturbative supersymmetry is only an *approximate* symmetry which is broken by (at least) nonperturbative effects, this is potentially dangerous. Perhaps the main lesson from the perturbative renormalization-group flows found in Secs. III and V is that deviations from supersymmetry are relevant since the flows take the Hamiltonian into nonperturbative regimes.

Since we have argued that perturbative results near four dimensions are misleading, we must question the apparent similarity between the random-field and random-anisotropy models in this limit. Once the perturbative results have been discarded, it is likely that, even if both models have a ferromagnetic to nonferromagnetic transition in $4+\epsilon$ dimensions, the critical behavior will be different. We will return to related questions in the next section.

A. Random fields near six dimensions and dimensional reduction

The calculations discussed in this paper illustrate the danger of using formal methods to calculate exponents perturbatively. If a singly unstable perturbative fixed point had existed in $4+\epsilon$ dimensions, then its eigenvalues could have been calculated by formal methods provided all the important operators were included. Since such a fixed point does not exist, we are left with a situation analogous to that before perturbative renormalization-group methods were understood: It is not at all clear what sense, if any, can be made of formal divergent perturbation expansions such as those discussed in Sec. IV.

We now turn to the question of the behavior of random-field systems near six dimensions. In contrast to the behavior near four dimensions, there does exist a perturbative fixed point for the random-field ϕ^4 theory in $6-\epsilon$ dimensions.[2] Thus the formal expansions, which (since random anisotropies are irrelevant) need include only one marginal operator, should yield the correct expansions of the exponents. There is one caveat, however, namely, the possible effects of the dangerous irrelevancy of temperature have not been analyzed in general.

In a recent paper Klein, Landau, and Perez[33] have proven that for the ϕ^4 theory in a random field, once the first step of Parisi-Sourlas[4]—the replacement of thermal averages by averages of extrema of $\mathscr{H}$—has been made, then dimensional reduction follows rigorously, not just perturbatively. Thus it is clear that if dimensional reduction breaks down it must do so by the breakdown of the first step, i.e., because of many extrema of $\mathscr{H}$ whose existence breaks the supersymmetry. Since we have argued that the many extrema are likely to play an important role near four dimensions, it is natural to ask what their effect will be near six dimensions. If the coefficient r of the ϕ^2 term in the Ginzburg-Landau-Wilson $\mathscr{H}$ is positive, then there is a unique extremum. Unfortunately, the fixed-point value of r in $6-\epsilon$ dimensions is negative and of order ϵ. If the renormalization-group flows are smooth in the neighborhood of $r=0$, as is usually assumed, then it should be reasonable to extrapolate the flows from $r \gtrsim 0$ (where the recursion relations are presumably valid) to the desired region near the fixed point. However, since the fixed point is at zero temperature, which is inherently singular in disordered systems, there is reason to doubt the smoothness of the flows in the vicinity of $T=0, r=0$. This potential problem with the $6-\epsilon$ expansion is related to the possible effects of the dangerous irrelevancy of temperature mentioned above. We must thus leave an intriguing open question the validity of the $6-\epsilon$ expansion as an asymptotic expansion of the exponents.

At this stage it is unclear what is the extent, if any, of the validity of dimensional reduction. There are at least three possibilities: (i) The critical exponents could be exactly equal to those of the pure system in two dimensions less for a nonzero range of dimensions below six, but differ below some dimension d_s. Alternatively, the critical exponents could differ in *any* dimension below six; this could be the result of either (ii) a complete breakdown of the $6-\epsilon$ expansion at some order in ϵ or perhaps more likely (iii) the presence of different essential singularities in the exponents as the upper critical dimensions of the random and pure systems are approached. If d_s is between five and six dimensions, it is not obvious that there is a real distinction between possibilities (i) and (iii) since it may be that the continuation in dimensionality is not uniquely enough defined beyond perturbation theory to make sense of essential singularities in exponents at critical dimensions. One of the main conclusions of this paper is that dimensional reduction is likely to fail for systems with continuous symmetry as well as for Ising systems.

B. Spherical model limit: $n \to \infty$

Although no results for exponents have been derived by use of the functional renormalization group studied in Secs. III and V, it may be possible to obtain some in a different limit. In the spherical model limit $n \to \infty$, it can be shown from the results of Appendix C that the structure of the recursion relations simplifies somewhat. In particular, in this limit all the μth-rank anisotropies will not be equally important; naively, their importance will decrease with increasing μ. It might thus be possible to study the effects of just, for example, a random-field and random second-rank anisotropy in the large-n limit for $4<d<6$. If the difficulties appearing in this paper in the $4+\epsilon$ expansion also appear in the $1/n$ expansion, it might be possible to control the latter by introducing just a second-rank anisotropy at some order in $1/n$ (rather than *all* the anisotropies needed here) into a model which only has random fields initially. A careful analysis of the $1/n$ expansion should shed some additional light on the subject.

C. Infinite sets of marginal operators in other problems

The models considered in this paper are the first example of which we are aware of a problem for which an infinite number of marginal operators which are not related by symmetry play an important role. The features of the models which cause this to occur are the fixed-length spin constraint together with an extra index on the fields, here the replica index. These two features both occur generically for random systems at their lower critical dimensions and perhaps for other systems with complicated order parameters. Thus it is likely that infinite sets of marginal operators will play an important role in other random problems; for example, localization with interactions near two dimensions. For these problems standard methods for renormalizing the effective Hamiltonian will lead to misleading or erroneous results similar to those discussed in Appendix B for the model studied here. It should be interesting to investigate other problems for which an infinite number of operators play a role; in particular, ones for which the nonuniformity problems encountered here can be controlled.

VII. NATURE OF "DISORDERED" PHASES AND RELATION TO OTHER WORK

In this section we speculate on possible phases which may exist for random-field and random-anisotropy systems. Several of the suggestions in the literature for possible phases rely on perturbative results of the form discussed in this paper; we comment on these in light of the present conclusions concerning the failure of perturbation theory.

In more than four dimensions, it is likely that, despite the possible difficulty mentioned in Sec. III, a ferromagnetic phase exists for weak randomness at low temperatures. As the strength of the randomness increases, the ferromagnetism may disappear and it is then natural to ask whether or not the resulting phase is a simple paramagnet. In less than four dimensions, ferromagnetism will be destroyed in the presence of any random field or anisotropy,[9,19] however, one can again ask whether for weak randomness at low temperatures the system is just a simple paramagnet. In the presence of a random field the phase diagram is most likely quite simple (even if the critical behavior is not), with ferromagnetism in $d>4$ for weak disorder and low temperatures and simple paramagnetism elsewhere and for arbitrarily weak disorder in $d<4$.

There are various interesting suggestions in the literature, however, that spin-glass phases or phases with quasiferromagnetic tendencies exist in systems with random anisotropy. Although the results of this paper do not bear directly on these questions since they involve the behavior of flows *away* from the ferromagnetic fixed point, it is nevertheless useful to consider some of the suggestions in light of the present results. Since these suggestions apply only to models with random even-rank anisotropies but *no* random field, we will use here random anisotropy to mean random second-rank anisotropy with *no* random magnetic field.

Dotsenko and Feigelman[27] have used the weak-disorder perturbative results discussed in Sec. II to argue that in three dimensions X-Y magnets with random twofold anisotropy will have spin-spin correlation functions which fall off as simple exponentials; this follows directly from assuming that the perturbative result of simple Gaussian phase correlations is correct. In addition, they argue that the helicity modulus, i.e., the response of the *phase* to a long-wavelength perturbation, is nonzero so that the system is *not* simply paramagnetic. From the flows discussed in Sec. III one can see that nonperturbative effects should come in at a length scale at which the perturbative spin-spin correlation function is still of order 1. Thus there is no reason to believe that the perturbative results used by Dotsenko and Feigelman[27] are correct in the long-wavelength regime of interest. This work does, however, raise an interesting possibility that we will consider later: the possibility of a phase with a helicity modulus but no ferromagnetic long-range order.

Another use of perturbative results has been made by Aharony and Pytte[34] who argue that random-anisotropy magnets between two and four dimensions exhibit an infinite susceptibility phase (though no long-range order) at low temperatures. They calculate the equation of state perturbatively to first order in the disorder. However, nonuniformities of the expansion in the small applied field and magnetization limit cast serious doubts on their results (this potential problem is alluded to by the authors) and there appears to be no compelling argument against mere crossover behavior from weak to strong randomness.

Villain and Fernandez[35] (VF) have recently performed an approximate position-space renormalization-group calculation on the X-Y model in a random field (or, equivalently, random anisotropy) with vortices excluded, i.e., exactly the model analyzed in Secs. II and III. They find, for $2<d<4$, a stable zero-temperature fixed point at a finite value of the randomness which corresponds to a low-temperature phase which is quasiferromagnetic, i.e., with *power-law* decay of spin-spin correlation functions.

This result is certainly intriguing; however, in one limit it appears to disagree with the results of this paper: As d approaches four from below, VF's fixed point[35] approaches the ferromagnetic fixed point *linearly* in $4-d$. This is in contrast to the absence of a perturbative fixed point of the renormalization group of Sec. III. It is not clear at this stage whether a modification of VF's results could reconcile this discrepancy.

In two dimensions there are rather compelling arguments[21] that a quasiferromagnetic phase exists for the $\cos(m\theta)$ anisotropy X-Y case with $m > \sqrt{8}$. However, this phase, which is the Kosterlitz-Thouless[24] quasiordered spin-wave phase, exists for a range of temperatures away from zero. This is due to the irrelevance in this range of temperatures of the random $\cos(m\theta)$ anisotropy. Cardy and Ostlund[21] argue that, at least for $m \gtrsim \sqrt{8}$, both above *and* below this quasiferromagnetic phase, the system is simply paramagnetic. If this is correct, then one might speculate that (as for X-Y magnets with *uniform* sixfold anisotropy) the power-law phase disappears in any dimension more than two, leaving only paramagnetism for dimensions between two and four. This is, perhaps, in contrast to the intuition that order tends to increase with dimension; however, the level of experience with random systems is rather low. For $\cos(m\theta)$ anisotropy, with m much larger than $\sqrt{8}$, the Cardy-Ostlund calculations are no longer compelling and it is possible that when the anisotropy becomes relevant at low temperatures the flow is *not* to simple paramagnetic behavior—this possibility will be discussed further below.

A possibility which has been discussed quite extensively in the literature is spin-glass behavior for n-component magnets with random anisotropy. Pelcovits *et al.*[19] have suggested that in the presence of strong second-rank anisotropy in more than four dimensions, the nonferromagnetic phase at low temperatures is a spin glass with long-range Edwards-Anderson order.[36] In less than four dimensions this spin-glass phase, which has only short-range ferromagnetic correlations, is argued to persist all the way down to zero anisotropy. Schematic phase diagrams are shown in Fig. 5. Most of the arguments in favor of a spin-glass phase rely on results for the spherical model limit of an infinite number of components of the order parameter.[19,37] Recent work by Sompolinsky and this author[38] demonstrates, however, that it is likely that the spin-glass phase discussed in the literature is entirely an artifact of the spherical model limit, and does not exist at large but finite n.

In more than four dimensions various arguments[38] suggest that for *all* strengths of random second-rank anisotropy the system will be a ferromagnet at low temperature, resulting in the simple phase diagram shown in Fig. 6(a). This result, which would have been in conflict with the putative $4+\epsilon$ zero-temperature fixed point found for this case by Pelcovits,[14] is clearly not ruled out by the *absence* of such a fixed point demonstrated in Appendix C. The apparent occurrence of renormalization-group flows into the nonperturbative regime suggests, however, that if a ferromagnet exists for all strengths of the random twofold anisotropy in $d > 4$, it may still have nontrivial correlations at long distances. This question clearly merits further study, although the failure of perturbative methods may make the problem rather difficult.

In less than four dimensions it is possible that this low-temperature ferromagnetic phase becomes not a paramagnet but some kind of spin-glass-like phase—perhaps with algebraic decay of ferromagnetic order. We first consider the X-Y case. Weak disorder is relevant in the spin-wave system excluding vortices; however, if Villain and Fernandez[35] are correct, the flows may be to a nontrivial zero-temperature fixed point. We note that this fixed point has a nonzero Edwards-Anderson[36] spin-glass order parameter, $q_{EA} = [\langle S \rangle^2]_{av}$, in terms of the spin variables. If there is a random field, q_{EA} is always nonzero; however,

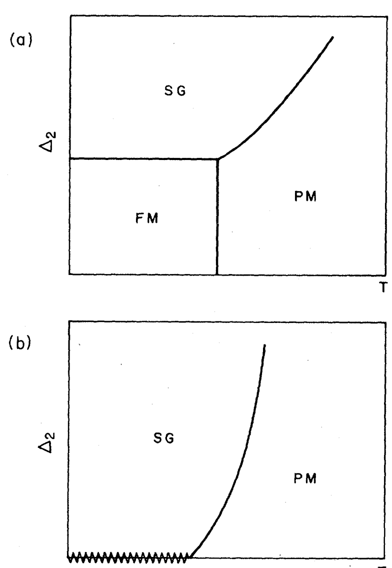


FIG. 5. Candidate phase diagrams in (a) $d > 4$ and (b) $d < 4$ for magnets with random second-rank anisotropy with strength Δ_2. Ferromagnetic (FM), spin-glass (SG), and paramagnetic phases are shown. In (b) ferromagnetism only exists on the jagged line at $\Delta_2 = 0$.

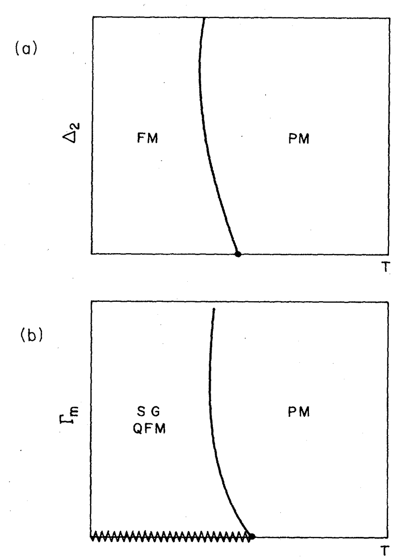


FIG. 6. Candidate phase diagrams for (a) random-anisotropy magnet in $d > 4$ and (b) an X-Y magnet with random m-fold anisotropy Γ_m in $2 < d < 4$. Paramagnetic (PM), ferromagnetic (FM), and spin-glass—quasiferromagnetic (SG QFM) phases are shown. In (b) ferromagnetism only exists on the jagged line at $\Gamma_m = 0$.

in the absence of a random field, a nonzero $q_{\rm EA}$ at low temperatures suggests a phase transition, and would definitely imply one in the *actual* X-Y model.

It is possible that for $\cos(m\theta)$ anisotropy with m sufficiently large, vortices are irrelevant at low temperatures in $d \lesssim 4$, so that the spin-glass phase could persist in the actual X-Y model with vortices. This could even occur in two dimensions for $m \gg \sqrt{8}$, although it is certainly more likely that vortex hyperplanes are irrelevant for $d \lesssim 4$ than that point vortices are irrelevant in $d=2$.

If this kind of spin-glass phase in which there are no free vortices on long length scales exists in, say, three dimensions, it is possible that it would have power-law falloff of ferromagnetic correlations as in the calculations of Aharony and Pytte[34] and Villain and Fernandez.[35] These correlations, which are not present in a conventional spin glass, would be observable by neutron scattering. In addition, due to the absence of vortices, the helicity modulus of such a phase would probably be nonzero. The occurrence of power-law decay of ferromagnetic correlations and a nonzero helicity modulus is often called quasiferromagnetism. A possible phase diagram which exhibits a spin-glass—quasiferromagnetic phase for a random-anisotropy X-Y model in $d<4$ is shown in Fig. 6(b).

The question of whether or not ferromagnetism, especially as manifested in the helicity modulus, can be destroyed without defects (the vortices in the X-Y case) has been discussed for pure systems by Halperin.[39] Defects are likely to be necessary to destroy the order completely only for small n; in particular, $n<d$.[39] Thus while a phase somewhat analogous to that suggested above for the X-Y case could conceivably exist for the $n=3$ random second-rank anisotropy Heisenberg model in three dimensions, such phases are much more unlikely for large n. Note that only for the X-Y case can large μ be an advantage as far as the possibility of suppressing defects: for $n \geq 3$ the non-Abelian character of the rotation group implies that any even-rank anisotropy will generate second-rank anisotropy under renormalization and any odd-rank anisotropy will generate a random field. This effect makes a spin-glass phase for $n \geq 3$ even less likely.

At this stage, it appears that, with the exception of intermediate temperatures in the two-dimensional X-Y model, the existence or lack thereof of quasiferromagnetism or spin-glass behavior in the presence of random anisotropies must be left as an open question. It may be possible to glean some additional information about the various possibilities by use of a nonperturbative approximate functional renormalization group (like that originally developed by Wilson[40] for pure systems) which reduces to the perturbative functional renormalization group introduced here for weak-disorder and low temperatures.

Note added. Since an earlier version of this paper was submitted, Imbrie[41] has proved rigorously that for weak disorder in more than two dimensions the random-field Ising model has a spontaneous magnetization at zero temperature. For the Ising case, dimensional reduction must therefore break down in low dimensions. Modified forms of dimensional reduction which satisfy the condition that the lower critical dimension is 2 for the Ising case have been recently proposed by two authors.[42,43] A recently studied example of a problem which is more easily describable in terms of a functional renormalization group than in terms of a few parameters is critical wetting in three dimensions.[44]

Note added in proof. Very recent work by this author on a related problem suggests that sense might be made of nonanalytic fixed points of Eq. (5.16). This will be discussed in a future paper.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix we prove that there is no nontrivial renormalization of the temperature in the X-Y case for δ-function-correlated disorder. As in Sec. III it is simple to work in angle variables. We must consider all graphs with only one replica index and two external lines which could renormalize the temperature. All such graphs fall into one of two categories: either (i) both of the external lines are attached to one pair of vertices connected by a dotted line, or (ii) there is a pair of vertices connected by a dotted line with exactly one external line attached to it.

Graphs in category (i) *cannot* depend on the external momentum and hence cannot contribute to the temperature renormalization. Graphs in category (ii), on the other hand, *can* have momentum dependence. However, they can each be paired with another graph which differs only by one of the external lines being moved to the other end of the dotted line to which it is attached. By the invariance of the replicated Hamiltonian under a uniform angular shift of *all* the replicas (corresponding to the rotational invariance of the distribution of the disorder), each of these pairs of graphs exactly cancel, establishing the desired result.

APPENDIX B

In this appendix we show how the need for considering generation of higher-rank anisotropies arises from a conventional graphical expansion for the nonlinear σ model in a random magnetic field with a replicated effective Hamiltonian given by

$$\overline{\mathscr{H}}=\int \frac{1}{2T}\sum_{\alpha}(\nabla S_\alpha\cdot\nabla S_\alpha)-\frac{1}{2T^2}\sum_{\alpha,\beta}R(S_\alpha\cdot S_\beta)\ . \qquad (5.1)$$

For the random-field case the bare R is given by $R_0(\chi)=\chi\Delta_1$. We write $S_\alpha=(\pi_\alpha,\sigma_\alpha)$ and $\sigma_\alpha=(1-\pi_\alpha^2)^{1/2}$, and then expand in the π_α's. The effective Hamiltonian then becomes

$$\overline{\mathscr{H}}=\int \frac{1}{2T}\sum_{\alpha}[\nabla\pi_\alpha\cdot\nabla\pi_\alpha+(\pi_\alpha\cdot\nabla\pi_\alpha)^2+\pi_\alpha^2(\pi_\alpha\cdot\nabla\pi_\alpha)^2]$$

$$-\frac{\Delta_1}{2T^2}\sum_{\alpha,\beta}(\pi_\alpha\cdot\pi_\beta+\tfrac{1}{4}\pi_\alpha^2\pi_\beta^2+\tfrac{1}{8}\pi_\alpha^2\pi_\beta^4)+O(\pi^8)\ , \qquad (\mathrm{B1})$$

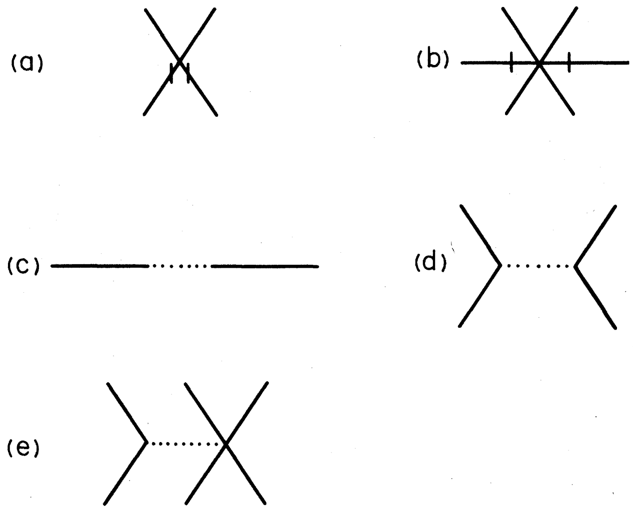

FIG. 7. Vertices for the nonlinear σ model in a random magnetic field but without higher-rank anisotropies. The slashes denote gradients. (a) and (b) carry factors $1/T$, while (c)—(e) carry factors Δ_1/T^2.

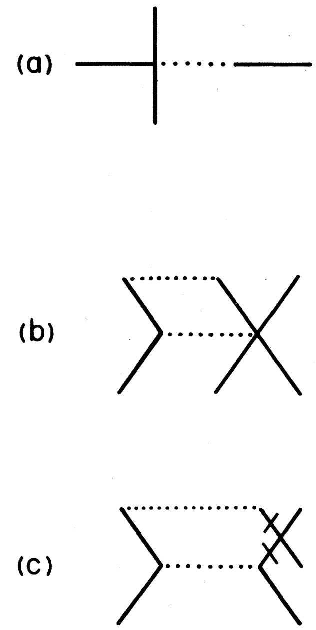

FIG. 8. (a) Vertex which does not occur with just a random field, but is generated under renormalization by the graphs shown in (b) and (c).

where we have dropped terms which vanish when the number of replicas p goes to zero. As noted in the text, to the desired order the Jacobian can be ignored. The small T and Δ_1 expansion is most conveniently generated by considering all the terms but the first to be interactions.

We denote the propagator $T\delta_{\alpha\beta}/q^2$ by a solid line which conserves replica index, gradients by slashes, and non-replica-conserving vertices by dotted lines which carry factors Δ_1/T^2. The vertices in Eq. (B1) are shown in Fig. 7. Graphs which contain closed loops will vanish under $p \to 0$, and those with more than two parts connected only by dotted lines will generate irrelevant three or more replica terms corresponding to correlations in the randomness.

If the form of the Hamiltonian with just a random field were to be preserved under renormalization, it is clear that no two-, four-, or six-point vertices other than those in Fig. 7 could be generated and, furthermore, that the coefficients of the renormalized vertices would have to be simply related.

It is straightforward to see that two-replica vertices which are *not* in Fig. 7 are generated with magnitudes of order Δ_1^2/T^2, i.e., the same magnitude as the desired $O(\Delta_1^2)$ renormalization of Δ_1. In Figs. 8(b) and 8(c) the two diagrams which generate, at this order, the vertex shown in Fig. 8(a) are shown; it can be verified straightforwardly that the combinatoric factors do not cancel. This vertex corresponds to a term of the form $\pi_\alpha \cdot \pi_\beta \pi_\alpha^2$ which does *not* occur in the expansion of the effective Hamiltonian with only a random field. It *will* occur, however, in the expansion of any higher-rank anisotropy; for example, the expansion of $(S_\alpha \cdot S_\beta)^2$ contains such a term. In addition, it can be seen by examining the renormalization of the four-point vertices, Figs. 7(a) and 7(d), the two-point vertex Fig. 7(c), and the propagator, that the renormalized coefficients are *inconsistent* with having been obtained from the expansion of a renormalized $S_{R\alpha} \cdot S_{R\beta}$ and $(\nabla S_{R\alpha})^2$.

Both of these problems should lead one to conclude that other operators of the same dimension are being generated. These are just the higher-rank anisotropies. However, it is not clear, without looking at the renormalization of graphs with arbitrary numbers of external legs, which combination of higher-rank anisotropies is being generated since all of these can give rise to similar terms. The functional renormalization group discussed in the text is equivalent to examining *all* the one-loop graphs of the desired order with one or two replica indices.

APPENDIX C

In this appendix we prove that the truncated functional recursion relation for the general n case (5.16) has no singly unstable fixed point in $d = 4+\epsilon$ with the function $R(\chi)$ having a Taylor-series expansion about $\chi = 1$. It is convenient to rescale R to eliminate ϵ and C_4 and change variables to $y = \chi - 1$. We thus define

$$P(y) \equiv \frac{C_4 R(1+y)}{\epsilon} , \tag{C1}$$

and rescale l by

$$t = \epsilon l . \tag{C2}$$

The recursion relation (5.15) then becomes, after rearranging terms for later convenience,

$$\frac{\partial P(y)}{\partial t}=-P(y)+2(n-2)P(y)P'(0)-\tfrac{1}{2}(n-1)[P'(0)]^2+\tfrac{1}{2}(n-1)[P'(y)-P'(0)]^2-(n-1)yP'(y)P'(0)$$
$$+\tfrac{1}{2}[P'(y)]^2(2y+y^2)+P''(y)[P'(y)-P'(0)]2y-P'(0)P''(y)y^2+\tfrac{1}{2}[P''(y)]^2(4y^2+4y^3+y^4)$$
$$+P'(y)P''(y)(3y^2+y^3)\ . \qquad \text{(C3)}$$

We now expand about $y=0$ ($\mathcal{X}=1$):

$$P(y)=\sum_k P_k y^k\ . \qquad \text{(C4)}$$

From the grouping of terms in Eq. (C3), it can be seen that, with the exception of the unimportant constant P_0, the recursion relation for each P_k *involves only* P_l for $l\leq k$. The flow and fixed-point equations can thus be solved by iteration.

We have

$$\frac{\partial P_1}{\partial t}=-P_1+(n-2)P_1^2\ ,$$
$$\frac{\partial P_2}{\partial t}=-P_2+6P_1P_2+(2n+14)P_2^2+\tfrac{1}{2}P_1^2\ , \qquad \text{(C5)}$$

and

$$\frac{\partial P_k}{\partial t}=-P_k+A_kP_1P_k+\sum_{i,j=1}^{k-1}B_{ijk}P_jP_i \quad \text{for } k\geq 3\ , \qquad \text{(C6)}$$

where

$$A_k=2k^2-k(n-1)+2n-4 \qquad \text{(C7)}$$

and all

$$B_{ijk}\geq 0\ . \qquad \text{(C8)}$$

We note that the initial condition is that all the derivatives of R at $\mathcal{X}=0$ are positive; this implies that the derivatives of $P(y)$ at $y=0$ are also positive. This positivity of the P_k is preserved by the flow because of condition (C8).

We now assume that there is a nontrivial fixed point P^* which must have

$$P_1^*=\frac{1}{n-2} \qquad \text{(C9)}$$

(P_1 is just proportional to $\widetilde{\Delta}$). The equation for P_2 has no fixed point unless $n>18$ and then has two fixed points, both positive. We have

$$P_2^{\pm *}=\frac{(n-8)\pm\sqrt{(n-2)(n-18)}}{(4n+28)(n-2)}>0\ . \qquad \text{(C10)}$$

The other P_k^* are simply obtained iteratively. Because of the form of the recursion relations, the eigenvalues of the linearized operator about the putative fixed point are also determined recursively. The fixed points with $P_2^{\pm *}$ given by Eq. (C10) have

$$\lambda_1=+1\ ,$$
$$\lambda_2^{\pm}=\pm\left[\frac{n-18}{n-2}\right]^{1/2}\ , \qquad \text{(C11)}$$

and

$$\lambda_k=\frac{A_k}{n-2}-1 \quad \text{for } k\geq 3\ .$$

For large k, A_k is greater than $n-2$ and hence *both* the fixed points will have $\lambda_k>0$ for large k and therefore be more than singly unstable. Note also that for high k, P_k^* will be *negative* and, hence, in the inaccessible unphysical regime. As long as they are initially positive, these P_k will strictly increase.

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multiple of m. Thus there are an infinite class of random $\cos(m\theta)$ anisotropy (m is often denoted p; see, e.g., Ref. 21) models with different symmetry. This is not true for general n; the only symmetry restriction on the generation of other Δ_μ's by one of them is parity; i.e., even μth-rank anisotropy cannot generate odd μth-rank anisotropy. For the X-Y case Δ_μ can be written as a combination of all the Γ_m with $\mu-m$ even and non-negative as in Eq. (2.4). Note that we have called the general anisotropies μth rank to avoid confusion with m-fold anisotropies for the X-Y case which are just the Γ_m. Twofold anisotropy for X-Y spins is (up to an important constant) the same as second-rank anisotropy and we will use the terms interchangeably for this case.

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