Renormalization-group study of the fixed points and of their stability for phase transitions with four-component order parameters

Jean-Claude Toledano
Centre National d'Etudes des Télécommunications, 92220-Bagneux, France

Louis Michel
Institut des Hautes Etudes Scientifiques, 91440-Bures sur Yvette, France

Pierre Toledano
Groupe de Physique Théorique, Université de Picardie, 80000-Amiens, France

Edouard Brezin
Département de Physique Théorique, Centre d'Etudes Nucléaires Saclay, 91190- Gif sur Yvette, France

(Received 23 August 1984)

The renormalization-group recursion relations are solved for the effective Hamiltonians relative to phase transitions with four-component order parameters. For this value of n there are 22 types of Hamiltonians which can be classified into two categories according to the action of their normalizer G_N on the corresponding parameter space (G_N is the symmetry group leaving globally invariant this space). In the first case, G_N generates a finite number of isolated fixed points whose characteristics can be deduced from the detailed investigation of five Hamiltonians only. In the second category, for which G_N is a continuous group, there are, in addition to isolated fixed points, continuous manifolds of physically equivalent fixed points (the dimension of the manifolds is either one or three). In the search for a stable fixed point, the continuous manifolds can be ignored, while the isolated points are related to the five former Hamiltonians. For $n=4$, it is necessary to solve the recursion relations to two-loop order. The only possible stable ones among the fixed points then arise from a splitting of points which coincide, to one-loop order, with the isotropic fixed point. Extending, to two-loop order, a result recently established to the preceding order, we show that if a stable fixed point exists, it is unique. For $n=4$, the stable fixed point has one of three possible symmetries : diicosahedral, hypercubic, or dicylindrical. Despite this anisotropy of the critical fluctuations, the exponents associated with any of the stable fixed points are identical to order ϵ^2 to the "isotropic" exponents corresponding to $n=4$. The cubic point is destabilized by any operator of symmetry lower than cubic. The dicylindrical one remains stable with respect to certain anisotropies of lower symmetry. We examine the available experimental data in light of the preceding theoretical results concerning the critical behavior and the thermodynamic order of the transitions. On the other hand, we establish two general symmetry conditions relative to the stable fixed points determined by the renormalization-group equations. The first one specifies group theoretically, for each value of n, the possible symmetries G_i^* of the stable fixed points. The second one formulates a necessary condition for the occurrence of an anisotropic stable fixed point: The normalizer G_N of the considered parameter space must fulfill the condition $G_N \subseteq G_i^*$ for one, at least, of the former G_i^* groups. These rules are shown to be very restrictive for $n=4$: On the basis of symmetry the lack of a stable fixed point can be asserted for 10 Hamiltonians out of 22, without solving the fixed-point equations.

I. INTRODUCTION

In the framework of the renormalization-group (RG) method in reciprocal space,[1] the critical behavior at a continuous phase transition can be investigated by means of an effective Hamiltonian density restricted to the n components of the transition's order parameter. This density contains a fourth-degree polynomial expansion (FDPE) of the order-parameter components, which is identical to the FDPE involved in the Landau free energy of the transition.[2]

With respect to the linear orthogonal transformations acting in the space carried by the n components of the order parameter, the FDPE is characterized by a symmetry group, a subgroup of O(n). This symmetry is usually higher than that of the order parameter itself.[3] As a consequence, for a given value of n, the same FDPE can be common to a wide variety of transitions. Thus, for $n=3$ only two types of FDPE are of interest. They possess respectively the isotropic O(n) symmetry and the cubic symmetry.[4–6] Likewise, three FDPE exist for $n=2$.[4–6] For these values of n, even a further increase of the symmetry is reached in the critical behavior, as the critical fluctuations erase the anisotropies and generate dynamically the O(n) symmetry at the critical point.[7]

For $n>3$ the situation is more complex. First, for each value of n, there are many distinct forms of FDPE's, and these have only been partly enumerated.[8–10] On the other hand, among the already identified FDPE's, only a few have been examined from the point of view of the RG method[11–14] (generally the ones corresponding to the highest symmetries). Finally, in the cases examined up to now, the results show that, in contrast to the case $n \leq 3$, one never has complete erasure of the anisotropies by the fluctuations. Depending on the form of the FDPE two possible situations have been pointed out:[11,13,15] (i) A continuous transition is possible and its critical behavior will reflect the preservation of some anisotropy (perhaps less asymmetric than that of the considered FDPE), characterized by a subgroup of $O(n)$. (ii) The effect of the fluctuations can be interpreted as the production of a first-order transition (conjectured to result from the lack of a stable fixed point).

As noticed by several authors the occurrence of either of the two former situations depends entirely on the symmetry of the FDPE. In particular, it has been suggested[15] that one should be able to elaborate a simple symmetry criterion for inferring the production of a first-order character by the fluctuations. Such a criterion would complete the well-known Landau rule[2] relying on the presence of a third-degree term in the transition's free energy.

Along these lines, it had been thought that the relevant symmetry indicator could be the number of linearly independent fourth-degree invariants contained in the FDPE. More precisely, it was tentatively suggested[16] that a first-order transition would arise if the FDPE contained more than three independent terms.

Recent investigations by Michel[17] and by Grinstein and Mukamel[18] have infirmed this conjecture. These authors have displayed examples of FDPE's containing an arbitrary number of terms and nevertheless compatible with a continuous transition. Thus, if a symmetry criterion predictive of first-order transitions exists, it has to rely on another common feature of the FDPE's than their number of independent terms.

In the aim of clarifying the possible existence of such a criterion, it seemed useful to undertake a systematic investigation of Hamiltonian densities for $n \geq 4$ by the RG method.

In this paper we examine the case $n=4$. Its relevance derives from the following considerations.

(i) It is the only case with $n>3$ for which a complete enumeration of the possible forms of the FDPE has been performed up to now.

(ii) A variety of situations is expected to be encountered since 22 distinct FDPE's exist with as many as 11 independent terms.

(iii) From the point of view of the RG method, it corresponds to a borderline situation for which the working out of the recursion relations of the ϵ expansion to one-loop order is never conclusive in contrast with the situation for other values of n.

(iv) A number of magnetic, structural, and incommensurate transitions in real systems are described by $n=4$ order parameters.

In the following section, we recall the procedure for enumerating the Hamiltonian densities which arise in the study of transitions with four-component order parameters. Section III is devoted to the discussion of the RG recursion relations, their symmetry properties, and their special features for $n=4$. In particular, on the basis of symmetry considerations, we achieve a reduction of the number of Hamiltonian densities for which the fixed-point equations of the ϵ expansion must be effectively solved. The uniqueness of the stable fixed point (FP) is studied, and the procedure used for solving the FP equations is presented. Section IV contains the results relative to the FP and their stability. These results are discussed in Sec. V from the standpoint of symmetry, and the working out of a symmetry criterion for the occurrence of a stable FP is achieved. Its effectiveness is illustrated by the case $n=4$. Finally, the experimental data pertaining to this order-parameter dimension are analyzed.

II. ENUMERATION OF HAMILTONIAN DENSITIES FOR $n=4$

Written in a standard form,[7] the effective Hamiltonian density we consider is

$$H(\mathbf{x}) = -\frac{r}{2}\left[\sum_{i=1}^{n} \phi_i^2(\mathbf{x})\right] + \left[\sum_{i=1}^{n} (\nabla\phi_i)^2\right] + \frac{\mu^{4-d}}{4!} P_4 , \quad (1)$$

where P_4 is the FDPE, whose general expression is

$$P_4 = \sum_{i,j,k,l} g_{ijkl}\phi_i\phi_j\phi_k\phi_l . \quad (2)$$

The n functions $\phi_i(\mathbf{x})$ are the local values of the n components of the order parameter, r and the g_{ijkl}, the coefficients of this polynomial expansion, μ a dimensional coefficient, and $d=(4-\epsilon)$ the space dimension.

For a given phase transition, the form of P_4 is specified by two properties.

(i) P_4 is the most general homogeneous polynomial of fourth-degree, invariant by the symmetry group G' of the high-symmetry phase adjacent to the transition.

(ii) The order-parameter (OP) components ϕ_i span a nontrivial representation Γ of G' which is irreducible on the real numbers (physically irreducible[19]). This irreducibility warrants that there is a single quadratic invariant in (1), and it implies the trace condition formulated by Brézin *et al.*[7] The transformation properties of the ϕ_i, as well as the form of P_4, are entirely determined by the set of distinct matrices of Γ. Due to the unitarity and reality of Γ, this set is isomorphic to a subgroup G of the full-orthogonal group $O(n)$ acting in the n-dimensional space of the order-parameter components. As a consequence, to work out the form of P_4, we can replace G' by G which is a point group in the n-dimensional space (while G' is a space group in the three-dimensional space) and construct P_4 as the most general G-invariant homogeneous fourth-degree polynomial. Let us show that this property of P_4 allows a systematic enumeration of the possible forms of P_4 for each value of n. The method used is to consider the various "irreducible" subgroups of $O(n)$ (i.e., the subgroups whose vector representation is irreducible[10]), and select those which describe the full symmetries of the different FDPE. This procedure leads, as shown by the subsequent discussion, to distinguish three types of full symmetries (the centralizers, the little groups, and the normalizers) which are all of interest in the application of the RG method.

A. Centralizers, little groups, and normalizers of P_4 polynomials

The most general homogeneous fourth-degree polynomial of n variables ϕ_i is a linear combination of $\binom{n+3}{4}$ distinct monomials. These monomials can be considered as the basis of a vector space having $\binom{n+3}{4}$ dimensions. For instance, if $n=4$, this vector space, which contains all fourth-degree polynomials of 4 variables has 35 dimensions. In this vector space, the G-invariant polynomials generate a subspace of dimension p, which we can denote E. Thus, the most general G-invariant polynomial P_4 can be written as a linear combination, with arbitrary coefficients, of p linearly independent, G-invariant polynomials $O_\nu(\phi_i)$ constituting a basis of E.

$$P_4=\sum_{i,j,k,l} g_{ijkl}\phi_i\phi_j\phi_k\phi_l=\sum_{\nu=1}^{p} u_\nu O_\nu(\phi_i) \ . \tag{3}$$

It can happen that several irreducible groups $G\subset O(n)$ have in common the same space $E=\{O_\nu\}$ of invariant polynomials. One of these groups, which we denote G_c, will contain all the others. G_c is the largest subgroup of $O(n)$ leaving simultaneously invariant all the $O_\nu \in E$. This group is the centralizer[6,10] of the space E. It represents the symmetry common to the entire set of polynomials defined by Eq. (3), and having arbitrary coefficients u_ν.

A second type of invariance group of P_4 is of interest in the RG method. It is the full invariance group of a given polynomial ($\sum u_\nu O_\nu$) with a specified set of coefficients u_ν (i.e., a specified vector direction in E). This group is the little group G^u of the considered polynomial, and it satisfies $G^u \supseteq G_c$. Two situations can occur[6] for the E space.

(i) In the first one, all the polynomials in E have little groups of strictly higher symmetry than G_c. In other words, the centralizer of E is not the full invariance group of any specified vector of this space. It only represents the intersection of the various little groups. This situation implies[6] that there exists, among the little groups, a continuous set $G^u(\alpha)$, having the centralizer as a common subgroup, and conjugate to each other in $O(n)$ [i.e., there exists a set of transformations $S(\alpha)\in O(n)$, such that $G^u(\alpha)=S(\alpha)G_0^u S^{-1}(\alpha)$].

(ii) In the second situation[6] the centralizer coincides with the little groups of a dense set of polynomials in E (these are the "general directions" of E which possess the lowest symmetry). We can note that the little group G^u of a given polynomial can be regarded as the centralizer of a subspace of E (possibly reduced to the sole polynomial direction considered).

An illustration of situation (i) is provided for $n=2$ by the polynomials

$$P_4=u_1(\phi_1^2+\phi_2^2)^2+u_2\phi_1^2\phi_2^2+u_3\phi_1\phi_2(\phi_1^2-\phi_2^2) \ , \tag{4}$$

for which $G_c \equiv C_4$ [Schoenflies notation[19] for this subgroup of O(2)]. For any specified values of the u_ν, the little group of P_4 has the higher symmetry $G^u(\alpha)=C_{4v}(\alpha)$, where the additional mirror symmetry v has an orientation (α) depending on the values of the u_ν coefficients. This higher symmetry corresponds to the fact that within a rotation of the reference frame in the two-dimensional space (ϕ_1,ϕ_2), P_4 can be brought to the standard form of polynomials with C_{4v} symmetry (i.e., the cubic symmetry in two dimensions):

$$P_4=u_1'(\phi_1'^2+\phi_2'^2)^2+u_2'(\phi_1'^4+\phi_2'^4) \ . \tag{5}$$

A third group of interest is the normalizer G_N defined as the largest group which preserves E as a whole. It transforms any polynomial $P_4\in E$ into a polynomial expanded as a function of the same basis $\{O_\nu\}$ but with, generally, different coefficients u_ν. The centralizer G_c is an invariant subgroup of G_N (however, G_N does not generally contain the little groups G^u of all the polynomials in E). In case (i), where G_c is not a little group, the continuous set of transformations $S(\alpha)$ establishing the correspondence between the conjugated little groups $G^u(\alpha)$, belong to G_N. In this case, G_N is necessarily a continuous subgroup of $O(n)$.

All the preceding groups, which have been defined by their action on the polynomial space, can also be considered as acting similarly on the space $\{u_\nu\}$ of the coefficients in Eq. (3).

B. Enumeration of physically distinct polynomials

It is worth pointing out that polynomials P_4 which are transformed from each other by a mere change of reference frame in the space of the ϕ_i components are physically equivalent and should not be distinguished. In general their centralizers do not coincide but are only conjugate to each other in $O(n)$. Conversely, if the centralizers are not conjugate, the polynomials cannot be brought to coincide by a change of reference frame and they must be considered as representing distinct physical situations.

In summary, the enumeration of physically distinct FDPE's consists in finding, up to a conjugation in $O(n)$, all the irreducible centralizers G_c. As shown in Ref. 6 and 10 a systematic group-theoretical method exists for selecting the centralizers G_c of polynomials of a given degree, among the irreducible subgroups of $O(n)$, and determining the form of the invariant polynomials associated with each centralizer. This method will also provide the irreducible little groups since these are centralizers of smaller spaces E. The knowledge of the various P_4 polynomials will permit the determination, in each case, of the corresponding normalizer by examining systematically the action on P_4 of the subgroups of $O(n)$ containing G_c.

Such an enumeration of the conjugation classes of irreducible centralizers and little groups of fourth-degree polynomials has recently been performed[10] for $n=4$. As mentioned in the introduction, a similar enumeration had been achieved before[4–6] for $n\le 3$. No complete results are available yet for $n>4$.

The results of the investigation corresponding to $n=4$ are recalled on Fig. 1 and in Table I. The notations of the subgroups of O(4) follow the convention adopted by Du Val[20] based on the homomorphisms existing between SU(2)×SU(2) and SO(4) on one hand, and between SO(4) and SO(3)×SO(3) on the other hand. This convention is

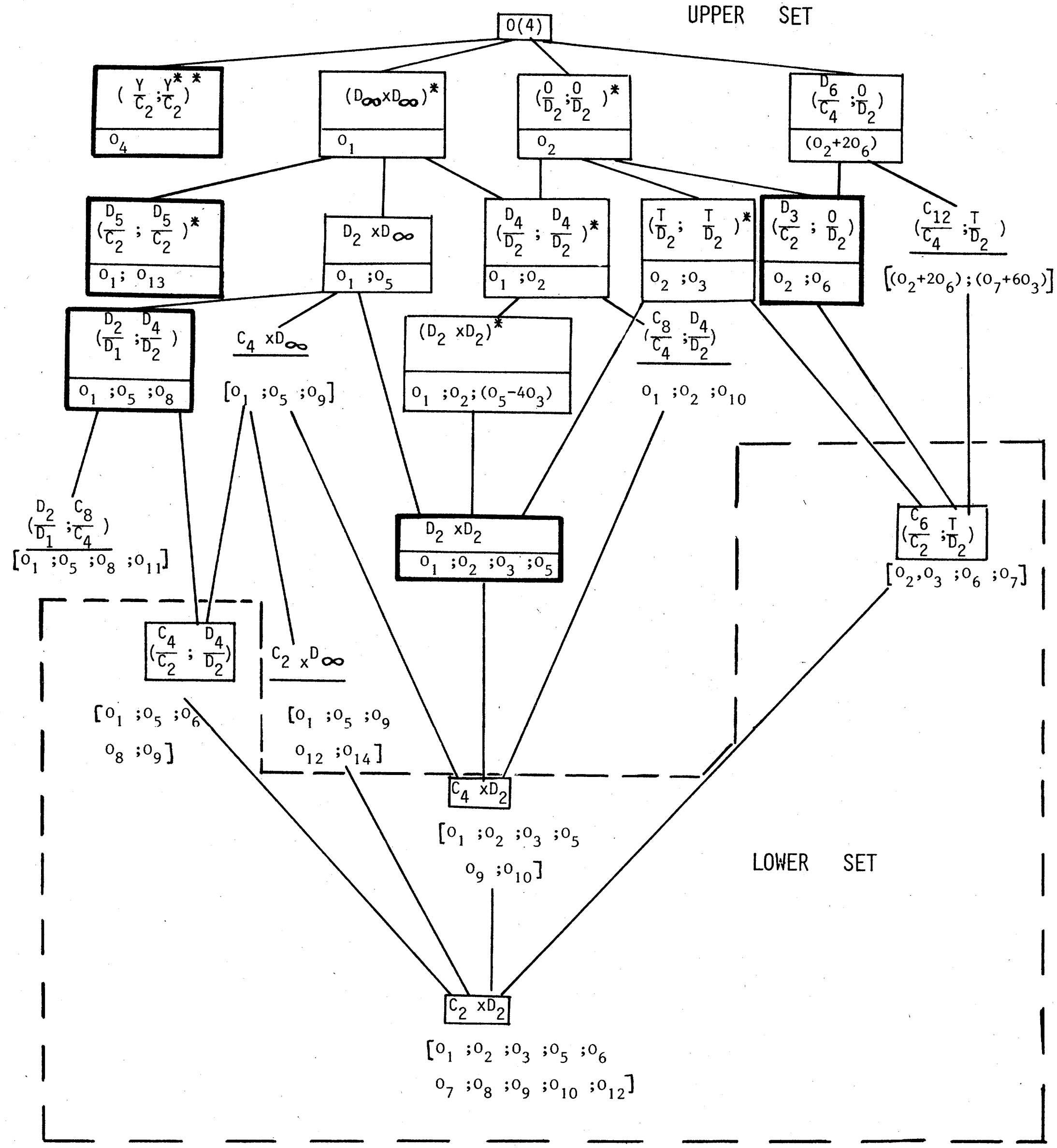

FIG. 1. Centralizers and little groups of fourth-degree polynomials of four variables, and group-subgroup relationship between them. Little groups are distinguished by a framed symbol. Among these, the five "minimal" groups discussed in the text have a thicker frame. Under each group are indicated the basic anisotropic invariants corresponding to it, whose expressions are supplied by Table I. The symbols identifying the groups follow the notation of Du Val (Ref. 20) (see Appendix A). An "upper set" and a "lower set" of little groups are distinguished according to the type of action of the corresponding normalizer (see text).

recalled in Appendix A.

In the following, besides this notation, we have used illustrative labels for the subgroups of O(4) emphasizing their connection with the better known three-dimensional point groups (e.g., "diorthorhombic" for $D_2 \times D_2$). Figure 1 shows that, up to a conjugation, there are 22 centralizers, and correspondingly, 22 possible forms of FDPE's. The P_4 polynomials are linear combinations of between 1 and 11 invariants $O_\nu(\phi_i)$. Among the centralizers, 17 are also little groups in the E space.[21] For the remaining ones, each polynomial of E possesses a little group which coincides with one of the centralizers of higher symmetry. Thus, for $C_4 \times D_\infty$, any polynomial of the form $(u_0 O_0 + u_1 O_1 + u_5 O_5 + u_9 O_9)$ is invariant by a group conjugate to $D_2 \times D_\infty$, while the particular polynomials O_0 and O_1 have even a higher symmetry, respectively O(4), and $(D_\infty \times D_\infty)^*$. The normalizers of the various E spaces were not considered before and will be discussed in Sec. IV.

III. RECURSION RELATIONS IN THE ϵ EXPANSION

The RG method associates with a given FDPE of the form indicated in Eq. (3) a flow of FDPE's having the same form and characterized by continuously varying u_ν values. Physically, all the FDPE's of a trajectory of the flow are associated with the same critical behavior. The

TABLE I. Basic polynomials O_ν appearing in the fourth-degree polynomial expansions P_4 which are invariant by irreducible subgroups of O(4) and represent all the possible fourth-degree contributions to the effective Hamiltonians involved in the renormalization-group method for $n=4$. $\phi_1,\phi_2,\phi_3,\phi_4$, are the components of the order parameter.

Label	Expression
O_0	$(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2)^2$
O_1	$(\phi_1^2+\phi_2^2)^2+(\phi_3^2+\phi_4^2)^2$
O_2	$(\phi_1^4+\phi_2^4+\phi_3^4+\phi_4^4)$
O_3	$\phi_1\phi_2\phi_3\phi_4$
O_4	$5(\phi_1^4+\phi_2^4+\phi_3^4)+\phi_4^4+\frac{60}{\sqrt{5}}\phi_1\phi_2\phi_3\phi_4+12\phi_4^2(\phi_1^2+\phi_2^2+\phi_3^2)$
O_5	$(\phi_1^2-\phi_2^2)(\phi_3^2-\phi_4^2)+4O_3$
O_6	$\phi_1\phi_2(\phi_3^2-\phi_4^2)-\phi_3\phi_4(\phi_1^2-\phi_2^2)+\phi_1\phi_3(\phi_2^2-\phi_4^2)+\phi_2\phi_4(\phi_1^2-\phi_3^2)+\phi_2\phi_3(\phi_1^2-\phi_4^2)-\phi_1\phi_4(\phi_2^2-\phi_3^2)$
O_7	$-\phi_1\phi_2(\phi_1^2-\phi_2^2)+\phi_3\phi_4(\phi_3^2-\phi_4^2)+\phi_1\phi_3(\phi_1^2-\phi_3^2)+\phi_2\phi_4(\phi_2^2-\phi_4^2)+\phi_1\phi_4(\phi_1^2-\phi_4^2)-\phi_2\phi_3(\phi_2^2-\phi_3^2)$
O_8	$\phi_1\phi_4[3(\phi_2^2-\phi_3^2)-(\phi_1^2-\phi_4^2)]-\phi_2\phi_3[3(\phi_1^2-\phi_4^2)-(\phi_2^2-\phi_3^2)]$
O_9	$\phi_1\phi_2(\phi_3^2-\phi_4^2)-\phi_3\phi_4(\phi_1^2-\phi_2^2)$
O_{10}	$\phi_1\phi_2(\phi_1^2-\phi_2^2)-\phi_3\phi_4(\phi_3^2-\phi_4^2)$
O_{11}	$\phi_1\phi_3(3\phi_2^2+3\phi_4^2-\phi_1^2-\phi_3^2)+\phi_2\phi_4(3\phi_1^2+3\phi_3^2-\phi_2^2-\phi_4^2)$
O_{12}	$(\phi_1\phi_3+\phi_2\phi_4)(\phi_1^2+\phi_2^2-\phi_3^2-\phi_4^2)$
O_{13}	$\phi_1\phi_4(3\phi_2^2+3\phi_3^2-\phi_1^2-\phi_4^2)-\phi_2\phi_3(3\phi_1^2-3\phi_4^2+\phi_3^2-\phi_2^2)$
O_{14}	$O_8+2(O_6+O_7+O_{10}-O_9-O_{12})$
X_1	$\phi_1^2\phi_2^2+\phi_3^2\phi_4^2$
X_2	$\phi_1^2\phi_3^2+\phi_2^2\phi_4^2$
X_3	$\phi_1^2\phi_4^2+\phi_2^2\phi_3^2$

RG transformation determines the flow of the u_ν coefficients by means of recursion relations whose general form is expressed through a transformation equation[7] of the g_{ijkl} coefficients:

$$\frac{dg_{ijkl}}{d(\ln\lambda)}=\beta_{ijkl}(g_{pqrs}) \tag{6}$$

where $\lambda\to 0$ corresponds to the critical singularities.

In the ϵ expansion, the functions β_{ijkl} are double expansions as functions of the small parameters $\epsilon=(4-d)$, and g_{ijkl}. In a zero-momentum renormalization scheme, and to two-loop order $(\epsilon g^2, g^3)$ the form of these functions are:[7]

$$\beta_{ijkl}=-\epsilon g_{ijkl}+(\tfrac{1}{2})\left[1+\frac{\epsilon}{2}\right]\left[\sum_{p,q}(g_{ijpq}g_{pqkl}+g_{ikpq}g_{pqjl}+g_{ilpq}g_{jkpq})\right]$$
$$-(\tfrac{1}{4})\left[\sum_{p,q,r,s}(g_{ipqr}g_{jprs}g_{klqs}+g_{ipqr}g_{kprs}g_{jlqs}+g_{ipqr}g_{lprs}g_{jkqs}+g_{jpqr}g_{kprs}g_{ilqs}+g_{jpqr}g_{lprs}g_{ikqs}+g_{kpqr}g_{lprs}g_{ijqs})\right]$$
$$+(\tfrac{1}{48})\left[\sum_{p,q,r,s}(g_{sjkl}g_{spqr}g_{ipqr}+g_{sikl}g_{spqr}g_{jpqr}+g_{sijl}g_{spqr}g_{kpqr}+g_{spqr}g_{sijk}g_{lpqr})\right]. \tag{7}$$

If we take into account the total symmetry of the g_{ijkl} coefficients under a permutation of the indices,[7] Eq. (3) provides linear relationships between the g_{ijkl} and the u_ν coefficients thus allowing us to draw from Eqs. (6) and (7) a set of p recursion relations determining the flow of the u_ν coefficients:

$$\frac{du_\nu}{d(\ln\lambda)}=\beta_\nu(u_{\nu'}) . \tag{8}$$

The critical behavior at a continuous transition associated with a given G-invariant polynomial P_4, is determined by the characteristics of the stable FP of the considered flow (8). A FP of the flow, denoted u_ν^*, is defined by

$$\beta_\nu(u_{\nu'}^*)=0 . \tag{9}$$

In the parameter space carried by the u_ν coefficients, the stability of a FP will correspond to the fact that the trajectories defined by Eq. (8) and terminating at u_ν^* must flow towards the FP when $\lambda\to 0$. If α_ν $(\nu=1,2,\ldots,p)$ are the eigenvalues of the matrix,

$$\left.\left\{\frac{\partial\beta_\nu}{\partial u_{\nu'}}\right\}\right|_{u_\nu^*} , \tag{10}$$

the stability condition is expressed by the positiveness of the real part of the α_ν for all ν.

It has been shown[22,23] that fixed points with real coordinates can always be classified as stable or unstable (i.e., no more complex pattern of the flow will occur, such as limit-cycle behavior) due to the fact that the β functions Eq. (7), are a gradient field, and that consequently, the preceding eigenvalues are real.

A. Symmetry properties of the recursion relations

The RG transformations defined by Eqs. (6) and (8) are covariant by the operations of O(n). This mathematical property relies on the physical requirement that these transformations should not decrease the symmetry of the system considered. More precisely the little group of the FDPE's will be preserved as a minimal symmetry along a trajectory. These invariance properties have been expressed in various forms by Wegner,[24] Zia and Wallace,[5] Korzhenevskii[25] and more recently by Jaric[26] and by Michel.[17] Actually G^u will be strictly preserved along a trajectory except possibly at a FP where the symmetry can increase.[26] Indeed the preservation of G^u as a minimal symmetry warrants that if a higher symmetry than G^u is realized at a nonfixed point, then, starting the trajectory at this point we can see that the higher symmetry will be a minimal symmetry along the trajectory, downstream, but also upstream by changing $\lambda \to 1/\lambda$ in Eq. (8).

Thus, each of the three symmetry groups defined in Sec. II has a specified action on the flow of the u_ν coefficients. The centralizer G_c leaves invariant each point of the flow, while the little group G^u is the invariance group associated with a trajectory. The normalizer G_N, being the covariance group of P_4 leaves the flow globally invariant and establishes a correspondence between physically equivalent trajectories.[25,26] Two consequences can be inferred from the action of G^u.

(i) Let us consider the case for which the centralizer G_c of the $\{u_\nu\}$ space is not a little group. According to the above property, the flow in the $\{u_\nu\}$ space will consist of trajectories all confined to smaller dimensional subspaces invariant by the various little groups G^u. Moreover, owing to the existence of a continuous set of conjugate little groups $G^u(\alpha)$ containing the centralizer, each of the preceding subspaces will contain the complete characteristics of the flow. The other subspaces corresponding to conjugated little groups represent identical physical situations, and display a pattern of fixed points deduced by a mere change of reference frame. As we can generate the subspaces from one of them by applying the continuous set of transformations $S(\alpha)$ belonging to the normalizer G_N of the space $\{u_\nu\}$, the flow will generally contain lines, surfaces, or more complex manifolds of FP's, each such manifold being constituted by physically equivalent FP's. Hence, it is not useful to perform the RG calculations for the centralizers which are not little groups. Their pattern of fixed points is physically equivalent to the pattern of FP's relative to any of the little groups $G^u(\alpha)$. Five FDPE's represented on Fig. 1 can be discarded on this basis, such as, for instance $C_4 \times D_\infty$. A similar remark was previously made by Zia and Wallace[5] and by Jaric[27] for the example with $n=2$ in Eq. (4). These authors, relying on the work of Wegner[24] have stressed that the additional u_3 coefficient introduced by consideration of Eq. (4) instead of Eq. (5) had no physical consequence.

(ii) Consider two FDPE denoted P_4 and P'_4, with respective little groups G^u and G'^u such that $G^u \subset G'^u$. In the p-dimensional parameter space $\{u_\nu\}$ associated with P_4, the set of P'_4 polynomials will generate a subspace of dimension $p' < p$. The preservation of G'^u by Eq. (8) means that the trajectories passing by a point of the subspace relative to P'_4 are entirely contained in this subspace. The FP associated with the set P'_4, will then also be the FP for the set P_4. Furthermore, the FP among these which are unstable in the p'-dimensional subspace will remain unstable in the p-dimensional space (however the stable ones will not necessarily remain stable).

Hence, the flow associated with a given G-invariant polynomial will possess, at least, all the FP's corresponding to the entire set of supergroups of G. In particular any flow will display the Gaussian FP ($u_\nu^*=0$) and the isotropic FP, which are the two FP's of the O(n)-invariant FDPE.[7] As there is only one isotropic FP for the latter FDPE, there will only be one FP with the O(n) symmetry in any flow.

Referring to Fig. 1 for the case $n=4$, we illustrate the former property by noting for instance that the FDPE having the invariance group labeled $D_2 \times D_2$ will possess, among others, the FP of the generalized cubic FDPE $(\mathrm{O}/D_2;\mathrm{O}/D_2)^*$ and those of the dicylindrical FDPE $(D_\infty \times D_\infty)^*$.

Conversely the investigation of the FP's and of their stability for an FDPE with invariance group G will also determine the characteristics of the pattern of FP's for all the cases corresponding to supergroups of G. The latter FP's are a fraction of the FP's found for G, and the eigenvalues of matrix (10), relative to the supergroup, are part of the eigenvalues calculated for G.

Thus we can concentrate on the little groups with minimal symmetries. Inspecting Fig. 1 we see that we can, in principle, restrict the calculations to two groups only, labeled $C_2 \times D_2$ and $(Y/C_2, Y^*/C_2)^*$, since the first of these groups is a subgroup of 20 centralizers. However, as will be shown later, the pattern of fixed points of $C_2 \times D_2$ is a very complex one, and the set of 11 recursion relations relative to it would be very difficult to handle. We have preferred to take advantage in a more limited way of the reduction of the number of cases allowed by the symmetry considerations. Referring to Fig. 1, we have distinguished two sets of little groups. The "upper" set is composed of 13 FDPE's. In this set, there are five "minimal" groups which contain the pattern of fixed points of the entire set. Their normalizers are finite subgroups of O(4). These groups are defined up to a conjugation in O(4). The five classes of conjugation are listed in the first column of Table II. The form of the corresponding P_4 polynomials is indicated in this table for a given representative of the class. One of the five cases requires consideration of two u_ν parameters. Two cases correspond to three parameters, and the two last cases correspond respectively to 4 and to 5 parameters. There is, on

TABLE II. Specification of the FDPE relative to the five "minimal" little groups which contain the results for $n=4$. Column 1 contains notation of the symmetry group referred to the convention of Ref. 10 and 20. Column 3 is the form of the FDPE expressed as a function of the invariants O_ν indicated on Table I. Column 4 is the relationship between the coefficients u_ν and the g_{ijkl} coefficients, as defined by Eq. (3). The g_{ijkl} not specified in the table (within a permutation of the indices) are equal to zero.

FDPE symmetry	Label	Form	Relation between coefficients
$\left[\frac{Y}{C_2};\frac{Y^*}{C_2}\right]^*$	di-icosahedral	(uO_0+vO_4)	$g_{iiii}(i\neq 4)=(u+5v);\ g_{4444}=(u+v)$ $g_{iijj}(i\neq 4, j\neq 4, i\neq j)=\frac{u}{3}$ $g_{ii44}(i\neq 4)=\left[\frac{u}{3}+2v\right];\ g_{1234}=\frac{\sqrt{5}}{2}v$
$\left[\frac{D_5}{C_2};\frac{D_5}{C_2}\right]^*$	dipentagonal	$(uO_0+vO_1+wO_{13})$	$g_{iiii}=(u+v);g_{1122}=g_{3344}=\frac{u+v}{3}$ $g_{1133}=g_{1144}=g_{2233}=g_{2244}=\frac{u}{3}$ $g_{2214}=g_{3314}=g_{4423}=g_{2223}=\frac{w}{4}$ $g_{1123}=g_{1114}=g_{3332}=g_{4441}=-\frac{w}{4}$
$\left[\frac{D_3}{C_2};\frac{O}{D_2}\right]$	trigonal-cubic	$(uO_0+vO_2+wO_6)$	$g_{iiii}=(u+v);\ g_{iijj}(i\neq j)=\frac{u}{3}$ $g_{1123}=g_{1124}=g_{2213}$ $=g_{2234}=g_{3312}=g_{3314}=\frac{w}{12}$ $g_{1134}=g_{2214}=g_{3324}$ $g_{4412}=g_{4413}=g_{4423}=-\frac{w}{12}$
$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right]$	orthotetragonal	$(uO_0+vO_1+wO_5+pO_8$	$g_{iiii}=(u+v);\ g_{1122}=g_{3344}=\frac{u+v}{3}$ $g_{1133}=g_{2244}=\frac{u}{3}+\frac{w}{6};\ g_{1234}=\frac{w}{6}$ $g_{1144}=g_{2233}=\frac{u}{3}-\frac{w}{6}$ $g_{2214}=g_{4423}=g_{2223}=g_{4441}=\frac{p}{4}$ $g_{3314}=g_{1123}=g_{1114}=g_{3332}=-\frac{p}{4}$
$D_2\times D_2$	diorthorhombic	$(tO_2+6u_1X_1+6u_2X_2$ $+6u_3X_3+24\mu O_3)$	$g_{iiii}=t;\ g_{1122}=g_{3344}=u_1$ $g_{1133}=g_{2244}=u_2;\ g_{1144}=g_{2233}=u_3$ $g_{1234}=\mu$

the other hand, a "lower set" of 4 little groups. Each is a subgroup of a centralizer which is not a little group. Their pattern of FP's is therefore expected to display continuous lines of FP's and to be difficult to determine directly. In the same way as the centralizers which contain them, these groups are associated with a normalizer which is a continuous subgroup of O(4). We show in Sec. IV that sufficient information can be drawn from the investigation of the five preceding cases to infer unambiguously for them the possible existence of a stable FP.

B. General characteristics of the flow at one-loop order

It has been shown recently[23] that the set of functions β_{ijkl} in Eq. (7) can be given a condensed form, to one-loop order, by means of introducing two mathematical operations defined on the set of the g_{ijkl} coefficients (or, equivalently, on the set of the polynomials P_4 they define). Thus let $g^{(1)}=\{g^{(1)}_{ijkl}\}$ and $g^{(2)}=\{g^{(2)}_{ijkl}\}$ be two such sets, associated with two polynomials $P_4^{(g_1)}$ and $P_4^{(g_2)}$ of the form specified by Eq. (3). The two considered operations, respectively denoted $(g^{(1)},g^{(2)})$ and $g^{(1)}\vee g^{(2)}$ are defined by

$$(g^{(1)},g^{(2)})=\sum_{i,j,k,l} g^{(1)}_{ijkl}g^{(2)}_{ijkl}\ , \tag{11}$$

and

$$g^{(1)}\vee g^{(2)}=g^{(3)}\ , \tag{12}$$

with

$$P_4^{(g_3)}=\frac{1}{144}\left[\sum_{i,k}\frac{\partial^2 P_4^{(g_1)}}{\partial\phi_i\partial\phi_k}\frac{\partial^2 P_4^{(g_2)}}{\partial\phi_i\partial\phi_k}\right]. \qquad (12')$$

$(g^{(1)},g^{(2)})$ has the properties of an O(n)-invariant scalar product in the polynomial space, and $g^{(1)}\vee g^{(2)}$ defines[6] a symmetric nonassociative algebra whose properties have been described before.[6,23]

By means of the symmetric product (12), the set of β_{ijkl} functions can be expressed, to one-loop order, by

$$\beta(g)=-\epsilon g+\tfrac{3}{2}g\vee g. \qquad (13)$$

It was shown in Ref. 23 that, owing to the form of this second-degree gradient field, some general characteristics of the flow of g coefficients were valid for any value of n:

(i) There is at most one stable FP, and all the other FP's are situated on the boundary of its attractor basin.

(ii) If it exists, the stable FP g^* is the fixed point having the largest "length" (g^*,g^*).

On the other hand, by using the β functions in their explicit form of Eq. (7), Brézin *et al.*[7] have shown that, to one-loop order,

(iii) The FP's with symmetry other than O(n) can be grouped by pairs. Each pair is located in a plane also containing the isotropic FP and the origin (Gaussian FP).

(iv) For $n\le 3$ the isotropic FP is the stable one, while for $n\ge 5$ this point is unstable for any system with lower symmetry than O(n).

The case $n=4$ of interest here represents a borderline situation for which some of the preceding statements hold in a special form. As established by Brézin *et al.*,[7]

(v) To the order $(\epsilon g,g^2)$ all the FP's are unstable except the isotropic one which has marginal stability. More precisely for this FP all the eigenvalues of the matrix (10) verify $\alpha_\nu=0$ except one which is strictly positive.

(vi) The FP's can still be grouped by pairs. However in this case one member of each pair will coincide with the isotropic FP to order ϵ. Thus the isotropic FP will be, to this order, a multiple solution of Eqs. (8) and (13).

Consideration (v) shows that for $n=4$, the use of the recursion relations at the usual lowest order are inconclusive since the stability of the isotropic FP is not determined. One has to solve Eq. (8) to the next order. As ϵ and g are considered vanishingly small, each anisotropic FP found unstable at order ϵ will remain unstable at higher order. Besides, as these FP's are nondegenerate solutions of Eq. (8) at order ϵ, each will give rise to a single FP with the same symmetry at order ϵ^2. For these FP's it is actually not useful to perform the calculations at two-loop order.

By contrast, for the isotropic FP, the latter order is essential to consider. This order is likely to lift the degeneracy of FP's encountered at order ϵ, and induce a splitting of the isotropic FP into several FP's with different symmetries and stabilities. Only one of these FP's will retain the isotropic symmetry.

C. Characteristics of the flow in the vicinity of the isotropic FP

Because we have to deal with a bifurcation of the solutions of Eqs. (9) occurring at the order $(\epsilon g^2,g^3)$, a linearization of these equations in the vicinity of the isotropic FP cannot be done in the first place, in spite of the assumed smallness of the corrections at this order. In order to solve this bifurcation, we have successively applied two different methods thus allowing a checking of the results.

First we have selected the various paths of the bifurcation on the basis of their symmetry. Indeed, for a given FP, the symmetry relationship between the coefficients u_ν^* does not depend on the order of approximation used. Accordingly we have combined in an exact way Eqs. (9) (i.e., without considering the order of magnitude of the successive terms) in order to isolate the solutions corresponding to the various relevant symmetries. This step leads to rigorous equalities or cancellations applicable to some of the u_ν^* coefficients. Taking then into account the former constraints [which are trivially obeyed by the O(4) symmetry] we linearize the Eq. (9) to determine the coefficients remaining, in the neighborhood of the isotropic FP.

In the second method, denoting by $g_s^*=\epsilon s/2$ the one-loop order coordinate of the isotropic FP, with $s=\{s_{ijkl}\}$ defined by[23]

$$s_{ijkl}=\tfrac{1}{3}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \qquad (14)$$

we have put $g=\epsilon(s/2+h)$ where $h=\{h_{ijkl}\}$ goes to zero with ϵ. The β functions can be expanded in the vicinity of the isotropic FP as a function of the small parameters h and ϵ up to the quadratic terms. Due to the limitation to this degree, the notations of Eqs. (11)–(13) can be used. One can write, to order $(\epsilon h)^2$,

$$\epsilon^{-3}\beta(h)=-\epsilon\frac{s}{48}+\frac{(s,h)s}{8}+\frac{3}{2}h\vee h-\frac{5\epsilon}{24}h -\frac{\epsilon}{12}(s,h)s+O((\epsilon H)^2). \qquad (15)$$

Depending on the order of h with respect to ϵ this equation yields different results. The cancellation of the lowest order terms leads either to $h^*\propto\epsilon$, or to $h^*\propto\epsilon^{1/2}$.

In the first case, we project Eq. (15) on the direction of s and on the perpendicular directions. The dominant term in each equation yields

$$(s,h^*)=\frac{\epsilon}{6}, \qquad (16)$$

$$\beta'(h)=-\epsilon h^*+\frac{36}{5}h^*\vee h^*-K(h^*)s=0, \qquad (16')$$

where $K(h^*)$ is a function of the length (h^*,h^*) of the fixed point.

Due to the linear dependence of a given FDPE, $(\sum u_\nu O_\nu)$, on the basic invariants O_ν, the use of Eqs. (16) and (16′) requires the sole knowledge of the scalar products $(O_\nu,O_{\nu'})$ and of the symmetric products $(O_\nu\vee O_{\nu'})$ involved in the considered FDPE.[3] These quantities are tabulated in Appendix B for the five Hamiltonians singled out in Sec. III A. The same quantities allow solution of the one-loop order Eqs. (13).

We can note that the $\beta'(h)$ functions in Eq. (16′) have a form similar to that of the one-loop $\beta(g)$ functions in Eq. (13). Actually, the investigation of the properties of $\beta'(h)$ by the same method previously[23] applied to $\beta(g)$ allows us, as shown in the Appendix, to establish that the flow of

h coefficients in the vicinity of the isotropic FP has common characteristics with the one-loop-order flow determined by $\beta(g)$:

(i) At order $(\epsilon g^2, g^3)$ if a stable FP exists in the vicinity of the isotropic FP, it is unique.

(ii) Among the fixed points of the form $g^* = \epsilon(s/2 + h^*)$, the stable one is characterized by the largest value of (h^*, h^*).

(iii) The isotropic FP, with coordinates $g^* = (\epsilon/2) \times [s + (\epsilon/24)]$ at order ϵ^2, is always unstable. The last property was also previously established in Ref. 7.

In Eq. (15) it remains to consider the case $h = h_1 \epsilon^{1/2} + \cdots$. This case is analyzed in Appendix C. It is shown that h_1 is a pure imaginary number and that, accordingly, the associated FP has complex coordinates. Such an FP must be considered as unphysical since no real Hamiltonian will flow to it through the RG transformations which are also real. Examples illustrating this situation are discussed in Sec. IV).

D. Critical exponents

Using the scaling-law relationships, the values of critical exponents can be determined from the knowledge of the two exponents η and ν respectively associated with the pair correlation function, and with the correlation length.[28]

In the framework of the ϵ expansion, the form of these two exponents has been indicated by Brézin *et al.*[7] up to the order ϵ^3 for η and ϵ^2 for ν (these two orders correspond to the working out of the fixed-point coordinates to order ϵ^2). From their formulas we draw

$$\eta = \frac{1}{24}\left[1 + \frac{3\epsilon}{4}\right]\left[\epsilon\left[1 - \frac{\epsilon}{2}\right]U^* - \frac{U^{*2}}{2}\right] \times \left[1 + \frac{2\epsilon}{3} + \frac{5U^*}{12}\right], \tag{17}$$

and

$$\left[\frac{1}{\nu} - 2\right] = -\frac{U^*}{2}\left[1 + \frac{\epsilon}{2}\right] + \frac{5}{24}\left[\epsilon U^* - \frac{U^{*2}}{2}\right], \tag{18}$$

where U^* is equal to

$$U^* \delta_{kl} = \sum_{i=1}^{n} g^*_{iikl}, \tag{19}$$

g^*_{ijkl} being the FP coefficients of the FDPE written in the form of Eq. (2). Thus it appears that the critical exponents do not depend on all the coordinates of a FP but only on their combination in the form of the trace U^*. Using the definition (14) of the tensor s, and that of the scalar product (11) it is straightforward to show that

$$U^* = \tfrac{1}{4}(s, g^*). \tag{20}$$

Consistently, the critical exponents are expressed as functions of a scalar product, invariant by any coordinate change in the space g, or in the order-parameter space.

Let us now concentrate on the FP's with real coordinates lying in the vicinity of the isotropic FP. These are the only FP's which are likely to be stable and have therefore to be considered for $n = 4$. As stressed in the preceding paragraph, these FP are of the form $g^* = \epsilon(s/2 + h^*)$, with $(s, h^*) = (\epsilon/6)$. Thus

$$U^* = \epsilon\left[1 + \frac{(s, h^*)}{4}\right] = \epsilon\left[1 + \frac{\epsilon}{24}\right]. \tag{21}$$

The value of U^* is therefore the same to order ϵ^2 for all the considered FP's. These FP's are associated with the same critical exponents whose values can be drawn from Eqs. (17) and (18) and from the scaling laws:[28]

$$\begin{aligned} &\eta = \frac{\epsilon^2}{48}\left[1 + \frac{5\epsilon}{6}\right], \quad \nu = \frac{1}{2} + \frac{\epsilon}{8} + \frac{7\epsilon^2}{96}, \\ &\gamma = 1 + \frac{\epsilon}{4} + \frac{13\epsilon^2}{96}, \quad \alpha = -\frac{\epsilon^2}{6}, \\ &\beta = \frac{1}{2} - \frac{\epsilon}{8} + \frac{\epsilon^2}{64}, \quad \delta = 3 + \epsilon + \frac{11\epsilon^2}{24}. \end{aligned} \tag{22}$$

Thus, with respect to the values of the critical exponents to order ϵ^2, no new result is obtained for $n = 4$, as compared to $n \leq 3$: If a stable FP exists, its critical exponents are the same as those of the isotropic FP.

IV. CHARACTERISTICS OF THE FIXED POINTS FOR $n = 4$

We have stated in the preceding section that the characteristics of the flows associated with the 22 types of FDPE's encountered for $n = 4$ can be deduced from the study of five FDPE's. In the paragraphs below we first describe the results for these FDPE's and then deduce them for the other little groups and centralizers.

A. Results for the "upper set" of five minimal groups

For each of the five cases the explicit forms of the $\beta_\nu(u_{\nu'})$ functions governing the flow of the u_ν coefficients are indicated on Table III. Using the recursion relations and following the methods outlined in Sec. III, we have determined the fixed points and their stabilities. Let us examine the results obtained in each case.

1. FDPE with $(Y/C_2; Y^*/C_2)^*$, "di-icosahedral" symmetry

There are two recursion relations acting on the coefficients u and v. The coefficient u is associated with the isotropic part of the FDPE, while $v \neq 0$ reflects the presence of a di-icosahedral anisotropy. To the $(\epsilon g, g^2)$ order, the calculation provides two nontrivial (i.e., non-Gaussian) FP's (Table IV). One has the minimal di-icosahedral symmetry and is unstable. In agreement with the general considerations in Sec. III B, the other is isotropic [O(4) symmetry], has marginal stability, and is an m-fold $(m = 2)$ solution of Eq. (9). At the next order $(\epsilon g^2, g^3)$, it splits into two real FP's (Table V). The first one keeps the isotropic symmetry. It is unstable in the v direction. The second one, S', has the di-icosahedral symmetry and is stable. Figure 2 shows the structure of the flow in the

TABLE III. Form of the β functions in Eq. (9) for the five effective Hamiltonians corresponding to minimal little groups. Columns 1 and 2 specify the label of the little group and the notation of the coefficients u_ν in Eq. (3). The function β_ν has the form $\beta_\nu = -\epsilon u_\nu + [1+(\epsilon/2)]\beta_\nu^1(u_\lambda) - \beta_\nu^2(u_\lambda) + u_\nu \Delta(u_\lambda)$ to two-loop order $(\epsilon g^2, g^3)$. To one-loop order $(\epsilon g, g^2)$ it reduces to $[-\epsilon u_\nu + \beta_\nu^1(u_\lambda)]$.

FDPE symmetry	u_ν	$\beta_\nu^1(u_\lambda)$	$\beta_\nu^2(u_\lambda)$	$\Delta(u_\lambda)$
$\left[\frac{Y}{C_2};\frac{Y^*}{C_2}\right]^*$	u	$\frac{1}{2}(4u^2+14uv+27v^2)$	$\frac{1}{2}(\frac{14}{3}u^3+28u^2v+\frac{197}{2}uv^2+\frac{321}{2}v^3)$	$\frac{1}{12}(2u^2+14uv+\frac{89}{2}v^2)$
	v	$v(2u+6v)$	$3v(u^2+\frac{19}{3}uv+\frac{45}{4}v^2)$	$\frac{1}{12}(2u^2+14uv+\frac{89}{2}v^2)$
$\left[\frac{D_5}{C_2};\frac{D_5}{C_2}\right]^*$	u	$\frac{1}{3}(6u^2+4uv+\frac{9}{4}w^2)$	$\frac{1}{3}(7u^3+8u^2v+2uv^2+6uw^2+3vw^2)$	$\frac{1}{36}(6u^2+4v^2+8uv+3w^2)$
	v	$\frac{1}{3}(5v^2+6uv-\frac{9}{8}w^2)$	$\frac{1}{9}(27u^2v+42uv^2+16v^3-\frac{27}{4}uw^2-\frac{9}{4}vw^2)$	$\frac{1}{36}(6u^2+4v^2+8uv+3w^2)$
	w	$w(2u+v)$	$\frac{w}{3}(9u^2+10uv+3v^2+\frac{9}{16}w^2)$	$\frac{1}{36}(6u^2+4v^2+8uv+3w^2)$
$\left[\frac{D_3}{C_2};\frac{O}{D_2}\right]$	u	$2u^2+uv+\frac{3w^2}{16}$	$\frac{1}{6}\left[14u^3+12u^2v+3uv^2+\frac{7w^3}{32}+3uw^2+\frac{9}{8}vw^2\right]$	$\frac{1}{48}(8u^2+4v^2+8uv+w^2)$
	v	$\frac{1}{2}\left[3v^2+4uv-\frac{w^2}{4}\right]$	$\frac{1}{2}\left[3v^3+6vu^2+8uv^2-\frac{w^3}{16}-\frac{uw^2}{2}-\frac{vw^2}{4}\right]$	$\frac{1}{48}(8u^2+4v^2 8uv+w^2)$
	w	$\frac{w}{2}\left[4u+v+\frac{w}{2}\right]$	$\frac{w}{32}(96u^2+64uv+8v^2+3w^2+16uw+2vw)$	$\frac{1}{48}(8u^2+4v^2 8uv+w^2)$
$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right]$	u	$2u^2+\frac{4uv}{3}+\frac{w^2}{3}+\frac{3p^2}{4}$	$\frac{1}{3}(7u^3+8u^2v+2uv^2+\frac{5}{2}uw^2+\frac{4}{3}vw^2+6up^2+3vp^2+\frac{3}{2}wp^2)$	$\frac{1}{12}\left[2u^2+\frac{8}{3}uv+\frac{4}{3}v^2+\frac{w^2}{3}+p^2\right]$
	v	$\frac{5}{3}v^2+2uv-\frac{1}{4}w^2-\frac{3p^2}{8}$	$(3u^2v+\frac{16}{9}v^3+\frac{14}{3}uv^2-\frac{1}{2}uw^2-\frac{1}{3}vw^2-\frac{3}{4}up^2-\frac{1}{4}vp^2-\frac{3}{8}wp^2)$	$\frac{1}{12}\left[2u^2+\frac{8}{3}uv+\frac{4}{3}v^2+\frac{w^2}{3}+p^2\right]$

TABLE III. (*Continued*).

FDPE symmetry	u_ν	$\beta^1_\nu(u_\lambda)$	$\beta^2_\nu(u_\lambda)$	$\Delta(u_\lambda)$
	w	$2uw+\frac{2}{3}vw+\frac{3}{4}p^2$	$(3u^2w+\frac{4}{9}wv^2+\frac{8}{3}uvw+\frac{3}{2}up^2+\frac{1}{2}vp^2+\frac{3}{4}wp^2)$	$\frac{1}{12}\left[2u^2+\frac{8}{3}uv+\frac{4}{3}v^2+\frac{w^2}{3}+p^2\right]$
	p	$p\left[2u+v+\frac{w}{2}\right]$	$p(3u^2+v^2+uw+\frac{10}{3}uv+\frac{1}{4}w^2+\frac{1}{3}vw)$	$\frac{1}{12}\left[2u^2+\frac{8}{3}uv+\frac{4}{3}v^2+\frac{w^2}{3}+p^2\right]$
$D_2\times D_2$	t	$\frac{3}{2}\left[t^2+\sum u_i^2\right]$	$\frac{3}{2}t\left[t^2+\sum u_i^2\right]+2\sum u_i^3+2\mu^2\left[\sum u_i\right]$	$\frac{1}{12}\left[t^2+3\sum u_i^2+6\mu^2\right]$
	u_i $(i=1,3)$ $(j,k\neq i)$	$tu_i+u_i^2+u_ju_k+2\mu^2$	$\frac{1}{2}u_i(t^2+6tu_i+5u_i^2+u_j^2+u_k^2+4u_ju_k+4\mu^2)+\mu^2(t+5u_j+5u_k)+u_ju_k^2+u_ku_j^2$	$\frac{1}{12}\left[t^2+3\sum u_i^2+6\mu^2\right]$
	μ	$2\mu\left[\sum u_i\right]$	$\mu\left[t\left[\sum u_i\right]+\sum u_i^2+5(u_iu_j+u_iu_k+u_ju_k)+3\mu^2\right]$	$\frac{1}{12}\left[t^2+3\sum u_i^2+6\mu^2\right]$

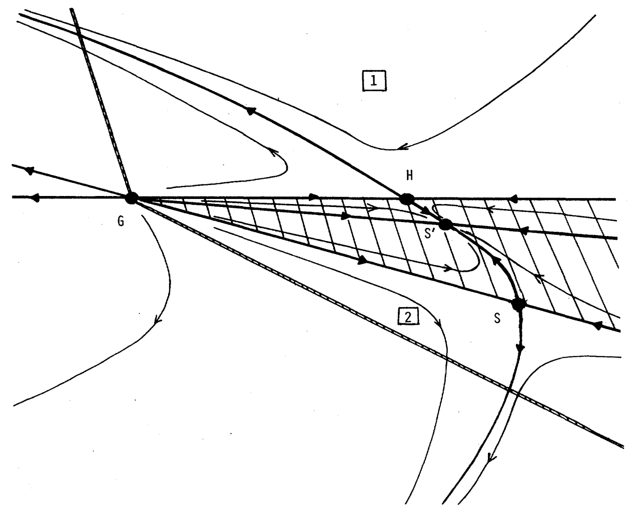

FIG. 2. Pattern of fixed points and trajectories of the flow corresponding to the "di-icosahedral" symmetry (Y/C_2; Y^*/C_2), represented in the plane of the two coefficients (u,v). G and H are, respectively, the Gaussian FP and the isotropic (Heisenberg) FP. S and S' have the di-icosahedral symmetry, the latter one being the stable FP. The shaded area is the attractor basin of S'. The double lines passing by G indicate the limits of positivity of the FDPE (i.e., outside this sector, no continuous transition is possible in the Landau theory). The framed numbers 1 and 2 refer to the ranges of stability of the two possible low-symmetry phases obtained in the Landau theory. The attractor basin of S' is entirely within the range of phase 2.

plane of the two coefficients u and v. The attractor basin of S' is defined by the conditions ($-2/27u<v<0$; $u>0$). By comparison, a minimization of the Landau free energy indicates the possibility of continuous transitions toward two low-symmetry phases, 1 and 2, stable for $(u+v>0;v>0)$ and $(u+5v>0;v<0)$, respectively.

2. FDPE with the $(D_5/C_2;D_5/C_2)^*$ "dipentagonal" symmetry

There are three recursion relations acting on (u,v,w). These coefficients are respectively associated with the isotropic part of the FDPE, a "dicylindrical" $(D_\infty \times D_\infty)^*$ anisotropy, and the considered dipentagonal anisotropy.

At the $(\epsilon g,g^2)$ order, we find four nontrivial FP's. The isotropic one has a fourfold multiplicity. A second one corresponds to the decoupling of two pairs of order-parameter components $(u=w=0)$. Its symmetry can be denoted O(2)×O(2) (isotropic symmetry in two dimensions). The two last FP's, S_1 and S_2, correspond to the minimal dipentagonal symmetry.

At the $(\epsilon g^2,g^3)$ order, the marginal isotropic FP decomposes into four distinct FP's all unstable. One keeps the isotropic symmetry and is unstable in the (v,w) plane. One has the dicylindrical symmetry and is unstable in the w direction. The two last FP's with $w^*\neq 0$, S_1' and S_2', have the dipentagonal symmetry. They are both associated with one negative eigenvalue of matrix (10).

We can note that S_1 and S_2 are associated to identical eigenvalues. This is also the case of S_1' and S_2'. This cir-

TABLE IV. Fixed points and stability for the five FDPE with minimal little group, to one-loop order ($\epsilon g, g^2$). In column 3, H represents the O(4) isotropic symmetry (Heisenberg-type FP). S or S_i is a fixed point with minimal symmetry (i.e., identical to that of the FDPE. The multiplicity of H is indicated between brackets after this symbol. For the case $D_2 \times D_2$, "perm" means that other FP can be obtained by permutation of the (u_1, u_2, u_3) indices. The Gaussian FP is omitted from the table.

FDPE symmetry	$\{u_\nu\}$	Fixed-point symmetry	Fixed-point coordinates	Fixed-point eigenvalues
$\left[\frac{Y}{C_2};\frac{Y^*}{C_2}\right]^*$	u,v	$H(2)$	$\left[\frac{\epsilon}{2},0\right]$	$(\epsilon,0)$
		S	$\left[\frac{9\epsilon}{14},-\frac{\epsilon}{21}\right]$	$\left[\epsilon,-\frac{\epsilon}{21}\right]$
$\left[\frac{D_5}{C_2};\frac{D_5}{C_2}\right]^*$	u,v,w	$H(4)$	$\left[\frac{\epsilon}{2},0,0\right]$	$(\epsilon,0,0)$
		O(2)×O(2)	$\left[0,\frac{3\epsilon}{5},0\right]$	$\left[\epsilon,-\frac{\epsilon}{5},-\frac{2\epsilon}{5}\right]$
		$S_1;S_2$	$\left[\frac{4\epsilon}{7},\frac{-\epsilon}{7},\frac{\pm 4\epsilon}{21}\right]$	$\left[\epsilon,-\frac{4\epsilon}{21},-\frac{\epsilon}{21}\right]$
$\left[\frac{D_3}{C_2};\frac{O}{D_2}\right]$	u,v,w	$H(5)$	$\left[\frac{\epsilon}{2},0,0\right]$	$(\epsilon,0,0)$
		Ising	$\left[0,\frac{2\epsilon}{3},0\right]$; $\left[\frac{4\epsilon}{9},-\frac{2\epsilon}{9},\frac{2\epsilon}{9}\right]$	$\left[\epsilon,\frac{-\epsilon}{3},-\frac{2\epsilon}{3}\right]$
$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right]$	u,v,w,p	$H(10)$	$\left[\frac{\epsilon}{2},0,0,0\right]$	$(\epsilon,0,0,0)$
		$[\mathrm{O}(2)\times\mathrm{O}(2)]_1$	$\left[0,\frac{3\epsilon}{5},0,0\right]$	$\left[\epsilon,\frac{-\epsilon}{5},\frac{-3\epsilon}{5},\frac{-2\epsilon}{5}\right]$
		$[\mathrm{O}(2)\times\mathrm{O}(2)]_2$	$\left[\frac{3\epsilon}{5},\frac{-3\epsilon}{10},\frac{3\epsilon}{5},0\right]$	$\left[\epsilon,-\frac{\epsilon}{5},-\frac{3\epsilon}{5},+\frac{\epsilon}{5}\right]$
		$[\mathrm{O}(2)\times\mathrm{O}(2)]_3$	$\left[\frac{3\epsilon}{5},\frac{-3\epsilon}{10},-\frac{3\epsilon}{5},0\right]$	$\left[\epsilon,-\frac{\epsilon}{5},-\frac{3\epsilon}{5},\frac{-2\epsilon}{5}\right]$
		$S_1;S_2$	$\left[\frac{\epsilon}{2},-\frac{\epsilon}{4},+\frac{\epsilon}{2},\pm\frac{\epsilon}{3}\right]$	$\left[\epsilon,-\frac{\epsilon}{3},-\frac{\epsilon}{3},\frac{-2\epsilon}{3}\right]$

TABLE IV. (*Continued*).

FDPE symmetry	$\{u_\nu\}$	Fixed-point symmetry	Fixed-point coordinates	Fixed-point eigenvalues
$D_2 \times D_2$	$t, u_1, u_2, u_3,$	$H(16)$	$\left[\frac{\epsilon}{2},\frac{\epsilon}{6},\frac{\epsilon}{6},\frac{\epsilon}{6},0\right]$	$(\epsilon,0,0,0,0)$
		Ising	$\left[\frac{2\epsilon}{3},0,0,0,0\right]$ $\left[\frac{\epsilon}{6},\frac{\epsilon}{6},\frac{\epsilon}{6},\frac{\epsilon}{6},\pm\frac{\epsilon}{6}\right]$	$\left[\epsilon,-\frac{\epsilon}{3},-\frac{\epsilon}{3},-\frac{\epsilon}{3},-\epsilon\right]$
		$O(2)\times O(2)$	$\left[\frac{3\epsilon}{5},0,0,\frac{\epsilon}{5},0\right]_{\text{perm}}$ $\left[\frac{3\epsilon}{10},\frac{\epsilon}{10},\frac{\epsilon}{10},\frac{3\epsilon}{10},\pm\frac{\epsilon}{10}\right]_{\text{perm}}$	$\left[\epsilon,\frac{\epsilon}{5},-\frac{\epsilon}{5},-\frac{3\epsilon}{5},-\frac{3\epsilon}{5}\right]$
		$B(2)\times B(2)$	$\left[\frac{\epsilon}{3},0,0,\frac{\epsilon}{3},0\right]_{\text{perm}}$	$\left[\epsilon,-\frac{\epsilon}{3},-\frac{\epsilon}{3},-\frac{\epsilon}{3},-\epsilon\right]$

TABLE V. Splitting of the m-fold isotropic fixed point to two-loop, $(\epsilon g^2, g^3)$ order. The FP with isotropic symmetry $u^*=(\epsilon/2)[1+(\epsilon/24)]$, always unstable, and occurring for any FDPE, has been omitted from the table. The conventions are the same as in Table IV. $a=(\epsilon/2)[1+(3\epsilon/8)]$; $b=(\epsilon/6)[1-(5\epsilon/8)]$; $c=(\epsilon/2)[1-(\epsilon/8)]$; $d=(\epsilon/6)[1+(7\epsilon/8)]$; $e=(\epsilon/2)[1+(\epsilon/24)]$; $f=(\epsilon/6)[1-(7\epsilon/24)]$; $g=(\epsilon/6)[1+(17\epsilon/24)]$; $h=(\epsilon/6)[1+(5\epsilon/24)]$.

FDPE	Fixed points: Symmetry	Fixed points: Coordinates	Fixed points: Eigenvalues
$\left[\frac{Y}{C_2};\frac{Y^*}{C_2}\right]^*$	S'	$\left[\frac{\epsilon}{2}\left[1+\frac{29\epsilon}{24}\right],-\frac{\epsilon^2}{6}\right]$	$\left[\epsilon,\frac{\epsilon^2}{6}\right]$
$\left[\frac{D_5}{C_2};\frac{D_5}{C_2}\right]^*$	$(D_\infty\times D_\infty)^*$	$\left[\frac{\epsilon}{2}\left[1-\frac{5\epsilon}{8}\right],\frac{\epsilon^2}{2},0\right]$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{3}\right]$
	$S_1';S_2'$	$\left[\frac{\epsilon}{2}\left[1+\frac{17}{24}\epsilon\right],-\frac{\epsilon^2}{2},\frac{\pm 2}{3}\epsilon^2\right]$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{2\epsilon^2}{3}\right]$
$\left[\frac{D_3}{C_2};\frac{O}{D_2}\right]$	$\left[\frac{O}{D_2};\frac{O}{D_2}\right]^*$	$\left[\frac{\epsilon}{2}\left[1-\frac{7\epsilon}{24}\right],\frac{\epsilon^2}{3},0\right]$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{3}\right]$
	$\left[\frac{O}{D_2};\frac{O}{D_2}\right]^*$	$\left[\frac{\epsilon}{2}\left[1+\frac{11}{72}\epsilon\right],-\frac{\epsilon^2}{9},\frac{4}{9}\epsilon^2\right]$	
$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right]$	$(D_\infty\times D_\infty)^*$	$\left[\frac{\epsilon}{2}\left[1-\frac{5}{8}\epsilon\right],\frac{\epsilon^2}{2},0,0\right]$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2},-\frac{\epsilon^2}{3}\right]$
	$(D_\infty\times D_\infty)^*$	$\left[\frac{\epsilon}{2}\left[1+\frac{3}{8}\epsilon\right],-\frac{\epsilon^2}{4},+\frac{\epsilon^2}{2},0\right]$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2},+\frac{\epsilon^2}{6}\right]$
	$(D_\infty\times D_\infty)^*$	$\left[\frac{\epsilon}{2}\left[1+\frac{3}{8}\epsilon\right],-\frac{\epsilon^2}{4},-\frac{\epsilon^2}{2},0\right]$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2},-\frac{\epsilon^2}{3}\right]$
	$S_1';S_2'$	$\left[\frac{\epsilon}{2}\left[1+\frac{5}{24}\epsilon\right],-\frac{\epsilon^2}{8},\frac{\epsilon^2}{4},\pm\frac{\epsilon^2}{6}\right]$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{3},-\frac{\epsilon^2}{6}\right]$

TABLE V. (*Continued*).

FDPE	Fixed points Symmetry	Coordinates	Eigenvalues
$D_2\times D_2$	$(D_\infty\times D_\infty)^*$	$\left[a,b,b,\frac{a}{3},0\right]_{\text{perm}}$; $\left[c,\frac{c}{3},\frac{c}{3},d,\pm\frac{\epsilon^2}{12}\right]_{\text{perm}}$	$\left[\epsilon,\frac{\epsilon^2}{6},\frac{\epsilon^2}{6},\frac{-\epsilon^2}{2}\right]$
	$\left[\frac{O}{D_2};\frac{O}{D_2}\right]^*$	$(e,f,f,g,0)_{\text{perm}}$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2},-\frac{\epsilon^2}{2}\right]$
	$\left[\frac{O}{D_2};\frac{O}{D_2}\right]^*$	$\left[c,h,h,h,\pm\frac{\epsilon^2}{12}\right]$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{6},-\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2}\right]$
		$(a,f,f,f,0)$	

cumstance suggests that the FP's in each pair are symmetry related. Actually, a correspondence is achieved between them by the action of the normalizer of the space (u,v,w) which is the "didecagonal" group labeled $G_N=(D_{10}/C_2;D_{10}/C_2)^*$. G_N contains the considered dipentagonal group and is, on the other hand a subgroup of $(D_\infty\times D_\infty)^*$. The action of G_N on the FDPE with coefficients (u,v,w) transforms it into an FDPE defined by $(u,v,-w)$. Thus any FP with $w_1^*\neq 0$ has a symmetric FP with $w_2^*=-w_1^*$ and identical eigenvalues. This is the situation realized by the pairs (S_1,S_2) and (S_1',S_2').

3. *FDPE with the $(D_3/C_2;O/D_2)$, "trigonal-cubic" symmetry*

This FDPE also involves three coefficients (u,v,w). The coefficient u is again relative to the isotropic invariant, while v and w are, respectively, associated to the presence of a "cubic" anisotropy and to the trigonal-cubic anisotropy.

There are three nontrivial FP's to lowest order (Table IV), among which the isotropic FP displaying a fivefold multiplicity. The two additional FP's are symmetry related "Ising-type" points (i.e., FP's representing four decoupled systems with $n=1$). The normalizer G_N' which establishes the correspondence between them is the "hexagonal-cubic" group $(D_6/C_4;O/D_2)$ appearing in Fig. 1. Its action is to transform (u,v,w) into $(u+2v/3-w/3,-v/3+2w/3,4v/3+w/3)$. Thus each FP with $w\neq 2v$ will posses an equivalent FP with the preceding transformed coordinates.

The decomposition of the isotropic point at higher order is more complicated than for subsections 1 and 2. It yields two types of FP's.

The first type is in the line of the preceding results. It comprises three FP's with real coordinates, differing by the term of order ϵ^2. One has the isotropic symmetry and the two others are "cubic" FP's, symmetry related by the normalizer G_N'. All are unstable in the (u,v,w) space.

The second type of FP, which possesses the same hexagonal-cubic symmetry as the normalizer G_N', illustrates a situation pointed out in Sec. III C. Thus, each FP has coordinates containing an imaginary part. Also, the coordinates differ from those of the isotropic FP by corrections of order $\epsilon^{3/2}$ (Table VI). We can note that a similar type of coordinates had been encountered by Khmelnitskii[29] and by Grinstein *et al.*[30] in the study of disordered systems, whose critical behavior was found to be governed by a FP with coordinates of order $\epsilon^{1/2}$.

TABLE VI. Coordinates of fixed points with complex coordinates, arising in several FDPE with $n=4$.

Fixed point symmetry	Fixed point coordinates
$\left[\frac{D_6}{C_4};\frac{O}{D_2}\right]$	$\left[\frac{\epsilon}{2}\left[1\pm\frac{i}{3}\sqrt{\epsilon}\right];\mp\frac{\epsilon^{3/2}}{3}\right]$
$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right]$	$\left[\frac{\epsilon}{2}\left[1+\frac{i}{2}\sqrt{\epsilon}\right];\frac{-3i\epsilon^{3/2}}{8};\frac{-i\epsilon^{3/2}}{4};\frac{\pm i\epsilon^{3/2}}{2\sqrt{3}}\right]$
	$\left[\frac{\epsilon}{2}\left[1-\frac{i}{2}\sqrt{\epsilon}\right];\frac{3i\epsilon^{3/2}}{8};\frac{i\epsilon^{3/2}}{4};\frac{\pm i\epsilon^{3/2}}{2\sqrt{3}}\right]$

4. FDPE with the $(D_2/D_1;D_4/D_2)$ "orthotetragonal" symmetry

Four recursion relations govern the flow in a space carried by the coefficients (u,v,w,p). u has the same meaning as before. The three coefficients v, w, and p are, respectively, associated with the dicylindrical anisotropy already encountered in subsection 2, with an "orthocylindrical" anisotropy, denoted $D_2 \times D_\infty$, and with the orthotetragonal anisotropy.

At lowest order $(\epsilon g, g^2)$, the isotropic FP has a tenfold multiplicity. There are five other nontrivial FP's. The normalizer of the (u,v,w,p) space is the group $D_2 \times D_4$.[10,20] Its action is to transform p into $(-p)$ and to leave the other coefficients unchanged. Thus, it relates two of the five FP's, which are denoted S_1 and S_2 on Table IV. The three other FP's which have the same O(2)$\times$O(2) symmetry (decoupled $n=2$), are nevertheless unequivalent.

At higher order, a situation similar to that of subsection 3 occurs. The isotropic FP is split into six real FP's and four complex ones. The real FP's are three unequivalent dicylindrical FP's, and two orthotetragonal ones symmetry related through the action of the normalizer. These five FP's differ by terms of order ϵ^2. The four complex FP's have the same characteristics as in subsection 3: their coordinates involve corrections of order $\epsilon^{3/2}$. They comprise two pairs of FP's, each pair being constituted by FP symmetry related by the action of $D_2 \times D_4$.

5. FDPE with the $D_2 \times D_2$ diorthorhombic symmetry

This FDPE requires solving a system of five FP equations relative to the coefficients (t,u_1,u_2,u_3,μ). For the sake of convenience, we have made a different choice of basis O_v than in the preceding cases. Thus, the isotropic symmetry is not associated with the vanishing of certain coefficients, but with a relationship between them $(t=3u_1=3u_2=3u_3)$.

At the order $(\epsilon g, g^2)$ one finds that the isotropic FP has a 16-fold multiplicity. There are 15 other nontrivial FP's which correspond to some decoupling of the four components of the OP. These FP's can be classified into only three sets of unequivalent FP's, each set being composed of symmetry-related points. The three sets, respectively, correspond to Ising $(n=1)$ FP's, to O(2)$\times$O(2) $(n=2)$ FP's and to decoupled $(n=2)$ FP's with cubic symmetry. The transformations relating equivalent FP's are generated by the permutations of the three coefficients u_i, and by the transformation

$$\{t,u_1,u_2,u_3,\mu\} \to \left\{ \frac{t}{4}+\frac{3}{4}(u_1+u_2+u_3)-\frac{3}{2}\mu, \frac{t}{4}+\frac{3u_1}{4}-\frac{u_2+u_3}{4}+\frac{\mu}{2}, \frac{t}{4}+\frac{3u_2}{4}-\frac{u_1+u_3}{4}+\frac{\mu}{2}, \frac{t}{4}+\frac{3u_3}{4}-\frac{u_1+u_2}{4}+\frac{\mu}{2}, \frac{t}{4}-\frac{u_1+u_2+u_3}{4}-\frac{\mu}{2} \right\}. \tag{23}$$

They belong to the normalizer G_N'' which is a subgroup of O(4) denoted (O$\times$O)* having 2304 elements.[10,20]

To the next order there are no complications of the kind encountered in the two preceding paragraphs. Thus, the multiplicity of the isotropic FP is entirely lifted and gives rise to 16 FP's with real coordinates differing from each other by corrections of order ϵ^2. These FP's can be classified into three unequivalent sets. One comprises 9 FP's with symmetries conjugated to $(D_\infty \times D_\infty)^*$. The two other sets contain three FP's each with the cubic symmetry. All these FP's are unstable in the considered parameter space.

B. Results for the remaining little groups of the "upper set"

These little groups are supergroups of the five preceding groups. Their fixed points can be found by simple inspection of the results already worked out. In each case, we have to retain the FP's whose symmetries are equal to, or greater than, that of the considered supergroup. The stability of a FP can be inferred on the basis of the information explicitly contained in Tables IV and V and in Fig. 1 (i.e., number of positive eigenvalues, number of u_v parameters, group-subgroup relationship).

In order to illustrate this deduction, consider the FDPE with the orthocylindrical symmetry $D_2 \times D_\infty$. This is a supergroup of $(D_2/D_1;D_4/D_2)$ and $D_2 \times D_2$ (subsections 4 and 5 considered above). Inspection of Tables IV and V reveals that there are seven nontrivial FP's with symmetries equal to or higher than $D_2 \times D_\infty$. In addition to the isotropic FP there are three equivalent FP's corresponding to the $(D_\infty \times D_\infty)^*$ symmetry and, also, three FP's with the O(2)$\times$O(2) symmetry. This pattern of FP's, constituted by triads of FP's with identical symmetries, derives from the fact that $D_2 \times D_\infty$ is a subgroup of three conjugated $(D_\infty \times D_\infty)^*$ groups. One $(D_\infty \times D_\infty)^*$ group has both its C_∞ axes parallel to each other (i.e., $C_\infty^z \times C_\infty^z$), while for the others, the "left" C_∞ axis and the "right" one are taken along perpendicular directions of the three-dimensional space, respectively, (C_∞^x, C_∞^z) and (C_∞^y, C_∞^z). The normalizer establishing the correspondence between the three symmetry-related FP's is a subgroup of O(4) denoted O$\times D_\infty$.[10,20] On the other hand, Tables IV and V show that for $(D_2/D_1;D_4/D_2)$ the six preceding FP's involve at most two positive eigenvalues. As $D_2 \times D_\infty$ is associated to three u_v coefficients, we can conclude that none of these FP's is stable in the considered three-dimensional $\{u_v\}$ space.

Table VII summarizes the results obtained along the preceding lines for all the nonminimal little groups. Part of these results can be compared to the ones worked out previously. Thus, Mukamel *et al.*[11] had examined the two FDPE's with dicylindrical and ditetragonal anisotropies. They had found that in both cases, the same FP with dicylindrical symmetry was stable. Our results agree with theirs, and further show that any anisotropy other than the ditetragonal one destabilizes the dicylindrical FP. This had been previously checked for the ditetrahedral anisotropy $(T/D_2;T/D_2)^*$,[11] and conjectured[12] for the orthocylindrical one $D_2 \times D_\infty$. Similarly, we find that the stable "cubic" FP found by Aharony[31] for the cubic $(O/D_2;O/D_2)^*$ FDPE, is destabilized by any anisotropy with lower symmetry than cubic.

C. Investigation of the centralizers which are not little groups

As apparent on Fig. 1 there are five centralizers of fourth-degree polynomials which are not little groups. We have explained in Sec. III B that the flow relative to them can be generated from the flows associated to the little groups by application of continuous sets of transformations belonging to the normalizer G_N of the space $\{u_v\}$. With respect to the action of G_N, the fixed points can be classified into two categories. Either a FP will be invariant under the action of G_N, and it will then constitute an isolated FP of the flow (this is always the case of the isotropic FP) or a FP is not invariant by this action, and it will generate lines or higher-dimensional manifolds of physically equivalent FP. The dimension of a manifold will depend both on the symmetry of the considered FP and on the number of parameters α defining the continuous group G_N.

As an example, let us examine the case of the FDPE with $C_4^z \times D_\infty^z$ symmetry (Fig. 1). The C_4 axis of the left group is assumed to lie along z. There exists a continuous set of little groups, denoted $D_2^\alpha \times D_\infty^z$ (conjugated to $D_2 \times D_\infty$), containing $C_4^z \times D_\infty^z$ as common subgroup, and differing by the orientation of the binary axis in the left (x,y) plane. The relevant normalizer is $G_N = (D_\infty^z \times D_\infty^z)^*$. The orientation of the set of $D_2^\alpha \times D_\infty^z$ groups is defined by a single parameter. As stressed above, the pattern of FP's relative to $C_4^z \times D_\infty^z$ can be found by applying the action of $G_N = (D_\infty^z \times D_\infty^z)^*$ to the FP of $D_2^z \times D_\infty^z$. The latter FP's have been enumerated in the preceding paragraph. They comprise the FP's corresponding to the three symmetries $(D_\infty^z \times D_\infty^z)^*$, $(D_\infty^x \times D_\infty^z)^*$, and $(D_\infty^y \times D_\infty^z)^*$. The first group coincides with G_N and its FP's are thus invariant by G_N. These FP's are the Gaussian and isotropic FP as well as one $(D_\infty \times D_\infty)^*$ FP and one $O(2) \times O(2)$ FP. They are isolated FP's of $C_4^z \times D_\infty^z$. The second and third dicylindrical groups also give rise to one $(D_\infty \times D_\infty)^*$ and one $O(2) \times O(2)$ FP each, which are not invariant by G_N. As those two groups are conjugate with respect to G_N, each pair of FP's with the same symmetry will generate a single line of FP's under the action of G_N. In summary, the flow relative to $C_4^z \times D_\infty^z$ contains four isolated FP's and two lines of FP's.

Table VIII reproduces the results obtained by the same type of derivation for all the centralizers appearing on Fig. 1. As the little groups containing these centralizers depend at most on three continuous parameters, we obtain manifolds of equivalent FP's having at most three dimensions. It is worth noting that a single one of the centralizers, denoted $(C_8/C_4;D_4/D_2)$ is associated with a stable FP (with dicylindrical symmetry). However, as mentioned above, the lowest symmetry preserving the stability of the former FP is actually the ditetragonal one since for the considered centralizer with symmetry $(C_8/C_4;D_4/D_2)$ all the trajectories possess the ditetragonal symmetry.

D. Absence of a stable fixed point for the "lower set" of little groups

Let us examine the four little groups whose respective labels are $(C_6/C_2;T/D_2)$, $(C_4/C_2;D_4/D_2)$, $C_4 \times D_2$, and $C_2 \times D_2$. The peculiarities of their pattern of FP's can be discussed on the example of $C_4^z \times D_2$.

First, we can note that, in contrast to the case of the "upper set" of little groups, $C_4^z \times D_2$ has a continuous normalizer $G = D_\infty \times O$, though being itself a discrete subgroup of O(4). $C_4^z \times D_2$ is the common subgroup of a one-parameter continuous set of groups $D_2^\alpha \times D_2$ conjugate to $D_2 \times D_2$. As shown by Fig. 1, the FDPE relative to $C_4^z \times D_2$ is a linear combination of seven independent fourth-degree terms, respectively, labeled $O_0, O_1, O_2, O_3, O_5, O_9, O_{10}$. The five first of these invariant terms span a space whose little group is $D_2 \times D_2$. Each one of the remaining two terms O_9 and O_{10} can be shown to invariant by a group of the form $D_2^\alpha \times D_2$. Thus the pattern of FP's relative to $C_4^z \times D_2$ will contain the fixed points generated by the action of the normalizer G_N on each FP of $D_2 \times D_2$. This action provides isolated FP's as well as lines of FP's.

Aside from the former sets of FP's, others can exist lying in general directions of the $\{u_v\}$ space (i.e., in directions whose little group is $C_4 \times D_2$) since all the "high-symmetry" directions have at least the symmetry $D_2 \times D_2$ already considered. For FP's having such a "minimal" symmetry, the action of the normalizer G_N necessarily generate lines of physically equivalent FP's. Thus, there are no isolated FP's with symmetry $C_4 \times D_2$. As shown in Sec. II D the stable FP, if it exists, is unique. Accordingly it cannot belong to a line of physically equivalent FP's. Thus, no FP with $C_4 \times D_2$ symmetry can be stable. Since, on the other hand, the FP's corresponding to $D_2 \times D_2$ are known to be all unstable (Tables IV and V), we can conclude that the FDPE with symmetry $C_4 \times D_2$ does not have a stable FP.

The same type of argument allows to show that no stable FP exists for the little groups of the "lower set". This argument had previously been used by Korzhenevskii[25] to conjecture that, more generally, low-symmetry Hamiltonians were unlikely to display stable FP, due to the existence for most FP's of continuous lines of equivalent FP generated by a continuous normalizer.

V. DISCUSSION

When dealing with a phase transition described by a four-component order parameter the effective Hamiltonian determining the critical behavior can have one of 22 possible forms. The above investigation shows that aside from the "isotropic" case, there are only five "anisotropic" effective Hamiltonians associated with the occurrence of a stable FP. Three of the preceding five cases,

TABLE VII. Real fixed points for eight FDPE whose little groups belong to the "upper set" of little groups and are supergroups of the "minimal groups" considered in Tables IV and V. The conventions are the same as in these tables. The choice of the parameters u_ν for the groups in the right column is the same as for the group $D_2 \times D_2$ (see Table II).

FDPE	Symmetry	Fixed points Coordinates	Eigenvalues
O(4)	O(4)	$\frac{\epsilon}{2}\left(1+\frac{\epsilon}{24}\right)$	ϵ
$(D_\infty \times D_\infty)^*$	$O(2)\times O(2)$	$\left(0,\frac{3\epsilon}{5}\right)$	$\left(\epsilon,-\frac{\epsilon}{5}\right)$
	$(D_\infty \times D_\infty)^*$	$\left(\frac{\epsilon}{2}\left(1-\frac{5\epsilon}{8}\right),\frac{\epsilon^2}{2}\right)$	$\left(\epsilon,\frac{\epsilon^2}{6}\right)$
$\left(\frac{D_6}{C_4};\frac{O}{D_2}\right)$		No real fixed point aside from the Gaussian and isotropic ones	
$\left(\frac{O}{D_2};\frac{O}{D_2}\right)^*$	Ising	$\left(0,\frac{2\epsilon}{3}\right)$	$\left(\epsilon,-\frac{\epsilon}{3}\right)$
	$\left(\frac{O}{D_2};\frac{O}{D_2}\right)^*$	$\left(\frac{\epsilon}{2}\left(1-\frac{7\epsilon}{24}\right),\frac{\epsilon^2}{3}\right)$	$\left(\epsilon,\frac{\epsilon^2}{6}\right)$
$D_2 \times D_\infty$	$O(2)\times O(2)$	$\left\{\begin{array}{l}\left(0,\frac{3\epsilon}{5},0\right)\\ \left(\frac{3\epsilon}{5},-\frac{3\epsilon}{10},\pm\frac{3\epsilon}{5}\right)\end{array}\right.$	$\left(\epsilon,-\frac{\epsilon}{5},-\frac{3\epsilon}{5}\right)$
	$(D_\infty \times D_\infty)^*$	$\left\{\begin{array}{l}\left(\frac{\epsilon}{2}\left(1-\frac{5\epsilon}{8}\right),\frac{\epsilon^2}{2},0\right)\\ \left(\frac{\epsilon}{2}\left(1+\frac{3\epsilon}{8}\right),-\frac{\epsilon^2}{4},\pm\frac{\epsilon^2}{2}\right)\end{array}\right.$	$\left(\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2}\right)$
$\left(\frac{D_4}{D_2};\frac{D_4}{D_2}\right)^*$	$O(2)\times O(2)$	$\left(\frac{3\epsilon}{5},0,\frac{\epsilon}{5}\right)$	$\left(\epsilon,\frac{\epsilon}{5},-\frac{\epsilon}{5}\right)$
	$B(2)\times B(2)$	$\left\{\begin{array}{l}\left(\frac{2\epsilon}{3},0,0\right)\\ \left(\frac{\epsilon}{3},0,\frac{\epsilon}{3}\right)\end{array}\right.$	$\left(\epsilon,-\frac{\epsilon}{3},-\frac{\epsilon}{3}\right)$
	$(D_\infty \times D_\infty)^*$	$\left(a,b,\frac{a}{3}\right)$	$\left(\epsilon,\frac{\epsilon^2}{6},\frac{\epsilon^2}{6}\right)$
	$\left(\frac{O}{D_2};\frac{O}{D_2}\right)^*$	$\left\{\begin{array}{l}(e,f,g)\\ (a,f,f)\end{array}\right.$	$\left(\epsilon\frac{\epsilon^2}{6},-\frac{\epsilon^2}{6}\right)$
$\left(\frac{T}{D_2};\frac{T}{D_2}\right)^*$	Ising	$\left\{\begin{array}{l}\left(\frac{2\epsilon}{3},0,0\right)\\ \left(\frac{\epsilon}{6},\frac{\epsilon}{6},\pm\frac{\epsilon}{6}\right)\end{array}\right.$	$\left(\epsilon,-\frac{\epsilon}{3},-\epsilon_3\right)$
	$\left(\frac{O}{D_2};\frac{O}{D_2}\right)^*$	$\left\{\begin{array}{l}\left(c,h,\pm\frac{\epsilon^2}{12}\right)\\ (a,f,0)\end{array}\right.$	$\left(\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2}\right)$

TABLE VII. (*Continued*).

FDPE	Symmetry	Fixed points Coordinates	Eigenvalues
$(D_2 \times D_2)^*$	Ising	$\left[\frac{2\epsilon}{3},0,0,0\right]$	$\left[\epsilon,-\frac{\epsilon}{3},-\frac{\epsilon}{3},-\frac{\epsilon}{3}\right]$
	$O(2)\times O(2)$	$\left[\frac{3\epsilon}{5},0,0,\frac{\epsilon}{5}\right]_{\text{perm}}$	$\left[\epsilon,\frac{\epsilon}{5},-\frac{\epsilon}{5},-\frac{3\epsilon}{5}\right]$
	$B(2)\times B(2)$	$\left[\frac{\epsilon}{3},0,0,\frac{\epsilon}{3}\right]_{\text{perm}}$	$\left[\epsilon,-\frac{\epsilon}{3},-\frac{\epsilon}{3},-\frac{\epsilon}{3}\right]$
	$(D_\infty \times D_\infty)^*$	$\left[a,b,b,\frac{a}{3}\right]_{\text{perm}}$	$\left[\epsilon,\frac{\epsilon^2}{6},\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2}\right]$
	$\left[\frac{O}{D_2};\frac{O}{D_2}\right]^*$	$(e,f,f,g)_{\text{perm}}$	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{6},-\frac{\epsilon^2}{2}\right]$
	$\left[\frac{O}{D_2};\frac{O}{D_2}\right]^*$	(a,f,f,f)	$\left[\epsilon,\frac{\epsilon^2}{6},-\frac{\epsilon^2}{6},-\frac{\epsilon^2}{6}\right]$

respectively characterized by the FDPE symmetrics $(D_\infty \times D_\infty)^*$, $(D_4/D_2;D_4/D_2)^*$, and $(C_8/C_4,D_4/D_2)$ possess the same type of "dicylindrical" stable FP. For any of these FDPE, the fluctuations generate the same $(D_\infty \times D_\infty)^*$ symmetry at the critical point. The fourth case of interest is the thoroughly discussed[31] cubic FDPE whose symmetry is preserved by the fluctuations. The last one is the di-icosahedral $(Y/C_2;Y^*/C_2)^*$ FDPE. Its symmetry is also preserved by the fluctuations since the corresponding stable FP has itself the di-icosahedral symmetry. This type of stable FP was not previously detected. Its symmetry contains both threefold and fivefold rotations.

The structure of the flow of $\{u_\nu\}$ coefficients around the di-icosahedral FP (Fig. 2) has been discussed in Sec. IV A. In Fig. 3 we have represented the flow associated to both the cubic and dicylindrical FDPE's: these two flows are identical, within a change of the FP labels. Examination of Figs. 2 and 3 reveals that all the unstable FP's lie on the boundary of the attractor basin of the stable FP's. The general validity of this property had been established previously[23] to one-loop order. In the considered examples, we find that this validity is preserved to the next order. It can also be noted in Figs. 2 and 3 that, consistent with a trend generally found,[32] the effect of critical fluctuations is to decrease the range of occurrence of a continuous transition with respect to the range determined by use of Landau's theory. This effect is expressed in the cases considered here by the fact that the FP is contained in the range of $\{u_\nu\}$ coefficients associated to a single of the low symmetry phases found in the framework of Landau's theory.[8,33] Thus, fluctuations suppress the possibility of a continuous transition toward one of the two low-symmetry phases and reduce this possibility toward the second phase to a narrower range of the $\{u_\nu\}$ coefficients. (The converse effect, pointed out by Blankshtein and Aharony,[34] of an extension of the range of continuous transitions, was based on the consideration of terms of degree six in the FDPE.)

It is also of interest to plot the trajectories of the flow in the case, lacking a stable FP, defined by the $(D_6/C_4;O/D_2)$ symmetry as this is the simplest example involving the occurrence of FP's with complex coordinates. Figure 4 shows that this flow has a very simple structure, without any apparent pattern revealing the influence of the complex FP's near the isotropic FP.

A. Symmetry conditions for the stable fixed points

Let us now examine the symmetry conditions presiding over the occurrence of a stable fixed point, and investigate

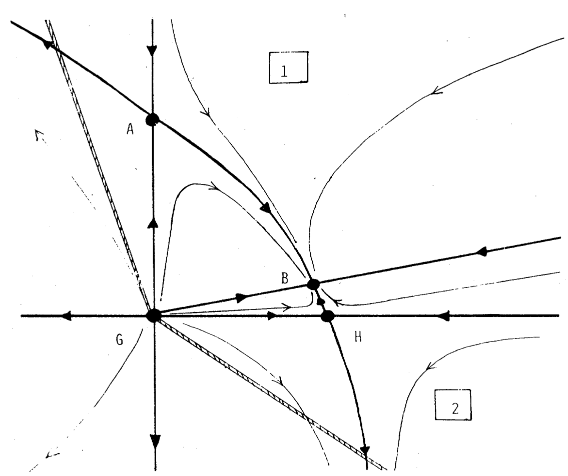

FIG. 3. Pattern of fixed points and trajectories common to both the "dicylindrical" and the hypercubic symmetries. The FP labeled A is, respectively, a decoupled isotropic $n=2$ FP for the dicylindrical case, or an "Ising" FP in the hypercubic case. The stable FP labeled B has the symmetry of the considered Hamiltonian (respectively, dicylindrical or hypercubic). The meaning of the other symbols are the same as in Fig. 2.

TABLE VIII. Pattern of fixed points for centralizers which are not little groups of the corresponding $\{u_\nu\}$ space. This pattern contains both isolated FP's and continuous manifolds of physically equivalent FP's. The symmetry of the FP, the number of points/manifolds, and the dimension of the manifold are indicated. 3D means three-dimensional

Centralizer (number of u_ν)	Conjugation class of little groups (number of u_ν)	Normalizer G_N (number of α parameters)	Quartic terms not invariant by G_N	Isolated anisotropic FP Symmetry	Isolated anisotropic FP Number	Continuous manifolds of anisotropic FP Type	Continuous manifolds of anisotropic FP Symmetry	Continuous manifolds of anisotropic FP Number	Existence of a stable FP?
$C_4^z \times D_\infty$	$D_2^\alpha \times D_\infty$	$(D_\infty^z \times D_\infty)^*$	O_5, O_9	O(2)×O(2)	1	line	O(2)×O(2)	1	no
(4)	(3)	(1)		$(D_\infty \times D_\infty)^*$	1	line	$(D_\infty \times D_\infty)^*$		
$\left[\frac{C_8^z}{C_4};\frac{D_4}{D_2}\right]$	$\left[\frac{D_4^\alpha}{D_2};\frac{D_4}{D_2}\right]^*$	$D_\infty^z \times D_4$	O_2, O_{10}	O(2)×O(2)	1	line	$B(2)\times B(2)$	1	yes
(4)	(3)	(1)		$(D_\infty \times D_\infty)^*$	1	line	Ising	1	
						line	$\left[\frac{O}{D_2};\frac{O}{D_2}\right]^*$	1	
						line	$\left[\frac{D_4}{D_2};\frac{D_4}{D_2}\right]^*$	1	
$\left[\frac{D_2}{D_1};\frac{C_8}{C_4}\right]$	$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right]$	$D_2 \times D_\infty^z$	O_8, O_{11}	O(2)×O(2)	3	line	$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right] S_1, S_2$	1	no
(5)	(4)	(1)		$(D_\infty \times D_\infty)^*$	3	line	$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right] S_1', S_2'$	1	
$C_2 \times D_\infty$	$D_2 \times D_\infty$	O(3)×D_∞	O_1, O_5, O_9			3D	O(2)×O(2)	1	no
(6)	(3)	(3)	O_{12}, O_{14}			3D	$(D_\infty \times D_\infty)^*$	1	
$\left[\frac{C_{12}^u}{C_4};\frac{T}{D_2}\right]$	$\left[\frac{D_6}{C_4};\frac{O}{D_2}\right]$	$D_\infty^u \times$O	$(O_2 + 2O_6)$						
(3)	(2)	(1)	$(O_7 + 6O_3)$						no

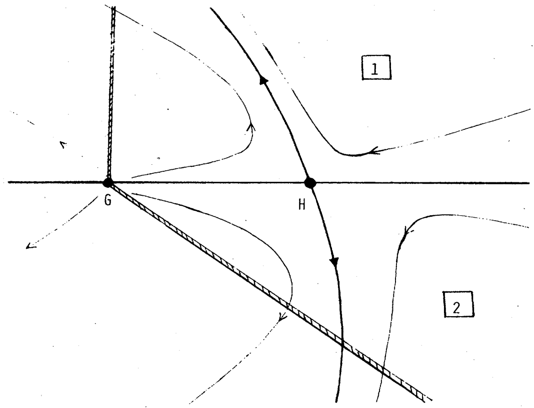

FIG. 4. Pattern of fixed points and trajectories of the flow corresponding to the symmetry $(D_6/C_4;O/D_2)$. The symbols have the same meaning as in Fig. 2.

the specific symmetry of these points. The symmetry of a given FP $\{u_v^*\}$ is defined as the little group G^* of the polynomial $(\sum u_v^* O_v)$. We can consider, the space E^*, containing the FP, and having G^* as a centralizer. For instance for the cubic FP, the space E^* is two dimensional and generated by the basic invariants O_0 and O_2. Any FDPE displaying $\{u_v^*\}$ as a FP is of lower symmetry than G^* and its parameter space E will contain E^* as a subspace. Thus, if $\{u_v^*\}$ is stable in a certain space, it is also stable in E^*. Invoking the unicity of the stable FP in E,* established in Sec. III C, we deduce that $\{u_v^*\}$ is invariant by the normalizer G_N^* of E^*. This requires $G_N^* \subseteq G^*$. On the other hand the general relation between the normalizer and the centralizer of E^* is $G^* \subseteq G_N^*$. *A stable fixed point is therefore necessarily characterized by the coincidence of the centralizer and the normalizer associated to it:* $G_c^* = G^* = G_N^*$.

This is a well-defined symmetry condition which allows a systematic enumeration of the possible symmetries of stable FP on the basis of group theory. In Table IX we have listed for $n=4$ the normalizers associated to each irreducible centralizer. We deduce from the comparison of the G_N and G_c reproduced on this table that there are only four possible symmetries of stable FP's: O(4), $(D_\infty \times D_\infty)^*$, $(O/D_2;O/D_2)^*$, and $(Y/C_2;Y/C_2)^*$. We know from general arguments (Secs. III B and III C) that the fixed point with the $O(n)$ symmetry is never stable for an anisotropic Hamiltonian with $n \geq 4$. This leaves three possibilities corresponding to 3 groups G_i^*. We have seen that they were effectively realized.

Furthermore, we can consider the unicity of the stable FP in the larger space E, corresponding to an FDPE with lower symmetry than G^*. This unicity requires $G_N \subseteq F^*$, where G_N is the normalizer of E. Referring to the list of possible symmetries G_i^* of the stable FP for the considered value of n ($i=1,3$ for $n=4$), we can formulate this condition as a necessary condition for the occurrence of a stable FP: *The normalizer G_N relative to the considered Hamiltonian symmetry must satisfy $G_N \subseteq G_i^*$ for one, at least, of the G_i^* groups. (Conversely, the nonfulfillment of this condition is a sufficient condition for the lack of a stable FP).*

The application of this rule allows to show, on a group-theoretical basis, that among the 21 anisotropic Hamiltonians arising for $n=4$, 10 cannot have stable FP's. For instance, the normalizer associated to $D_2 \times D_2$ is the group $(O \times O)^*$ contained in none of the G_i^* and accordingly the five-parameter space $\{u_v\}$ of symmetry $D_2 \times D_2$ cannot have a stable FP in agreement with the detailed working out of the recursion relations.

This rule can be used more generally for the search of stable FP's for phase transitions with an anisotropic order parameter with $n>4$. For such systems the $O(n)$-invariant FP is not stable. One has first to determine for each n the irreducible subgroups G_i^* of $O(n)$ which are identical to the normalizer of their FDPE. Those G_i^* represent the possible symmetries of stable FP's for the considered value of n. For a given Hamiltonian, one has then to check the condition $G_N \subseteq G_i^*$ for one of the G_i^* at least. Only in this case can a stable FP exist.

We can note that if the polynomial space considered $(\sum u_v O_v)$ has many dimensions, its global invariance group G_N is likely to be a large subgroup of $O(n)$ not contained in any of the G_i^*. Thus, we can expect, in agreement with the initial conjecture[16] stated in the introduction, that only Hamiltonians involving a small number of u_v coefficients will, generally, possess a stable FP. However, while the earlier conjecture only provided a general trend, the group-theoretical criteria established here define precise constraints.

B. Discussion of the experimental data

Let us now, analyze the experimental data relative to phase transitions having four-component OP's in light of the present theoretical results. Table X lists the available examples as well as the symmetries of the corresponding FDPE's. The systems undergoing a transition to an incommensurate phase (structural or magnetic) appear as always associated to an FDPE whose little group is a continuous subgroup of O(4). Indeed for these systems the order parameter describes the breaking of a continuous symmetry consisting in the translational invariance of the "phase angle" of the modulation.[35] For a similar reason, certain speculative types of transitions[36,37] between mesomorphic phases, will also be defined by a "continuous" FDPE. Ordinary structural or magnetic transitions corresponding to a "simple superstructure"[8] are, instead, associated with a finite subgroup of O(4).

The discussion of the experimental data can focus on two aspects. Whenever the theoretical results disclose the existence of a stable FP and when the transition is experimentally a continuous one, it is of interest to compare the experimental and theoretical values of the critical exponents. On the other hand if there is no stable FP, the relevant experimental result is the thermodynamic order of the transition since one expects a first-order transition.[38] This derives from the argument that the trajectories of the flow starting at any trivial point of the $\{u_v\}$ space will flow out of the range of positivity of the FDPE. An illustration is provided by the diagram in Fig. 4.

However the observation of a discontinuous transition does not necessarily constitute a convincing test of this

TABLE IX. Normalizers and little groups of the different polynomial spaces. The little groups and the invariant polynomial spaces are specified in columns 1 and 2. The action of the normalizers on the basic invariants are indicated for the relevant operations of the normalizer which are not contained in the little group. This action is not shown for the normalizers which are continuous groups. G^u in column 3 means that the normalizer coincides with the little group.

Little group G^u	Invariants	Normalizer G_N	Action of the normalizer	Stable fixed point?
O(4)	O_0	G^u		yes
$(Y/C_2;Y^*/C_2)$	O_0O_4	G^u		yes
$(D_\infty \times D_\infty)^*$	O_0O_1	G^u		yes
$(O/D_2;O/D_2)^*$	O_0O_2	G^u		yes
$\left[\frac{D_6}{C_4};\frac{O}{D_2}\right]$	$O_0(O_2+2O_6)$	$\left[\frac{D_{12}}{C_8};\frac{O}{D_2}\right]$		no
$\left[\frac{D_5}{C_2};\frac{D_5}{C_2}\right]^*$	O_0O_1	$\left[\frac{D_{10}}{C_2};\frac{D_{10}}{C_2}\right]^*$	$O_{13}\to(-O_{13})$	no
	O_{13}			
$D_2\times D_\infty$	O_0O_1	$O\times D_\infty$	$(C_3\mid 1)$ $O_1\to\left[O_0-\frac{O_1}{2}+O_5\right]$	no
	O_5		$O_5\to\left[\frac{O_0}{2}-\frac{3O_1}{4}-\frac{O_5}{2}\right]$	
			$(C_8^z\mid 1)$ $O_5\to -O_5$	
$\left[\frac{D_4}{D_2};\frac{D_4}{D_2}\right]$	O_0O_1	$(D_4\times D_4)^*$		yes
	O_2			
$\left[\frac{T}{D_2};\frac{T}{D_2}\right]^*$	O_0O_2	$\left[\frac{O}{T};\frac{O}{T}\right]^*$	$(C_3\mid 1)$ $O_2\to\left[\frac{3}{4}O_0-\frac{O_2}{2}+6O_3\right]$	no
	O_3		$O_3\to\left[\frac{O_2}{16}+\frac{3}{8}O_3\right]$	
$\left[\frac{D_3}{C_2};\frac{O}{D_2}\right]$	O_0O_2	$\left[\frac{D_6}{C_4};\frac{O}{D_2}\right]$	$O_2\to\frac{1}{3}(2O_0-O_2+4O_6)$	no
	O_6		$O_6\to\frac{1}{3}(-O_0+2O_2+O_6)$	
$\left[\frac{C_4}{C_2};\frac{D_4}{D_2}\right]$	$O_0O_1O_5$	$D_\infty\times D_4$	continuous	no
	$O_6O_8O_9$			
$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right]$	O_0O_1	$D_2\times D_4$	$(C_4^z\mid 1)$ $O_8\to(-O_8)$	no
	O_5O_8			
$(D_2\times D_2)^*$	O_0O_1	$\left[\frac{O}{D_2};\frac{O}{D_2}\right]^*$	$(C_8^z\mid C_8^z)$ $(O_5-4O_3)\to-(O_5-4O_3)$	no

TABLE IX. (*Continued*).

Little group G^u	Invariants	Normalizer G_N	Action of the normalizer	Stable fixed point?
$D_2\times D_2$	$O_0O_1O_2$ O_3O_5	$(O\times O)^*$	$(C_3\mid 1)$ $(O_1;O_2;O_3;O_5)$ $(C_8^z\mid 1)$ $O_2\rightarrow(\frac{3}{4}O_0-\frac{1}{2}O_2)$ $O_5\rightarrow -O_5$ $O_3\rightarrow(O_3-\frac{1}{4}O_5)$ $(C_8^z\mid C_8^z)$ $O_3\rightarrow -O_3$ $O_5\rightarrow -O_5$	no
$C_4\times D_2$	$O_0O_1O_2$ $O_3O_5O_9$ O_{10}	$D_\infty\times O$	continuous	no
$\left[\frac{C_6}{C_2};\frac{T}{D_2}\right]$	$O_0O_2O_3$ O_6O_7	$\left[\frac{D_\infty}{C_\infty};\frac{O}{T}\right]$	continuous	no
$C_2\times D_2$	O_0-O_{12}	$O(3)\times O$	continuous	no

theoretical inference since the discontinuity can have an origin other than the growth of the critical fluctuations. More specifically a minimal requirement is that the experimental results should contain simultaneously evidences for the existence of critical fluctuations in the neighborhood of the transition, and for its first order character.

Among the results reproduced in Table X, two sets of transitions are associated with the occurrence of a stable FDPE. In the two cases the symmetry of the FP is $(D_\infty\times D_\infty)^*$. The magnetic transitions in these sets were discussed previously by Mukamel *et al.*[11] These authors examined the available experimental data and pointed out that though the considered transitions were found continuous, either the data were insufficiently accurate to assign reliable values of critical exponents (e.g., for DyC_2 or $TbAu_2$), or the measured critical exponents differed signi-

TABLE X. Comparison of the available experimental data relative to the thermodynamical order of some transitions in real systems, to the theoretical results revealing the absence or presence of a stable fixed point.

FDPE symmetry	Nature of the transition	Chemical composition	Experimental thermodynamic order	Stable fixed point?	Ref.
$D_\infty\times D_\infty^*$	magnetic-incommensurate	$TbAu$;DyC_2 $DyAu_2$	2	yes	11
	structural-incommensurate	biphenyl	2	yes	39
		$Ba_2NaNb_5O_{15}$	2	yes	40
	nematic—multilayer-smectic	speculative		yes	37
$D_2\times D_\infty$	structural-incommensurate	$BaMnF_4$	1	no	45
			2		47
	nematic—multilayer-smectic	speculative		no	37
$\left[\frac{T}{D_2};\frac{T}{D_2}\right]^*$	antiferromagnetic	NdSe;NdTe; CeS;CeSe;CeTe	2	no	11
$\left[\frac{D_4}{D_2};\frac{D_4}{D_2}\right]^*$	structural	NbO_2	2	yes	41
		$PbNb_2O_6$	?	yes	9
$D_2\times D_2^*$	structural	VO_2	1	no	43
$C_4\times D_\infty$	magnetic-incommensurate	$Ca_3(Fe_2)(GeO_4)_3$	?	no	48
	nematic—multilayer-smectic	speculative		no	37

ficantly from the theoretically expected values (for Tb,Dy,Ho). For instance, in the case of the incommensurate "spiral" magnetic transition in Terbium,[38] the experimental values are $\beta=0.25$, $\gamma=1.33$ as compared to theoretical values of $\beta=0.39$ and $\gamma=1.39$. The structural transitions in biphenyl[39] and in barium sodium niobate[40] which pertain to the same theoretical scheme, are also continuous. In the first material the β exponent has, surprisingly, the classical value 0.5 in a wide temperature interval below the transition. However its value has not been probed in the close vicinity of T_c (i.e., for $\Delta T/T_C<2\times10^{-2}$). The case of barium sodium niobate is even less clear since the transition shows a diffused character with the order parameter vanishing with a horizontal tangent in the region of T_C.[40]

Only in the case of the structural transition in NbO_2 has an assignment been given of the critical exponent consistent with the expected theoretical value.[41] However the corresponding measurement is not very accurate ($0.33<\beta<0.44$) and the interpretation of the observations still requires clarification (existence of an unusually narrow crossover region to another critical regime,[41] temperature independence of the crystal periodicity inconsistent with the OP symmetry[41,42]).

Table X shows examples for 4 theoretical schemes associated with the lack of a stable FP. In this category, the transition in VO_2 is observed to be strongly of first order. This order is unlikely to be caused by the fluctuations. The behavior of physical quantities on approaching the transition show no precursor effects,[43] thus disclosing no apparent growth of the fluctuations. Besides a recent microscopic theory of this transition[44] shows that its strong first-order character can be understood even in the mean-field approximation. In the case of $BaMnF_4$, recent optical data have disclosed a slight first-order character.[45] Manifestations of the presence of critical fluctuations are also known.[46] However a number of previous measurements[47] had assigned to this transition a continuous character, and the first order cannot be considered yet as well established.

The most thorough discussion for systems with $n=4$ lacking a stable FP has been performed by Mukamel and Wallace[11] in the case of the family of magnetic transitions described by the $(T/D_2;T/D_2)^*$ FDPE. These authors have shown that for this symmetry the lack of a stable FP can be demonstrated beyond the approximations of the ϵ expansion, thus confirming that one is entitled to expect a first order transition. By contrast, careful experimental examination by two different techniques have shown that the transitions in CeSe, CeTe, CeS, appear as continuous with critical exponents which, within the experimental uncertainties, are consistent with the values of Eq. (22).

The preceding evaluation of the experimental data thus reveals an unsatisfactory situation. Either the experimental data contain some uncertainties or inaccuracies not yet resolved, or they are found to be in clear contradiction with the expectations. For systems in which the $\{u_\nu\}$ flow lacks a stable FP, this overall disagreement had already been pointed out in the broader framework of systems possessing on OP with $n\geq4$.[48]

Judging from the case probed in detail by Mukamel and Wallace,[11] the theoretical results drawn from the ϵ expansion do not seem questionable. Thus, we can rather infer from the experimental observations that the first-order character induced by fluctuations must be, in all cases, very slight. In the examples for which a pronounced first order had been convincingly attributed to the effect of fluctuations (e.g., in the magnetic transition in MnO,[48] corresponding to $n=12$), additional effects, such as a strong coupling of the OP to strains, can be considered at the origin of the magnitude of the discontinuity.

As for the matching of the measured critical exponents with the calculated ones, a systematic discrepancy has also been recently noted[49] in all the structural phase transitions in which the expected behavior is nonclassical. The reason underlying this situation has been assigned to the sensitivity of structural transitions to the influence of defects, always present in the real systems, and whose modification of the critical behavior can be drastic. The present analysis shows that such a discrepancy even extends to some examples of magnetic systems.

VI. CONCLUSIONS

In the present work we have completed the theoretical framework describing the critical behavior of phase transitions with n-component order parameters by specifying this framework for $n=4$. We have seen that in the same manner as for other values of $n>3$, one does not find in the available experimental data a satisfactory illustration of the predictions of the theory. The successes of the RG method for other classes of transitions warrant, nevertheless, that the theoretical results obtained here provide the correct situation of reference for systems with $n=4$. The observed deviations must be interpreted as disclosing the influence of a variety of possible effects in the real systems such as the presence of defects.

On the other hand, the examination of the complex set of effective Hamiltonians which arise for $n=4$ has led us to rely systematically on the symmetry properties of the RG recursion relations and on those of the pattern of fixed points. Along this line we have extended methods and specified inferences previously stated by a number of authors, and especially clearly by Korzhenevskii[25] and by Jaric.[26] In particular, we have based on the unicity of the stable fixed point, the group-theoretical enumeration of the possible symmetries of these points and the formulation of a necessary condition for their occurrence. This formulation specifies the group-theoretical condition relative to the absence or presence of a stable FP, which had been searched for previously.[15,16] Furthermore, we have illustrated the fact that symmetry considerations allow one to avoid, in many cases, the actual solving of the RG fixed-point equations.

APPENDIX A: NOTATION OF THE CLOSED SUBGROUPS OF O(4)

The notation used in the paper is due to Du Val.[20] It is based on the following properties.

(i) SO(3) and SO(4) are homomorphic images of SU(2) and SU(2)×SU(2), respectively, with a two-element kernel. The first homomorphism allows to use for the subgroups of SU(2) the same label as that of their image in SO(3): O, T, and D_m [which have, respectively, 48, 24, and $4m$ elements as subgroups of SU(2)]. Du Val[20] does not apply the same labeling rule for the cyclic groups. Thus, C_n is an n-element cyclic subgroup of SU(2). The second homomorphism allows to label the elements of SO(4) by pairs (g_1,g_2) of elements of SU(2), with the general relation $(g_1,g_2)\sim(-g_1,-g_2)$.

(ii) Let L, L_k, R, R_k be four subgroups of SU(2) satisfying the conditions that L_k (R_k) is an invariant subgroup of L (R) and that the two quotient groups (L/L_k) and (R/R_k) are isomorphic. We can write

$$L=\bigcup_{i=1}^{m} g_i\cdot L_k \ \left(R=\bigcup_{i=1}^{m} g_i'\cdot R_k\right),$$

where $g_i\cdot L_k$ and $g_i'\cdot R_k$ are isomorphic elements of the quotient groups. Then, the elements of SO(4) of the form $(g_i\cdot L_k, g_i'\cdot R_k)$ (same value of i for g and g'), constitute a subgroup G of SO(4) denoted $(L/L_k,R/R_k)$. Besides, one generates in this way all the subgroups of SO(4). In the case $(L=L_k;R=R_k)$ the group is the direct product $L\times R$, and we use this simpler notation. The order of G is half the product of three terms: the order of L_k, the order of R_k, and the order of (L/L_k). For instance $D_2\times D_2$ has 32 elements $(\frac{1}{2}\times 8\times 8)$.

(iii) The subgroups of O(4) comprise, besides those of SO(4), additional groups of the form $G^*=G+SC\times G$, where S is an element of SO(4), C is the "axial rotation" (i.e., the transformation represented by the matrix $C_{ij}=c_i\delta_{ij}$, with $c_i=1$ for $i=1,2,3$ and $c_i=-1$ for $i=4$), and G is a subgroup of SO(4) satisfying $L_k=R_k$, $L=R$. The subgroups G^* are denoted $G^*=(L/L_k,R/R_k)^*$.

APPENDIX B: SCALAR PRODUCTS AND SYMMETRIC PRODUCTS FOR THE BASIC INVARIANTS O_ν

These products are required to solve the set of fixed-point Eqs. (13), (16), and (16′) for the five Hamiltonians investigated in Secs. III and IV. From left to right, we specify the FDPE symmetry, the table of scalar products $(O_\nu,O_{\nu'})$ and the table of symmetric products $(O_\nu \vee O_{\nu'})$.

$(Y/C_2;Y^*/C_2)^*$	O_0	O_4	O_0	O_4
	8	28	$\frac{4}{3}O_0$	$\frac{7}{3}O_0+\frac{2}{3}O_4$
		178		$9O_0+4O_4$

$\left[\frac{D_5}{C_2};\frac{D_5}{C_2}\right]^*$	O_0	O_1	O_{13}	O_0	O_1	O_{13}
	8	$\frac{16}{3}$	0	$\frac{4}{3}O_0$	$\frac{4}{9}O_0+\frac{2}{3}O_1$	$\frac{2}{3}O_{13}$
		$\frac{16}{3}$	0		$\frac{10}{9}O_1$	$\frac{1}{3}O_{13}$
						$\frac{1}{2}O_0-\frac{1}{4}O_1$

$\left[\frac{D_3}{C_2};\frac{O}{D_2}\right]$	O_0	O_2	O_6	O_0	O_2	O_6
	8	4	0	$\frac{4}{3}O_0$	$\frac{1}{3}O_0+\frac{2}{3}O_2$	$\frac{2}{3}O_6$
		4	0		O_2	$\frac{1}{6}O_6$
			1			$\frac{1}{8}O_0-\frac{1}{12}O_2+\frac{1}{6}O_6$

$\left[\frac{D_2}{D_1};\frac{D_4}{D_2}\right]$	O_0	O_1	O_5	O_8	O_0	O_1	O_5	O_8
	8	$\frac{16}{3}$	0	0	$\frac{4}{3}O_0$	$\frac{4}{9}O_0+\frac{2}{3}O_1$	$\frac{2}{3}O_5$	$\frac{2}{3}O_8$
		$\frac{16}{3}$	0	0		$\frac{10}{9}O_1$	$\frac{2}{9}O_5$	$\frac{1}{3}O_8$
			$\frac{4}{3}$	0				
				4			$\frac{2}{9}O_0-\frac{1}{6}O_1$	$\frac{1}{6}O_8$
								$\frac{1}{2}O_0-\frac{1}{4}O_1+\frac{1}{2}O_5$

$D_2\times D_2$	O_2	O_3	X_1	X_2	X_3	O_2	O_3	X_1	X_2	X_3
	4	0	0	0	0	O_2	0	$\frac{1}{3}X_1$	$\frac{1}{3}X_2$	$\frac{1}{3}X_3$
		$\frac{1}{24}$	0	0	0					
			$\frac{1}{3}$	0	0		$\frac{1}{72}(X_1+X_2+X_3)$	$\frac{1}{9}O_3$	$\frac{1}{9}O_3$	$\frac{1}{9}O_3$
				$\frac{1}{3}$	0			$\frac{1}{36}(O_2+8X_1)$	$\frac{1}{18}X_3$	$\frac{1}{18}X_2$
					$\frac{1}{3}$				$\frac{1}{36}(O_2+8X_2)$	$\frac{1}{18}X_1$
									$\frac{1}{36}(O_2+8X_3)$	

APPENDIX C: FIXED-POINT COORDINATES IN THE VICINITY OF THE ISOTROPIC FIXED POINT, AND UNIQUENESS OF THE STABLE FIXED POINT

1. Uniqueness of the stable FP of the form $\epsilon[(s/2)+b\epsilon]$

As mentioned in Sec. III A, the β functions involved in the recursion relations are the gradient field of a potential F whose general form at two-loop order has been indicated in Ref. 23. In the vicinity of the isotropic FP, for $g=\epsilon[(s/2)+h]=\epsilon[(s/2)+b\epsilon]$, F takes the form

$$\epsilon^{-3}F=\text{const}+\frac{\epsilon^2}{16}\left[(s,b)^2-\frac{(s,b)}{3}\right]+\epsilon^3\left[\frac{(b_{\vee}b,b)}{2}-\frac{(s,b)^2}{24}-\frac{5(b,b)}{48}\right] . \tag{C1}$$

If g^* is a FP, we can apply the set of Eqs. (16) and (16′) and eliminate $(s,b)^*$ and $(b^*{}_{\vee}b^*)$ from Eq. (C1). We obtain

$$\epsilon^{-3}F=\text{const}-\frac{\epsilon^2}{2^6 3^2}-\frac{19\epsilon^3}{2^9 3}-\frac{\epsilon^3}{2^2 3^2}(b^*,b^*) . \tag{C2}$$

The stable FP corresponds to the lowest value of F and thus to the largest value of the "length" (b^*,b^*).

Let us assume that the stable FP is not unique and consider two stable FP's associated with the values b_1^* and b_2^*. Necessarily, $(b_1^*,b_1^*)=(b_2^*,b_2^*)$. Along the line joining the two FP's, defined by $b=\{\lambda b_1^*+(1-\lambda)b_2^*\}$, the potential (C1) becomes a third degree scalar function of λ having two equal minima for $\lambda=0$ and $\lambda=1$ and is therefore independent of λ. As a consequence, if we consider the matrix of second derivatives $F^{(2)}=\partial^2F/\partial^2 b_{ijkl}$, its average value taken for the vector $(b_1^*-b_2^*)$ vanishes along the line. However one can show that $(b_1^*-b_2^*)$ is not an eigenvector of $F^{(2)}$ corresponding to the zero eigenvalue. Indeed if the opposite was true, we could draw from Eqs. (16)and (16′):

$$\epsilon^{-4}F^{(2)}(b_1^*-b_2^*)\,|_{b_1^*}=3b_1^*{}_{\vee}b_2^*-\frac{5\epsilon}{24}(b_1^*+b_2^*)-K(b_1^*)s=0 . \tag{C3}$$

Equation (C3) is symmetric with respect to the permutation of b_1^* and b_2^*, since $K(b)$ only depends on the length of b. Projecting (C3) successively on b_1^* and on b_2^* and subtracting the resulting equations we find

$$(b_1^*,b_1^*)=(b_1^*,b_2^*)=(b_2^*,b_2^*) , \tag{C4}$$

which implies that $b_1^*=b_2^*$ in contradiction with the initial assumption.

The zero average of $F^{(2)}$ for a vector which is not one of its eigenvectors, requires then that this symmetric matrix possesses both positive and negative eigenvalues for b_1^* or b_2^*. This is also in contradiction to the assumed stability of these FP which implies that all the eigenvalues are non-negative. Therefore the stable fixed point is unique.

2. Fixed-point equations for h^* of order $\epsilon^{1/2}$

We proceed as in Sec. III C and project Eq. (15) on the direction of s and on the perpendicular directions. Putting $h=(h_1\epsilon^{1/2}+h_2\epsilon)$ and retaining the two lowest orders in the equations, we obtain

$$(s,h_1)=0 , \tag{C5}$$

$$(s,h_2)+(h_1,h_1)=\tfrac{1}{6} , \tag{C5′}$$

$$\tfrac{3}{2}(h_1{}_{\vee}h_1)=\frac{(h_1,h_1)s}{8} , \tag{C5″}$$

$$\tfrac{3}{2}(h_1{}_{\vee}h_2)-\tfrac{5}{48}h_1=\frac{(h_1,h_2)s}{8} . \tag{C5‴}$$

Projecting (C5‴) on h_1 and using (C5′) and (C5″), it is easy to find

$$(h_1,h_1)=-\tfrac{2}{3} , \tag{C6}$$

which can only be achieved if h_1 is imaginary.

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