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Fluctuation effects near H_{c2} in type-II superconductors

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Abrikosov's mean-field theory of type-II superconductors predicts an unusual continuous freezing transition to a triangular flux lattice at H_{c2}. We show that fluctuations, which are negligible above $d=6$, drive this transition first order to lowest order in an expansion in $\epsilon=6-d$.

I. INTRODUCTION

In 1958 Abrikosov published[1] a remarkable mean-field analysis of type-II superconductors using the Ginzburg-Landau phenomenological free energy

$$H=\int d^3r\left[\frac{1}{2m^*}\,|(\nabla+ie^*\mathbf{A})\psi|^2+a\,|\psi|^2+b\,|\psi|^4+\frac{1}{2\mu_0}(\nabla\times\mathbf{A}-\mathbf{H})^2\right]. \qquad (1.1)$$

We have used the standard notation[2] for the Ginzburg-Landau parameters in this functional of the superconducting order parameter $\psi(\mathbf{r})$ and have set $\hbar=c=1$. We shall be interested here in the behavior near H_{c2}, where fluctuations in the magnetic field can be neglected. Upon taking $\langle H\rangle\,||\hat{z}$, the free energy[1] simplifies in this limit to

$$H=\int d^3r\left[\frac{1}{2m^*}\,|(\nabla+ie^*\mathbf{A}_0)\psi|^2+a\,|\psi|^2+b\,|\psi|^4\right], \qquad (1.2a)$$

where the vector potential is fixed (up to a gauge transformation) to be

$$\mathbf{A}_0(\mathbf{r})=\tfrac{1}{2}H(-y,x,0)\,. \qquad (1.2b)$$

Although Abrikosov's main contribution was, of course, to provide a description of an entirely new class of superconducting materials, his analysis is also directly relevant to several problems in the modern theory of phase transitions.

The free energy (1.2) is in fact the continuum limit of a class of "uniformly frustrated" XY models, used by several authors as simplified models of spin glasses.[3,4] It also describes superfluidity in a sample of He^4 rotated at a constant frequency $\Omega\sim H$ about the $\hat{z}$ axis.[5] The term "frustration" is used because it is impossible to make the gradient term vanish everywhere. One cannot minimize the gradient energy by imposing the "obvious" phase relation,

$$\psi(\mathbf{r})=\psi(\mathbf{0})\exp\left[-ie^*\int_0^{\mathbf{r}}\mathbf{A}_0\cdot d\mathbf{l}\right], \qquad (1.3)$$

because the resulting order parameter field is multivalued for nonzero H. The frustration introduces an extra length scale, $(e^*|H|)^{-1/2}$, which limits the growth of order-parameter correlations in the plane perpendicular to the magnetic field.[6] Within mean-field theory,[1,2] this frustration is eventually accommodated with decreasing temperatures (or decreasing fields, at fixed T) by a second-order phase transition at $H_{c2}(T)$ to a triangular Abrikosov flux lattice of vortices in the order parameter. Uniformly frustrated models similar to Eq. (1.2) with a *non-Abelian* order parameter have been proposed recently to describe the statistical mechanics of glass.[7]

Abrikosov's calculation is also interesting because it describes a very unusual continuous freezing transition. Crystallization of atoms and molecules is accompanied by modulations in the particle density which are similar to the modulations in $|\psi(\mathbf{r})|^2$ in the triangular Abrikosov flux lattice state. A general argument, due to Landau, shows that such transitions must be first order within mean-field theory.[8] We shall argue below that the mean-field transition to the Abrikosov flux lattice is continuous only because the density which becomes modulated [i.e., $|\psi(\mathbf{r})|^2$] is itself only appreciable below $H_{c2}(T)$.

In this paper we study the effect of fluctuations on the continuous freezing transition found by Abrikosov. Although fluctuation effects are difficult to observe experimentally in ordinary type-II superconductors, they may be more important in the recently discovered "heavy fermion" materials.[9] We shall restrict our attention here to s-wave pairing, and not consider the more exotic proposals for p-wave superconductivity. Fluctuations play an important role near $H_{c2}(T)$ below six dimensions. The leading order fluctuation contribution to the specific heat above H_{c2}, for example, is readily shown using the methods of Sec. II to be given in d dimensions by

$$C(\tau)=\int d^d r[\langle |\psi(\mathbf{r})|^2|\psi(\mathbf{0})|^2\rangle-\langle|\psi(\mathbf{0})|^2\rangle^2]$$
$$\sim\int\frac{d^{d-2}q}{(q^2+\tau)^2}$$
$$\sim\tau^{-(6-d)/2}\,, \qquad (1.4)$$

where τ is a parameter which vanishes on the mean-field transition line $H_{c2}(T)$. Below $d=6$, this fluctuation correction to the specific heat appears to diverge, invalidating Abrikosov's results in a way we can study via a perturbation expansion in $\epsilon=6-d$. To lowest order in ϵ, we find that the transition is always driven first order by the fluctuations. This result can be understood by noting that Eq. (1.4) shows that the "condensate density"

$$\rho_c(\mathbf{r})=|\psi(\mathbf{r})|^2 \qquad (1.5)$$

starts to develop long-range correlations when d drops below 6. The quantity $\rho_c(\mathbf{r})$ is then like the particle density in Landau's theory[8] of first-order freezing, which we expect to be rather reliable in $6-\epsilon$ dimensions. Although this point of view neglects the phase degrees of freedom, we do not expect phase modes to be important, since they are decorrelated by the applied magnetic field. We should emphasize, however, that our starting point is quite far from Landau's theory of freezing. The calculations are nontrivial, requiring that we generalize standard renormalization group methods[10] to allow for an infinite number of marginal operators near $d=6$.

Because of the physical arguments given above, our qualitative prediction of a first-order transition at H_{c2} in the presence of fluctuations is probably correct even in $d=3$. In two dimensions, however, fluctuations drive the superconducting transition quite far below its mean-field value, and it may be appropriate to think of the normal phase as a disordered liquid of point vortices. As discussed, e.g., by Fisher,[11] continuous transformations to an Abrikosov flux lattice are again possible, mediated by disclination and dislocation pairing transitions.

In Sec. II we present the renormalization-group analysis of the free energy (1.2) in $6-\epsilon$ dimensions. We show that fluctuations in the magnetic field do not change our results in Sec. III. Several technical calculations are described in the Appendix.

II. LARGE-κ LIMIT

A. The Landau-Ginzburg model near H_{c2}

We first consider the large-κ limit,[2] in which the spatial extent of the fluctuations of the microscopic magnetic field is large compared to the correlation length of the superconducting order parameter. In such materials, in the normal phase (H larger than H_{c2}) we can ignore the fluctuations of the magnetic field. In this simplified limit the magnetic field may be regarded as uniform and, up to an additive constant, one may write the Landau-Ginzburg free energy (1.2), generalized to d dimensions. We choose the "symmetric" gauge in which

$$\mathbf{A}=\frac{H}{2}(-y,z;\underline{0})\,. \qquad (2.1)$$

This d-dimensional generalization of a constant field, perpendicular to the x-y plane, corresponds to an electromagnetic tensor F_{ij}, whose only nonzero components are $F_{12}=-F_{21}=H$. The order parameter ψ is a function of x,y and of $(d-2)$ transverse coordinates $\underline{\mathbf{r}}_\perp$. It is convenient to use the complete basis of functions of (x,y) which are the eigenfunctions of the two-dimensional Hamiltonian

$$h_0=\frac{1}{2m^*}(i\underline{\nabla}+e^*\underline{\mathbf{A}})^2 \qquad (2.2)$$

The eigenstates of h_0, the Landau levels, are simple harmonic oscillator wave functions:[12]

$$h_0U_{n,m}(x,y)=(n+\tfrac{1}{2})\frac{e^*H}{m^*}U_{n,m}(x,y),\quad n,m\geq 0\,. \qquad (2.3)$$

The index n labels the energy eigenvalues of the Landau levels and m labels their degeneracy, which is proportional to the area of the system in the (x,y) plane.

If we expand the order parameter in this basis,

$$\psi(x,y;\underline{\mathbf{r}}_\perp)=\sum_{n,m}\varphi_{n,m}(\underline{\mathbf{r}}_\perp)U_{n,m}(x,y)\,, \qquad (2.4)$$

we diagonalize the quadratic part of the free energy $\mathscr{H}_0$:

$$\mathscr{H}_0=\int d^dx\left[\frac{1}{2m^*}|(i\nabla+e^*\mathbf{A})\psi|^2+a|\psi|^2\right]=\sum_{n,m\geq 0}\int d^{d-2}r_\perp\left\{\frac{1}{2m^*}|\underline{\nabla}_\perp\varphi_{n,m}|^2+\left[a+\left[n+\frac{1}{2}\right]\frac{e^*H}{m^*}\right]|\varphi_{n,m}|^2\right\}. \qquad (2.5)$$

Below the zero-field normal-to-superconducting transition, the parameter $a(T)$ is negative, but for large enough magnetic field the coefficients of all the $|\varphi_{n,m}|^2$ in (2.5) remain positive. When the field is decreased and reaches the value $H_{c2}(T)$ defined by

$$\frac{e^*H_{c2}}{2m^*}=-a(T)\,, \qquad (2.6)$$

the coefficients of the $|\varphi_{0,m}|^2$ modes vanish, whereas those of the nonzero Landau modes, i.e., $\varphi_{n,m}$ with $n\geq 1$, remain positive. Therefore in the critical region of the normal—to—type-II superconducting transition (if indeed a second-order transition persists when fluctuations are included), one can ignore completely all the nonzero Landau levels and replace the order parameter ψ by

$$\psi \to \psi_0 = \sum \varphi_m(\underline{r}_\perp) U_{0,m}(x,y) \; . \tag{2.7}$$

In the gauge that we have chosen, the $n=0$ Landau levels are spanned by the normalized eigenfunctions

$$U_{0,m} = (\pi m!)^{-1/2} (\mu^2/2)^{(m+1)/2} (x+iy)^m \times \exp[-\tfrac{1}{4}\mu^2(x^2+y^2)], \quad m \geq 0 \tag{2.8}$$

$$\mu^2 \equiv e^* H \; . \tag{2.9}$$

The critical order parameter ψ_0 is therefore an arbitrary linear combination of the $U_{0,m}$, or equivalently up to the Gaussian factor $\exp[-\frac{1}{4}\mu^2(x^2+y^2)]$, an arbitrary function of the variable $z=x+iy$. Therefore the restriction to the $n=0$ Landau level is fully implemented by the condition

$$\psi_0(x,y;\underline{r}_\perp) = \varphi(z,\underline{r}_\perp) \exp(-\tfrac{1}{4}\mu^2 z^* z) \; , \tag{2.10}$$

in which φ is an holomorphic function of z (i.e., $\partial\varphi/\partial z^*=0$). This representation of the $n=0$ subspace is more convenient,[13] especially for the quartic terms of the free energy, than the explicit decomposition (2.7). The Landau-Ginzburg free energy now reads (after a rescaling of φ)

$$\mathscr{H} = \int d^{d-2}\underline{r}_\perp \int dz \int dz^* [(|\underline{\nabla}_\perp \varphi|^2 + \tau|\varphi|^2) \exp(-\tfrac{1}{2}\mu^2 z^* z) + \tfrac{1}{4} g |\varphi|^4 \exp(-\mu^2 z^* z)] \; , \tag{2.11}$$

in which τ, defined as

$$\tau = 2m^* a(T) + e^* H \; , \tag{2.12}$$

is proportional to $H - H_{c2}$, and $\int dz \int dz^*$ means an integral over the (x,y) plane.

In the following we consider the vicinity of H_{c2} in the normal region, in which τ is small and positive. We shall see that fluctuations are important below six dimensions and take them into account in the vicinity of six dimensions thanks to the renormalization-group formalism.

B. Perturbation theory and renormalization—the upper critical dimension

Following Wilson[10] we now regard the Landau-Ginzburg free energy $\mathscr{H}$ as a Boltzmann weight and imagine expanding in powers of the quartic terms in the order parameter. This yields a series of generalized ϕ^4 Feynman diagrams. The propagator is the inverse of the quadratic form in φ contained in $\mathscr{H}$. For the $d-2$ transverse directions we perform as usual a Fourier decomposition, but for the (x,y) (or rather z-z^*) plane we note that for an arbitrary holomorphic function one has the identity

$$\int dz \int dz^* e^{-\mu^2|z|^2/2} \left[\frac{\mu^2}{2\pi} \exp(\tfrac{1}{2}\mu^2 z' z^*) \right] f(z) = f(z') \; , \tag{2.13}$$

easily verified by expanding f in a power series of z.

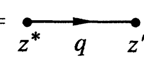
Defining the inverse of an operator $K(z^*,z)$ acting on the space of holomorphic functions by

$$\int dz \left[\int dz^* K^{-1}(z',z^*) K(z^*,z) \right] f(z) = f(z') \; , \tag{2.14}$$

we see that the identity (2.13) proves that the inverse of the kernel $K(z^*,z) = \exp(-\frac{1}{2}\mu^2 z^* z)$ is simply equal to

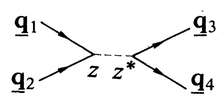
$$K^{-1}(z',z^*) = \frac{\mu^2}{2\pi} \exp(\tfrac{1}{2}\mu^2 z' z^*) \; . \tag{2.15}$$

Consequently the propagator $\Delta(\underline{q};z^*,z')$ is equal to

$\Delta(\underline{q}_\perp;z^*,z') =$ 

$$= (q_\perp^2 + \tau)^{-1} \frac{\mu^2}{2\pi} \exp(\tfrac{1}{2}\mu^2 z^* z') \tag{2.16a}$$

and the (bare) quartic vertex to



$$= -\tfrac{1}{4} g \delta^{(d-2)} \left[\sum_{i=1}^{4} q_i \right] \exp(-\mu^2 z z^*) \; . \tag{2.16b}$$

An arbitrary diagram consists therefore of a Gaussian integral over the variables (z,z^*) and of an integral over $(d-2)$-dimensional loop momenta $\underline{q}_\perp$. Near the transition, when (the renormalized) τ vanishes, long-distance singularities emerge out of these $\underline{q}_\perp$ integrations. The (z,z^*) integrations yield simple constants (related to the number of Euler trails of the diagrams[14]) which multiply the result of the $\underline{q}_\perp$ integrations. Therefore critical fluctuations are concentrated in the transverse $(d-2)$-dimensional modes, and the standard Wilson analysis shows that the upper critical dimension is given by

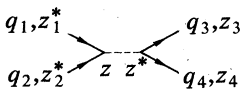
$$d_c - 2 = 4 \; . \tag{2.17}$$

In conclusion we have shown that in the fluctuating order parameter $\psi(z,\underline{r}_\perp)$, the variable z is simply an index labeling the field, whereas fluctuations take place in the $\underline{r}_\perp$ directions.

Above six dimensions the classical picture remains valid and one recovers Abrikosov's second-order transition. Below six dimensions fluctuations are essential and we performed a $6-\epsilon$ expansion in order to explore the nature of the transition. At first order in ϵ, this requires an analysis of the one-loop diagrams. There are several types of such diagrams and their contributions are given in the Appendix, but it is important to characterize the differences between the bare (tree-level) vertex and some of the diagrams. Let us compare two different one-loop diagrams to the bare four-point function.

From (2.16) we obtain for this function

$$G^{(4)}_{\text{bare}}(\underline{q}_i, z_i) =$$



$$= -g \prod_{\lambda=1}^{4} (q_i^2+\tau)^{-1} \delta \left[\sum_{i=1}^{4} \underline{q}_i \right] \left[\frac{\mu^2}{2\pi} \right]^4 \int dz\, dz^* \{ \exp \tfrac{1}{2}\mu^2[(z_1^* + z_2^*)z + (z_3+z_4)z^* - 2zz^*] \}$$

$$= -\tfrac{1}{2} g \prod_{1}^{4} (q_i^2+\tau)^{-1} \delta \left[\sum_{i=1}^{4} \underline{q}_i \right] \left[\frac{\mu^2}{2\pi} \right]^3 \exp[\tfrac{1}{4}\mu^2 (z_1^* + z_2^*)(z_3+z_4)] \ , \tag{2.18}$$

which we compare with the two one-loop diagrams shown in Fig. 1.

An elementary explicit calculation (see the Appendix) shows that the z dependence of Fig. 1(a) is proportional to $\exp[\frac{1}{4}\mu^2(z_1^*+z_2^*)(z_3+z_4)]$ which is the same as in (2.18) for the bare function, whereas that of Fig. 1(b) is proportional to $\exp\{\frac{1}{2}\mu^2[z_1^*(2z_3+z_4)+z_2^*(z_3+2z_4)]\}$. Therefore the simple Landau-Ginzburg Hamiltonian is not closed under renormalization; new marginal ϕ^4 interactions are generated by fluctuations and we have to allow for such operators in the interaction. These new interactions will also in turn generate other vertices; clearly a more general approach is needed.

C. The renormalized Hamiltonian and renormalization-group equations

The most general ϕ^4 operator, local in $\underline{r}_\perp$, taking into account holomorphy ($n=0$) and reality, may be written as

$$\int d^{d-2} r_\perp \int dz_1 dz_1^* \, dz_2 dz_2^* \, F(z_1, z_1^*, z_2, z_2^*) \times \varphi^*(z_1^*, \underline{r}_\perp) \varphi^*(z_2^*, \underline{r}_\perp) \varphi(z_1, \underline{r}_\perp) \varphi(z_2, \underline{r}_\perp) \ . \tag{2.19}$$

However, we obtain further restrictions from translational invariance. The gauge that we have chosen in Eq. (2.1) singles out the origin; therefore a translation $(x,y) \to (x+x_0, y+y_0)$ must be accompanied by a gauge transformation. It is straightforward to show from the behavior of ψ under a gauge transformation and the definition (2.10) of φ that the Hamiltonian is invariant under the transformation

$$\varphi(z, r_\perp) \to \varphi(z-a, r_\perp) \exp[\tfrac{1}{2}\mu^2(a^*z - \tfrac{1}{2}a^*a)] \ , \tag{2.20}$$

where $a = x_0 + iy_0$.

It is elementary to verify that the quadratic terms

$$\int dz\, dz^* \varphi^*(z^*) \varphi(z) \exp(-\tfrac{1}{2}\mu^2 z^* z)$$

are indeed invariant under (20). For the interaction (19) this invariance, together with reality, implies the restriction

$$F(z_1, z_1^*, z_2, z_2^*) = \{ \exp[-\tfrac{1}{2}\mu^2(z_1^* z_1 + z_2^* z_2)] \} \times g(|z_1 - z_2|^2) \ . \tag{2.21}$$

Before renormalization g is a delta function constraining z_2 and z_2^* to be equal to z_1 and z_1^*, respectively, but we must generalize the perturbative analysis to allow for an *arbitrary function* g of a single variable in the interaction. There is thus a continuum of marginal operators in $d=6$, and the renormalization-group equations in $d=6-\epsilon$ will involve nonlinear integral equations on the function g. For this renormalization analysis we thus start from

$$\mathscr{H} = \int d^{d-2}r \int dz\, dz^* \, e^{-\mu^2|z|^2/2} (|\underline{\nabla}_\perp \varphi|^2 + \tau|\varphi|^2) + \int d^{d-2}r \int dz_1 dz_1^* \, e^{-\mu^2|z_1|^2/2} dz_2 dz_2^* \, e^{-\mu^2|z_2|^2/2} \times g(|z_1-z_2|^2) |\varphi(z_1, \underline{r}_\perp)|^2 |\varphi(z_2, \underline{r}_\perp)|^2 \ . \tag{2.22}$$

We will use standard dimensional regularization and the minimal subtraction method.[15] In this method let us recall that we have a set of marginal coupling constants g_i^B, related to the renormalized ones g_i, by an expansion

$$g_i^B = g_i + \frac{1}{\epsilon} \gamma_{ijk} g_j g_k + O(g^3) \ . \tag{2.23}$$

We then obtain the one-loop β functions as

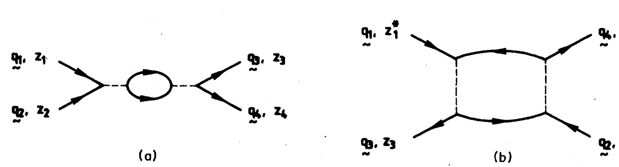


FIG. 1. Two different one-loop diagrams, which are compared to the bare four-point function.

$$\beta_i = -\epsilon g_i + \gamma_{ijk} g_j g_k + O(g^3) . \quad (2.24)$$

In our problem the index i is continuous, since there is a continuum of coupling constants $g(u)$, and the analog of (24) is

$$\beta[g(u)] = -\epsilon g(u) + \int dv\, dw\, \gamma[u,v,w] g(v) g(w) + O(g^3) . \quad (2.25)$$

It is more convenient to parametrize $g(|z-z'|^2)$ in terms of a weight $\rho(x)$, defined by

$$g(u) = \tfrac{1}{2} \int_0^\infty \frac{dx}{x} \rho(x) e^{-(\mu^2/4x)u} . \quad (2.26)$$

The function $g(u)$ reduces to the (unrenormalized) Landau-Ginzburg model when $\rho(x)$ has a vanishingly small support around $x=0$. The x interval in Eq. (26) is restricted to the positive axis (i) in order to give an interaction in the Hamiltonian (22) which falls off at large $|z_1|$ and $|z_2|$—this eliminates the range $-1 \le x \le 0$; (ii) in order to give a nondivergent shift of T_c (at finite lattice spacing). The interaction defined by (26) yields a well-defined perturbation theory provided $\int_0^\infty dx\, \rho(x)/(1+x) < \infty$; this condition is required in order to have a well-defined bare four-point function at coinciding points (as shown in the Appendix).

We obtain therefore functional renormalization-group equations of the form

$$\frac{d\rho_t(x)}{dt} = \beta[\rho_t(x)] \quad (2.27)$$

for a dilatation e^{-t} of the length scale (t is negative). At the one-loop level we obtain

$$\beta[\rho(x)] = -(6-d)\rho(x) + \int dx_1\, dx_2\, \gamma(x;x_1,x_2) \rho(x_1) \rho(x_2) + O(\rho^3) \quad (2.28)$$

with (after lengthy calculations summarized in the Appendix) the explicit expression

$$\begin{aligned}\gamma(x;x_1,x_2) = {} & \delta(x(1+x_1+x_2) - x_1x_2) + \delta(2x(x_1+\tfrac{1}{2})(x_2+\tfrac{1}{2}) + x_1x_2 - \tfrac{1}{4}) + \delta(x(1+x_1+x_2) - x_1x_2 + \tfrac{1}{4}) \\ & + 2\delta(x - x_1 - x_2 - \tfrac{1}{2}) + \delta(x(x_1+\tfrac{1}{2}) - (x_1x_2 + \tfrac{1}{2}x_2 + \tfrac{1}{4})) + \delta(x(x_2+\tfrac{1}{2}) - (x_1x_2 + \tfrac{1}{2}x_1 + \tfrac{1}{4})) . \end{aligned} \quad (2.29)$$

D. Numerical study of the renormalization-group equations

Fixed "points" $\rho^*(x)$, which here are fixed functions of one variable, are solutions of the nonlinear integral equation on the positive real line

$$(6-d)\rho^*(x) = \int_0^\infty dx_1 \int_0^\infty dx_2\, \gamma(x;x_1,x_2) \times \rho^*(x_1) \rho^*(x_2) \quad (2.30)$$

with ρ^* of order ϵ ($=6-d$), and γ given Eq. (2.29).

Stability requires that the eigenvalues of the operator

$$\Omega(x,x') = \left. \frac{\partial \beta[\rho(x)]}{\partial \rho(x')} \right|_{\rho=\rho^*} = -\epsilon\delta(x-x') + 2 \int dy\, \rho^*(y) \gamma(x;x',y) \quad (2.31)$$

should have positive real parts; this eliminates as usual the fixed point $\rho^*=0$ below six dimensions.

We have solved numerically the renormalization-group flow equations (2.28)—(2.29), in short-hand notation

$$\frac{1}{2} \frac{d\rho}{dt} = -\rho + \rho \cdot \rho , \quad (2.32)$$

by discrete "time" increments, letting t, which is the opposite of the logarithm of the rescaling factor, go to $-\infty$. We have tried several different initial conditions, including of course the bare interaction. In all cases, after a time which depends upon the initial conditions, the iteration leads to a negative $\rho_t(x)$. If fluctuations are such that $\rho_t(x)$ becomes negative (in fact, if its Laplace transform takes nonpositive values) the free energy becomes unbounded and negative for some field configurations. Higher order terms must be taken into account for stabilization and we conclude that, as usual, this implies a first-order transition. For explicit calculations of first-order transitions in similar cases in $d=4-\epsilon$, see Rudnick[16] and Chen *et al.*[17]

III. THE GENERAL MODEL

In real type-II superconductors order-parameter fluctuations are coupled to fluctuations in the magnetic field about its average value, as in the usual Landau-Ginzburg free energy. We shall show that near H_{c2}, assuming again a possible second-order transition in order to eliminate irrelevant variables, this model is identical to that of the preceding section. The argument consists of tracing out the fluctuating part of the electromagnetic field and of examining the resulting effective free energy for the order parameter.

Let us first recall that the total Gibbs free-energy density consists of several pieces:

$$G_{\text{tot}} = \frac{1}{2m^*} |(\underline{\mathbf{p}} - e^* \underline{\mathbf{A}}_{\text{tot}})\psi|^2 + a|\psi|^2 + b|\psi|^4 + \frac{1}{2\mu_0} (\underline{\nabla} \times \underline{\mathbf{A}}_{\text{tot}} - \underline{\mathbf{H}}_{\text{ext}})^2 , \quad (3.1)$$

where $\underline{\mathbf{p}} = -i\underline{\nabla}$. The vector potential $\underline{\mathbf{A}}_{\text{tot}}$ consists of a nonfluctuating part $\underline{\mathbf{A}}_{\text{ext}}$, associated with $\underline{\mathbf{H}}_{\text{ext}}$ and an additional piece A. We can then split G_{tot} into three pieces:

$$G_0=\frac{1}{2m^*}\,|(p-e^*A_{\text{ext}})\psi|^2+a\,|\psi|^2+b\,|\psi|^4\,, \tag{3.2a}$$

$$G_1=\frac{1}{2\mu_0}(\underline{\nabla}\times\underline{\mathbf{A}})^2\,, \tag{3.2b}$$

$$G_2=\frac{1}{2m^*}\{(e^*)^2a^2\,|\psi|^2 -e^*A[\psi^*(-i\nabla-e^*A_{\text{ext}})\psi+\text{c.c.}]\}\,. \tag{3.2c}$$

We will trace out the fluctuating part A and calculate $G_{\text{eff}}=G_0+\delta G$ as

$$\exp\left[-\int d^dx\,\delta G\right]=\int D\underline{\mathbf{A}}\exp\left[-\int d^dx(G_1+G_2)\right] \tag{3.3}$$

(with a gauge-fixing prescription). The generalization to d dimension of a fluctuating magnetic field may be defined in several ways. We can decide that (i) again the only nonzero components of the vector potential are in the $(x\text{-}y)$ plane or (ii) take an arbitrary d-dimensional vector potential. We will choose some gauge-fixing condition; for example, in the Feynman gauge we replace G_1 by

$$G_1\to G_1+\frac{1}{2\mu_0}(\text{div}\underline{\mathbf{A}})^2\,.$$

A. First d-dimensional generalization

The results will be identical for these two cases. Let us begin with the first case,

$$2\mu_0G_1=\tfrac{1}{2}\sum_{i,j}(F_{i,j})^2+(\text{div}\underline{\mathbf{A}})^2$$

with $F_{i,j}=\partial_iA_j-\partial_jA_i$; $i=1,\ldots,d$; $A_3=\cdots=A_d=0$. Thus

$$2\mu_0G_1=(\partial_1A_2-\partial_2A_1)^2+(\boldsymbol{\nabla}_\perp A_1)^2 +(\boldsymbol{\nabla}_\perp A_2)^2+(\partial_1A_1+\partial_2A_2)^2$$

and the propagators of the A_1 and A_2 fields are obtained by inverting this quadratic form. In Fourier space one obtains two modes with propagators equal to $\mu_0/(q_\perp^2+q_1^2+q_2^2)$. The free energy G_2 generates couplings between the electromagnetic field and the order parameter. An expansion in a power series of G_2 yields various diagrams built with the vertices of G_2 and the propagators of A_1 and A_2. In fact we will see below that again the only potential massless modes are the $n=0$ Landau modes. But the vertex $\psi^*(-i\nabla-e^*A_{\text{ext}})\psi+\text{c.c.}$ involves a coupling between $n=0$ to nonzero Landau modes which have no effect on the critical behavior. Therefore the only remaining vertex in G_2 is the $A^2|\psi|^2$ interaction. At one-loop order this yields a mass renormalization, i.e., a simple change of the critical temperature, new $|\psi|^4$ interactions, and irrelevant operators. In fact these new $|\psi|^4$ interactions fall into the class of generalized vertices considered in the previous nonfluctuating-field model. We have, for instance,

Indeed this diagram converges when the external lines have zero perpendicular momenta since

$$\int\frac{d^dq}{q^2(p+q)^2}$$

is infrared convergent when $\underline{\mathbf{p}}_\perp$ vanishes for any value of p_1 and p_2 in dimension greater than four, in particular near six dimensions. The dependence in p_1 and p_2 at $\underline{\mathbf{p}}_\perp$ equal to zero, after Fourier transformation, yields an interaction of the $n=0$ Landau mode of the form

$$\int\frac{dx\,\rho(x)}{x}\exp\left[-\frac{\mu^2}{4x}|z-z'|^2\right] \quad \text{with } \rho(x)\propto\frac{1}{x^{(d-2)/2}}$$

except in a tiny neighborhood of size μ^2/Λ^2 of $x=0$, where $\rho(x)$ is cut off by nonuniversal effects.

This interaction is indeed of the class considered earlier [since the integral $\int_0^\infty dx\,\rho(x)/(1+x)$ converges]. Consequently we are led, after integration over the fluctuating magnetic field, to the previous model with a generalized interaction. The analysis of the preceding section may then be repeated and the conclusions are unchanged.

B. Second generalization

For the second d-dimensional generalization of a fluctuating magnetic field, the Hamiltonian is again given by Eq. (3.1) but $\underline{\mathbf{A}}$ now has d components. These d modes in the Feynman gauge have a propagator $1/q^2$. The integration over these modes involve the vertices of G_2. The first vertex leads to the same diagram that we just considered with an overall factor $d/2$, counting the increase in the number of modes. For the second factor we must split $\underline{\mathbf{A}}$ into A_1, A_2, and $A_\perp$ components. The vertices with A_1 or A_2 were present before and we have already discarded them since they necessarily involve nonzero Landau levels. The vertex involving $\underline{\mathbf{A}}_\perp$, namely, $\underline{\mathbf{A}}_\perp[\psi^*(-i\underline{\boldsymbol{\nabla}}_\perp)\psi+\text{c.c.}]$, contains perpendicular derivative couplings and in the zero momentum these vertices vanish; in other words this vertex yields an irrelevant (derivative) coupling of the zero modes. Hence again we are led to the same conclusion.

We can thus conclude that in all cases for any κ for which a type-II phase exists the normal type-II superconducting transition is first order below six dimensions (within the ϵ expansion). It is interesting to examine the same problem with an N-component order parameter. One remembers that in the problem of the zero-field transition between a type-I superconductor and a normal metal, Halperin *et al.*[18] have shown that the transition, driven also to first order by fluctuations, becomes second order if N is greater than 365. In this problem the same numerical analysis of the renormalization-group equations in

$6-\epsilon$ dimensions leads for all values of N up to 10^8 to the same instability, and the transition seems to remain first order for any N.

We have attempted to solve the large-N limit at fixed dimension to see if one could confirm our conclusions. Unfortunately if translationally invariant solutions of the large-N limit are simple as usual, we have not yet found the solutions which would describe an Abrikosov lattice of vertices, and we cannot conclude what is the order of the transition at N equals infinity. We are convinced by the numerical evidence that the transition should be first order for any N near six dimensions and, unless new physics takes place at lower dimensions, the three-dimensional transition should also be first order.

ACKNOWLEDGMENTS

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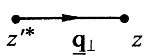
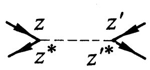
APPENDIX

Diagrams up to one loop. For the propagator, we have



$$= \frac{\mu^2}{2\pi} e^{\mu^2 z z^{*\prime}/2} (\underline{q}_\perp^2 + \tau)^{-1} .$$

For the vertex,

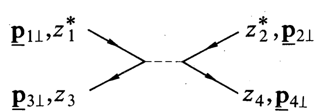


$$= -\tfrac{1}{2} e^{-\mu^2(|z|^2+|z'|^2)/2} \int_0^\infty \frac{dx}{x} \rho(x) e^{-(\mu^2/4x)|z-z'|^2} .$$

When the support of ρ is vanishingly small, this vertex reduces to

$$-\frac{2\pi}{\mu^2} e^{-\mu^2|z|^2} \delta(z-z')\delta(z^*-z'^*) \int_0^\infty dx\, \rho(x)$$

and one recovers the initial Landau-Ginzburg Hamiltonian. For the tree diagram,



$$= -\left[\frac{\mu^2}{2\pi}\right]^2 \exp[\tfrac{1}{4}\mu^2(z_1^*+z_2^*)(z_3+z_4)]\delta(p_1+p_2-p_3-p_4)$$
$$\times \int_0^\infty \frac{dx}{1+x} \rho(x) \exp\left[\frac{1}{4}\mu^2 \frac{x}{1+x} (z_1^*-z_2^*)(z_3-z_4)\right] .$$

One-loop diagrams. An arbitrary one-loop diagram may be written under the form

$$\left[\frac{\mu^2}{2\pi}\right]^2 \exp[\tfrac{1}{4}\mu^2(z_1^*+z_2^*)(z_3+z_4)] \int_0^\infty \frac{dx}{1+x} \exp\left[\frac{1}{4}\mu^2 \frac{x}{1+x}(z_1^*-z_2^*)(z_3-z_4)\right]$$
$$\times \int_0^\infty \gamma(x;u,v)\rho(u)\rho(v)du\,dv \int \frac{d^{d-2}q}{(2\pi)^{d-2}} \frac{1}{[(\underline{p}+\underline{q})^2+\tau](\underline{q}^2+\tau)} ,$$

in which $\underline{p}$ is an external $(d-2)$-dimensional momentum.

In the vicinity of six dimensions the integral is proportional to $1/\epsilon$ since

$$\int \frac{d^{d-2}q}{(2\pi)^{d-2}} (q^2+\tau)^{-1}[(\underline{p}+\underline{q})^2+\tau]^{-1} = \frac{K_{d-2}}{\epsilon} + F_{\text{finite}} ,$$

$$K_{d-2} = \frac{2^{3-d}\pi^{(2-d)/2}}{\Gamma((d-2)\frac{1}{2})} ,$$

where F_{finite} represents the finite part of the integral. The factor K_{d-2}, which may be included as usual in the coupling constant

$$\rho K_{d-2} \rightarrow \rho ,$$

has been omitted.

We may thus write the total interaction (i.e., the bare coupling constant)

$$\rho_B=\rho(x)+\delta\rho(x)$$

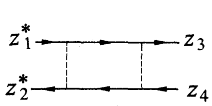
with

$$\delta\rho(x)=\frac{1}{\epsilon}\int \gamma(x,u,v)\rho(u)\rho(v)du\,dv\ .$$

A diagram with n vertices of the four-point function involves a Gaussian integral over $2n$ complex variables, at one loop $n=2$, and we have used the result

$$\int d^n\zeta\, d^n\zeta^* e^{-\mu^2(\zeta_i^* M_{ij}\zeta_j - a_i^*\zeta_i - \zeta_i^* b_i)} = \left[\frac{\pi}{\mu^2}\right]^n (\det M)^{-1}\exp[\mu^2(a_i^* M_{ij}^{-1} b_j)]\ .$$

One-loop diagrams. We have



(A)

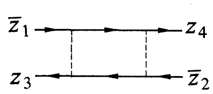
$$M_A=\left|\begin{matrix} \frac{2u+1}{4u} & -\frac{1}{4u} & -\frac{1}{2} & 0 \\ -\frac{1}{4u} & \frac{2u+1}{4u} & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{2v+1}{4v} & 0 \\ 0 & 0 & 0 & \frac{2v+1}{4v} \end{matrix}\right| ,$$

$$\det M_A=\frac{(1+u)(1+v)}{16uv},\quad a_i^*=\tfrac{1}{2}(z_1^*,z_2^*,0,0),\quad b_i=\tfrac{1}{2}(0,0,z_3,z_4)\ .$$

Hence

$$\gamma_A(x,u,v)=(1+u+v)^{-1}\delta(x-uv/(1+u+v))\ .$$

We have



(B)

$$M_B=\left|\begin{matrix} \frac{2u+1}{4u} & -\frac{1}{4u} & -\frac{1}{2} & 0 \\ -\frac{1}{4u} & \frac{2u+1}{4u} & 0 & 0 \\ 0 & 0 & \frac{2v+1}{4v} & -\frac{1}{4v} \\ 0 & -\frac{1}{2} & -\frac{1}{4v} & \frac{2v+1}{4v} \end{matrix}\right| ,$$

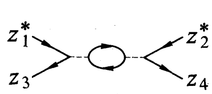
$$\det M_B=\frac{1}{64uv}(4uv+4u+4v+3)\ ,$$

$$a_i^*=\tfrac{1}{2}(z_1^*,0,0,z_2^*)\ ,$$

$$b_i=\tfrac{1}{2}(0,z_3,z_4,0)\ ,$$

$$\gamma_B=\frac{2}{(1+2u)(1+2v)}\delta(x-(1-4uv)/2(1+2u)(1+2v))+\frac{1}{1+u+v}\delta(x-(4uv-1)/4(u+v+1))\ .$$

We have



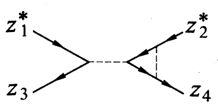
(C)

$$M_C=\left|\begin{matrix} \frac{2u+1}{4u} & -\frac{1}{4u} & 0 & 0 \\ -\frac{1}{4u} & \frac{2u+1}{4u} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{2v+1}{4v} & -\frac{1}{4v} \\ 0 & 0 & -\frac{1}{4v} & \frac{2v+1}{4v} \end{matrix}\right| ,$$

$$\det M_C=\frac{2u+2v+3}{64uv}, \quad a_i^*=\tfrac{1}{2}(z_1^*,0,0,z_2^*), \quad b_i=\tfrac{1}{2}(z_3,0,0,z_4)\ ,$$

$$\gamma_C=2\delta(x-(u+v+\tfrac{1}{2}))\ .$$

We have



(D)

$$+\begin{bmatrix} z_1^* & z_2^* \\ z_3 & z_4 \end{bmatrix} M_D=\begin{vmatrix} \frac{2u+1}{4u} & -\frac{1}{4u} & 0 & 0 \\ -\frac{1}{4u} & \frac{2u+1}{4u} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{2v+1}{4v} & -\frac{1}{4v} \\ 0 & 0 & -\frac{1}{4v} & \frac{2v+1}{4v} \end{vmatrix},$$

$$\det M_D=\frac{4uv+2u+4v+3}{64uv}, \quad a^*=\tfrac{1}{2}(z_1^*,0,z_2^*,0), \quad b=\tfrac{1}{2}(z_3,0,0,z_4)\ ,$$

$$\gamma_D=\frac{2}{1+2v}\delta(x-(2uv+u+\tfrac{1}{2})/(2v+1))+\frac{2}{1+2u}\delta(x-(2uv+v+\tfrac{1}{2})/(2u+1))\ .$$

This completes the derivation of Eq. (2.29).

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