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Fractional statistics and fractional quantized Hall effect

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We suggest that the origin of the odd-denominator rule observed in the fractional quantized Hall effect (FQHE) may lie in fractional statistics which govern quasiparticles in FQHE. A theorem concerning statistics of clusters of quasiparticles implies that fractional statistics do not allow coexistence of a large number of quasiparticles at fillings with an even denominator. Thus, no Hall plateau can be formed at these fillings, regardless of the presence of an energy gap.

I. INTRODUCTION

The fractionally quantized Hall effect (FQHE) in a two-dimensional electron gas in a strong magnetic field discovered by Tsui, Störmer, and Gossard[1,2] has stimulated a series of theoretical investigations. One of the most important and interesting discoveries in this effect is the odd-denominator rule: fractional quantization of Hall conductance is only found at filling factors with an odd denominator, such as $\frac{1}{3},\frac{1}{5},\frac{2}{5},\ldots$

Up until now this odd-denominator rule does not have any convincing explanation. The usual guess that this rule is due to the absence of an energy gap at fillings with an even denominator lacks evidence and proof. Laughlin's trial wave function[3] does not tell us anything about states at fillings with an even denominator. Physically, at these fillings there must be a ground state though it cannot be expressed in Laughlin's wave function. Especially at filling factor $\nu=\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{6},\ \frac{5}{6},\ \frac{3}{8},\ \frac{5}{8}$, the system is not in a charge-density wave (CDW) state;[4] why is there no FQHE? Unfortunately, Haldane's sequence does not provide any suggestions about the nature of the ground state at these fillings, either.[5] On the other hand, the many-body theory[6–8] for this effect, as a microscopic theory, suggests a systematic way to find the ground state for the filling q/p. The calculation in this theory yields an energy gap at fillings with an even denominator though it is very crude. Some recent numerical calculations[4] also indicate the possible presence of an energy gap at these fillings. Using a gauge-invariance argument, we showed elsewhere[9] that there may be a more profound reason for this odd-denominator rule. In this Rapid Communication, we further suggest that the origin of the odd-denominator rule may lie in fractional statistics which govern quasiparticles (elementary excitations) in FQHE, regardless of the presence of an energy gap.

Fractional statistics were first suggested in two specific models.[10] It recently has been shown[11] that because of the complicated topology of the configuration space for indistinguishable particles in two dimensions, the path-integral formalism in general allows fractional statistics in addition to Bose-Eistein and Fermi-Dirac statistics. Fractional statistics were characterized by an angular parameter θ. An interchange of two identical particles along a counterclockwise loop yields a phase factor $e^{i\theta}$ where $-\pi<\theta\leq\pi$. Halperin[12] suggested that quasiparticles in FQHE obey fractional statistics, based on the approach of Laughlin's wave function. Arovas, Schrieffer, and Wilczek recently proved this suggestion with the adiabatic theorem and showed that an interchange of two quasielectrons (quasiholes) along a counterclockwise loop yields a phase factor $e^{i\theta}$ with $\theta=\nu\pi$.[13] Their proof can be generalized to include $\nu=q/p$ cases. All quasiparticles discussed in FQHE possess the property that creating a cluster of p quasielectrons (quasiholes) is equivalent to creating a cluster of q electrons (holes) at $\nu=q/p$. From this, quasielectrons (quasiholes) have charge $-\nu e$ (νe) and the statistics of quasielectrons can be determined to be fractional with $\theta=\nu\pi$.[14]

In Sec. II, we further clarify the role of fractional statistics in FQHE. At fillings with an even denominator, fractional statistics forbid the coexistence of a large number of quasiparticles, so that even the presence of an energy gap at these fillings does not lead to condensation. This is a consequence of the following theorem: the statistics of a cluster of p quasielectrons can be the same as a cluster of q electrons only when p is odd. Since the formation of a Hall plateau and a dip in ρ_{xx} depends on coexistence of a large number of quasiparticles, fractional quantization of Hall conductance can only be found at fillings with an odd denominator.

II. ODD-DENOMINATOR RULE

Now let us exploit the property that the statistics of a cluster of p quasielectrons are the same as a cluster of q electrons. We want to prove the following theorem.

Theorem. Let p and q be mutual primes. If quasielectrons obey fractional statistics with $\theta=q\pi/p$ and p quasielectrons are equivalent to q electrons, then many quasielectrons can coexist only when p is odd.

We first consider a counterclockwise interchange of two identical clusters, each having p quasielectrons joined together. The sizes of these clusters are negligible in comparison with their separation. Because we are considering statistics, as with considering the statistics of atoms, the ordering of particles inside the cluster is fixed. There are p^2 interchanges of pairs of quasielectrons. After the interchange, the wave function gets a phase factor

$$e^{ip^2\theta}=e^{ipq\pi}\ . \tag{2.1}$$

On the other hand, we consider an exchange of two identical clusters, each having q electrons. From Fermi statistics, the phase factor should be $(-1)^{q^2}$. If a cluster of p quasielectrons is equivalent to q electrons in charge and in

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statistics, we must have

$$e^{ipq\pi}=(-1)^{q^2} . \tag{2.2}$$

Thus, if p were even, q would have to be even; then p and q could not be mutual primes.

This argument can be easily generalized to two identical clusters each having N_e electrons and p quasielectrons, which are not joined together. The separation of these two clusters is also much bigger than their own sizes. A counterclockwise exchange of these two clusters gives a phase factor

$$e^{i(\pi N_e^2+p^2\theta)}=e^{i\pi(N_e^2+pq)} , \tag{2.3}$$

where $e^{i\pi N_e^2}$ is from N_e^2 interchanges of pairs of electrons; $e^{ip^2\theta}$ is from p^2 interchanges of pairs of quasielectrons. There is not any other phase factor picked up because quasielectrons and electrons are not identical particles. On the other hand, these clusters are equivalent to two identical clusters, each having N_e+q electrons, so that the phase factor gained in this exchange should be

$$e^{i\pi(N_e+q)^2}=e^{i\pi(N_e^2+q^2)} . \tag{2.4}$$

These two must be equal. Comparing Eqs. (2.3) and (2.4), we have Eq. (2.2) again: if p is even, we always have a contradiction. Because in two-dimensional systems there are no other exotic statistics than the fractional ones,[11] this inconsistency implies that no more than $2p$ quasielectrons can coexist when p is even. The same conclusion is also true for quasiholes. Q.E.D.

The current picture for the origin of finite width of the Hall plateau is as follows. Suppose that at filling factor $\nu=q/p$ there is an energy gap separating the ground state from excited states. When ν is near q/p, $\nu=q/p+\Delta\nu$, the ground state is composed of the q/p state plus quasiparticles. These quasiparticles are localized by impurities; they do not contribute to any current, so σ_H is still given by $(q/p)e^2/h$ and ρ_{xx} has a dip. In order to produce a finite Hall plateau and a dip in ρ_{xx}, $\Delta\nu$ must have a finite value. For example, experiments[15] showed $\Delta\nu\approx\pm0.07$ at $\nu=\frac{2}{3}$. Therefore, the formation of FQHE depends on the coexistence of a large number of quasiparticles. When p is even, the above theorem tells us that only few quasiparticles are allowed to coexist; therefore, no Hall plateau and no dip in ρ_{xx} can be formed at these fillings. At $\nu=q/p$ with an odd p, fractional statistics do not forbid quasiparticles to coexist; only in these cases can the presence of an energy gap lead to condensation, and the strength of FQHE is determined by the calculation of an energy gap.

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