Implications of the chiral anomaly for quantum Hall-effect devices

A. Widom, M. H. Friedman, and Y. N. Srivastava
Physics Department, Northeastern University, Boston, Massachusetts 02115

(Received 29 November 1984)

The anomaly equation $\partial_\mu J_5{}^\mu = -(e^2/8\pi^2\hbar^2 c)\,(F^*_{\lambda\sigma}F^{\lambda\sigma})$ is used to derive a relationship between the filling factor (ν) and the Hall conductance on the gate capacitor of a field-effect-transistor device. In addition, the chiral phase (θ) states are shown to be related to those of the filling factor by $|\theta\rangle = |2\pi\nu\rangle$.

I. INTRODUCTION

In quantum electrodynamics electromagnetic fields are a source of chiral currents. This is exhibited in Schwinger's equation (5.24) of Ref. 1:

$$\frac{\partial \rho_5}{\partial t} + \text{div}\mathbf{J}_5 = -(e^2/2\pi^2\hbar^2 c)\mathbf{E}\cdot\mathbf{B} . \tag{1}$$

The implications of Eq. (1) for macroscopic systems have not yet been fully explored. However, in a recent letter[2] it was noted that the Hall effect is intimately connected to the chiral anomaly.[3] The purpose of this work is to provide further details of this relationship. Starting from Eq. (1) the fractionally quantized Hall conductance will be derived.

In Sec. II the integrated form of the chiral anomaly will be written in terms of bulk quantities (e.g., voltage, flux, etc.) valid for any capacitor in an externally applied magnetic field. In Sec. III a conserved, chiral current will be introduced and its spatially integrated form (suitable for engineering applications) will be discussed. The thermodynamic law for a capacitor in an applied magnetic field will be reviewed in Sec. IV.

A complementary formulation of the effects of the chiral anomaly involves the use of θ states. These are introduced in Sec. V. In Sec. VI it will be proved that the equilibrium θ state of a capacitor in an applied magnetic field is simply related to the charge on the capacitor.

Finally, in Sec. VII the θ state of a gate capacitor in a field-effect transistor will be shown to yield the Hall conductance of the electronic layer on the semiconductor surface. The results will be discussed in the concluding Sec. VIII.

II. MACROSCOPIC FORMULATION

Consider a parallel plate capacitor with plates of area Σ separated by a distance L. The voltage V across the capacitor is related to the electric field between the plates via

$$\mathbf{E} = -(V/L)\mathbf{n} , \tag{2}$$

where $\mathbf{n}$ is a unit vector normal to the plates. If the capacitor is subject to a magnetic field $\mathbf{B}$ uniform in space and time, then

$$\mathbf{E}\cdot\mathbf{B} = -(V/L)\mathbf{B}\cdot\mathbf{n} . \tag{3}$$

Since the volume of material (insulator) between the capacitor plates is given by

$$\Omega = L\Sigma , \tag{4}$$

and the magnetic flux through the capacitor plates is given by

$$\Phi = (\mathbf{B}\cdot\mathbf{n})\Sigma , \tag{5}$$

it is evident that

$$-\int_\Omega \mathbf{E}\cdot\mathbf{B}\, d^3r = V\Phi . \tag{6}$$

From Eqs. (1) and (6) it is clear that the chiral anomaly may be formulated in electrical engineering language, i.e., for a capacitor at voltage V with a magnetic flux Φ passing through the capacitor plates, the chiral production between the capacitor plates is given by

$$\dot{N}_5 = (e^2/2\pi^2\hbar^2 c)V\Phi . \tag{7}$$

Equation (7) is the integrated form of chiral anomaly to be used in what follows.

III. PSEUDOCHIRAL CURRENT

The electromagnetic field is given in terms of the potentials by

$$\mathbf{B} = \text{curl}\mathbf{A} , \tag{8a}$$

$$\mathbf{E} = -c^{-1}\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\phi . \tag{8b}$$

It is possible to introduce a conserved chiral current and density

$$\mathbf{J}_5^* = \mathbf{J}_5 - (e^2/4\pi^2\hbar^2 c)(\phi\mathbf{B} - \mathbf{A}\times\mathbf{E}) , \tag{9a}$$

$$\rho_5^* = \rho_5 - (e^2/4\pi^2\hbar^2 c^2)(\mathbf{A}\cdot\mathbf{B}) . \tag{9b}$$

By virtue of Eqs. (1), (8), and (9), one obtains

$$\frac{\partial \rho_5^*}{\partial t} + \text{div}\mathbf{J}_5^* = 0 . \tag{10}$$

Let the voltage across the capacitor be generated via Faraday's law:

$$V = -c^{-1}\frac{d\tilde{\Phi}}{dt} . \tag{11}$$

Then, defining

$$N_5^* = N_5 + (e^2/2\pi^2\hbar^2 c^2)(\Phi\tilde{\Phi}) \tag{12}$$

yields, by virtue of Eqs. (7) and (11),

$$\dot{N}_5^* = 0 , \tag{13}$$

so long as the applied flux Φ through the capacitor plates is fixed, i.e., for a fixed applied magnetic field

$$N_5^* = \text{const.} \tag{14}$$

This conservation law is important for introducing θ states.

IV. STATISTICAL THERMODYNAMICS

The thermodynamic law for capacitors in applied magnetic fields reads

$$dA = -S\,dT + V\,dQ - M\,d\Phi , \tag{15}$$

where Q is the capacitor charge and M is the magnetic moment per unit area of the capacitor plates.

From a statistical-mechanical viewpoint, the capacitor free energy is computed from a Hamiltonian $H(Q,\Phi)$ via

$$A(T,Q,\Phi) = -k_B T \ln \text{Tr}\exp[-H(Q,\Phi)/k_B T] . \tag{16}$$

From Eq. (16) one notes that

$$\left[\frac{\partial A}{\partial Q}\right]_{T,\Phi} = \left\langle \left[\frac{\partial H}{\partial Q}\right]_{\Phi} \right\rangle , \tag{17}$$

i.e., the thermodynamic voltage in Eq. (15) is given by

$$V = \left\langle \left[\frac{\partial H}{\partial Q}\right]_{\Phi} \right\rangle . \tag{18}$$

The voltage operator for the capacitor may then be constructed from the Hamiltonian operator via

$$\hat{V} = \left[\frac{\partial H}{\partial Q}\right]_{\Phi} , \tag{19}$$

which will be used in the quantum-electrodynamic circuit element theory to follow.

V. CHIRAL-PHASE REPRESENTATION

The simplest manner in which the chiral phase[4] can be introduced is by considering the Schrödinger equation for the capacitor

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi . \tag{20}$$

If one looks for a solution of Eq. (20) of the form

$$\psi = e^{i\theta N_5^*/2} \chi , \tag{21}$$

with a rotating phase

$$\frac{-d\theta}{dt} = \frac{2\mu_5^*}{\hbar} , \tag{22}$$

then Eqs. (20)—(22) read

$$i\hbar \frac{\partial \chi}{\partial t} = (H - \mu_5^* N_5^*)\chi , \tag{23}$$

where the conservation Eq. (13) has been invoked in the quantum-mechanical form

$$[H, N_5^*] = 0 . \tag{24}$$

Equation (23) is simply the Schrödinger equation in the chiral grand-canonical representation, and the chiral phase rotates at a rate determined by the chemical potential μ_5^*, as in Eq. (22).

Equations (11) and (19), in quantum-mechanical form, read

$$(i/\hbar c)[\tilde{\Phi}, H(Q,\Phi)] = \frac{\partial H(Q,\Phi)}{\partial Q} \tag{25}$$

for any model capacitor Hamiltonian. Thus, one deduces the commutation relation between the Faraday-law-generated flux and the charge of the capacitor

$$[Q, \tilde{\Phi}] = i\hbar c . \tag{26}$$

Equation (26) has also been derived elsewhere, directly from the canonical commutation relations of quantum electrodynamics.[5]

It is important to note that the chiral quantum number N_5 commutes with the capacitor charge

$$[N_5, Q] = 0 , \tag{27}$$

whereas, by virtue of Eqs. (12), (26), and (27), the chiral quantum number N_5^* does not commute with the capacitor charge

$$[N_5^*, Q] = -i(e^2/2\pi^2\hbar c)\Phi . \tag{28}$$

Thus, there is indeed charge production on the capacitor at the rate

$$\dot{Q} = (i/\hbar)[H(Q,\Phi) - \mu_5^* N_5^*, Q] \tag{29}$$

in the chiral grand-canonical representation of Eq. (23).

From Eqs. (28) and (29) it is evident that the chemical potential is simply related to the current through the capacitor

$$\frac{dQ}{dt} = -(e^2\Phi/2\pi^2\hbar^2 c)\mu_5^* . \tag{30}$$

VI. FILLING-FACTOR THEOREM

From Eqs. (22) and (30) we deduce that

$$\dot{\theta} = (4\pi^2\hbar c/e^2\Phi)\dot{Q} , \tag{31}$$

or equivalently, for fixed magnetic flux Φ through the capacitor,

$$d\theta = (4\pi^2\hbar c/e^2\Phi)dQ , \tag{32}$$

so that changes in the θ state are directly proportional to changes in the capacitor charge.

Introducing the filling factor ν by the usual definition

$$Q = (e^2/2\pi\hbar c)\Phi\nu , \tag{33}$$

one easily proves the following:

Theorem. In thermal equilibrium, the chiral phase state is related to the filling factor by

$$|\theta\rangle = |2\pi\nu\rangle \ . \quad (34)$$

The proof of Eq. (34) follows, for fixed magnetic flux Φ, from Eqs. (32) and (33), by writing

$$\frac{d\theta}{2\pi} = d\nu \ . \quad (35)$$

The importance of Eq. (34) becomes evident when the θ state is related to the Hall conductance. The action-principle formulation of the capacitor quantum dynamics provides the simplest proof of this relationship.

VII. CHIRAL ACTION

With $(p_1, \ldots, p_n, x_1, \ldots, x_n)$ representing the phase space of the Hamiltonian in Eq. (16), the capacitor action would read

$$S = \int \left[\sum_j p_j dx_j - H\, dt \right] . \quad (36)$$

On the other hand, Eqs. (23) and (26) yield a θ-state representation of the action

$$S^* = \int \left[\sum_j p_j dx_j + \tilde{\Phi}(dQ/c) \right] - \int (H - \mu_5^* N_5^*) dt \ . \quad (37)$$

From Eqs. (36) and (37) one obtains the chiral piece of the action

$$\Delta S = \frac{1}{c} \int \tilde{\Phi}\, dQ + \int \mu_5^* N_5^* dt \ . \quad (38)$$

From Eqs. (12) and (38), the chiral action may be written

$$\Delta S = \int \mu_5^* N_5 dt + \Delta S' \ , \quad (39a)$$

$$\Delta S' = \frac{1}{c} \int \tilde{\Phi}\, dQ + (e^2/2\pi^2\hbar^2 c^2) \int \mu_5^* \Phi \tilde{\Phi}\, dt \ . \quad (39b)$$

Equation (39b) simplifies by using Eq. (30), i.e.,

$$(e^2/2\pi^2\hbar^2 c^2)\mu_5^* \Phi \tilde{\Phi}\, dt = -c^{-1} \tilde{\Phi}\, dQ \quad (40)$$

yields, via Eqs. (39b) and (40),

$$\Delta S' = 0 \ , \quad (41)$$

so that Eqs. (39a) and (41) yield

$$\Delta S = \int \mu_5^* N_5 dt \ . \quad (42)$$

The physical interpretation of Eq. (42) is quite clear in thermodynamic terms, i.e., $\mu_5^* dN_5$ can formally be considered as the "chemical-work" term for "producing chirality." However, this is best described in θ-state language using Eqs. (22) and (42), i.e.,

$$\Delta S = \frac{\hbar}{2} \int \theta \dot{N}_5 dt - \frac{\hbar}{2} \int dt \frac{d}{dt}(\theta N_5) \ . \quad (43)$$

The second term on the right-hand side of the action, Eq. (43), can be gauged into the quantum-mechanical wave function so that the θ state action is given by

$$\Delta S_\theta = (\hbar/2) \int \theta \dot{N}_5 dt \ . \quad (44)$$

For an equilibrium θ state, Eq. (44) and the chiral anomaly equation (1) yield the central result of this section. It is the following.

Theorem. In an equilibrium θ state, the chiral action is given by

$$\Delta S_\theta = -(e^2/4\pi^2\hbar c)\theta \int dt \int d^3r\, \mathbf{E}\cdot\mathbf{B} \ . \quad (45)$$

As previously noted[2] (with θ normalized with a different factor), Eq. (45) implies a quantized Hall effect on the surface of the capacitor plates. Note that Eq. (45) is merely the θ state representation of Eq. (1).

The Hall conductance is deduced in the following manner: (i) From Eq. (8) it follows that

$$\mathbf{E}\cdot\mathbf{B} = \frac{1}{2} \left[\mathrm{div}(\mathbf{A}\times\mathbf{E} - \phi\mathbf{B}) - c^{-1} \frac{\partial}{\partial t}(\mathbf{A}\cdot\mathbf{B}) \right] . \quad (46)$$

(ii) From Eqs. (45) and (46) it follows that

$$\Delta S_\theta = -(e^2/8\pi^2\hbar c)\theta \int dt \oint d\mathbf{\Sigma}\cdot(\mathbf{A}\times\mathbf{E} - \phi\mathbf{B}) + (e^2/8\pi^2\hbar c^2)\theta \int dt \frac{d}{dt} \int d^3r\, \mathbf{A}\cdot\mathbf{B} \ , \quad (47)$$

where the volume integral over the region between the capacitor plates has been converted into a surface integral over the capacitor plate. (iii) The second term in Eq. (47) can be guaged into the wave function while the first term yields the two-spatial dimensional Hall effect action

$$\Delta W = -(e^2/8\pi^2\hbar c)\theta \int dt \oint d\mathbf{\Sigma}\cdot[\mathbf{A}\times\mathbf{E} - \phi\mathbf{B}] \ . \quad (48)$$

(iv) To see that ΔW does indeed describe a Hall surface current, one merely uses the action in Eq. (48) to compute the current

$$\mathbf{J}_H = c\, \delta\Delta W/\delta\mathbf{A} \quad (49)$$

to obtain

$$\mathbf{J}_H = (e^2/4\pi^2\hbar)\theta\mathbf{n}\times\mathbf{E} \ , \quad (50)$$

where $\mathbf{n}$ is a unit vector normal to the surface.

From Eq. (50), it is evident that the Hall conductance

$$\mathbf{J} = g\mathbf{n}\times\mathbf{E} \quad (51)$$

can be deduced from both Eq. (50) and the filling factor theorem, [Eq. (34)]. g is given by

$$g = (e^2/4\pi^2\hbar)\theta = (e^2/2\pi\hbar)\nu \ . \quad (52)$$

Equation (52) is the central result of this work which relates the θ states of the Abelian U(1) chiral anomaly of quantum electrodynamics to the Hall conductance of a gate capacitor in a field-effect transistor.

VIII. DISCUSSION AND CONCLUDING REMARKS

As stated in the Introduction the chiral anomaly found in quantum electrodynamics by Schwinger has been used in its integrated form for macroscopic (or engineering) systems. In particular, we have applied this essential ingredient to discuss an arbitrary capacitor in an externally applied magnetic field. It is indeed remarkable that our formalism leads to an unambiguous result for the Hall conductance of the electronic layer on a semiconductor surface from the most successful microscopic theory available, viz., quantum electrodynamics. No appeal to variational or nonrelativistic approximations needs to be made. Perhaps our results would appear less surprising if one would remember that there are other cases in which even to discuss static (or near static) situations, an underlying relativistic theory is not only of help but essential. Consider, for example, the g value of the electron. In nonrelativistic Pauli theory, this number is totally arbitrary. But for the *Dirac electron* g is fixed at 2. The situation for the quantization of the Hall conductance is somewhat similar. Here, the Schwinger anomaly (which has a fixed coefficient) present in $(3+1)$-dimensional QED leads, after dimensional reduction in $(2+1)$-dimensional QED, to the topological term producing just the right values for the Hall effect. Our derivation brings out clearly that such a quantization can only occur through a fundamental relativistic theory. On the other hand, derivations based on a nonrelativistic theory must involve *ad hoc* assumptions.

Let us briefly discuss a technical point concerning renormalization. Once again we proceed by analogy. Suppose we calculate the zero-frequency Thomson cross section, i.e., the cross section (σ) for scattering zero-frequency photons on any target of mass m and charge (Ne). It is only through QED that one discovers (Thirring theorem) that the answer depends only upon (Ne) and m—irrespective of any internal structure, spin, etc., of the target. The renormalized fine-structure constant α occurs in the result. For the Hall effect, a similar result holds. In $(3+1)$ dimensions QED has an adimensional α. In $(2+1)$-dimensional QED, the charge $\hat{e}=e/\sqrt{l}$ where l is the thickness. Once again renormalization does not affect the manner in which α enters into the Hall conductance itself. The equation $(g=e^2/2\pi\hbar)\nu$ remains unchanged whatever the material.

APPENDIX

In the above investigation we have presented (in concrete engineering terms) the formalism involved in relating the chiral anomaly of quantum electrodynamics to the Hall effect. The purpose of this appendix is to clarify the meaning of the formalism and exhibit the engineering equations in a relativistic form.

A detailed mathematical treatment of the electronic (Dirac) Green's functions and their relation to Hall steps is reserved for future work, since the above engineering equations are already quite lengthy.

The reader here may note that Eq. (52), relating g and ν, may be obtained from classical reasoning so that the chiral angle θ need not be invoked. Our reasons for invoking chirality are as follows: (i) At low temperature magnetoconductivity is a "Fermi-surface" property, while the Hall effect involves all states below the Fermi level. (ii) In the Dirac theory "all states below the Fermi level" means the inclusion of positron states. (iii) The inclusion of positron states means polarization of condensed matter and even "vacuum" polarization degrees of freedom.

One describes polarization by the tensor

$$P^{\mu\nu}=-P^{\nu\mu}\ , \tag{A1}$$

which produces a "bulk" current

$$J^{\mu}=\partial_{\nu}P^{\nu\mu} \tag{A2}$$

and a "surface" current

$$K^{\mu}=N_{\nu}\,\Delta P^{\nu\mu}\ , \tag{A3}$$

as in a capacitor. (N_{μ} is the unit normal to the surface.) For example, in a metal-oxide-semiconductor field-effect-transistor (MOSFET) device, the inversion layer is often modeled by a "surface current," while depletion regions are often modeled by a "bulk" current.

In the above notation, Eqs. (51) and (52) are represented as

$$K^{\mu}_{\text{Hall}}=(e^2/8\pi^2\hbar)\theta N_{\nu}\epsilon^{\nu\mu\lambda\sigma}F_{\lambda\sigma}\ , \tag{A4}$$

so that the θ-state polarization tensor reads

$$P^{\nu\mu}_{\theta}=(e^2/8\pi^2\hbar)\theta\epsilon^{\nu\mu\lambda\sigma}F_{\lambda\sigma}\ . \tag{A5}$$

From Eqs. (A2) and (A5) one obtains the θ-state bulk current

$$J^{\mu}_{\theta}=-(e^2/8\pi^2\hbar)\epsilon^{\mu\nu\lambda\sigma}(\partial_{\nu}\theta)F_{\lambda\sigma}\ . \tag{A6}$$

Equation (A5) has appeared in our engineering presentation as Eq. (31), and in our previous work[6] as Eqs. (6)—(8). However, Eqs. (A2) and (A5) are written here without restricting (for special cases) the components of $F_{\mu\nu}$.

[1]J. Schwinger, Phys. Rev. **82**, 664 (1951).

[2]M. H. Friedman, J. B. Sokoloff, A. Widom, and Y. N. Srivastava, Phys. Rev. Lett. **52**, 1587 (1984); Y. Srivastava and A. Widom, Nuovo Cimento Lett. **17**, 285 (1984).

[3]For the chiral anomaly in non-Albelian theories, see J. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969); S. Adler, Phys. Rev. **177**, 2426 (1969).

[4]R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976); C. Callan, R. Dashen, and D. Gross, Phys. Lett. **63B**, 334 (1976).

[5]A. Widom and T. D. Clark, Phys. Lett. **90A**, 280 (1982).

[6]M. H. Friedman, J. B. Sokoloff, A. Widom, and Y. N. Srivastava, Phys. Rev. Lett. **53**, 2592 (1984).