Generalized many-channel conductance formula with application to small rings

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The conductance of a sample scattering elastically and coupled to leads with many channels is derived. We assume that all the incident channels on one side of the sample are fed from the same chemical potential. The transmitted and reflected streams are determined by the incident streams through the multichannel scattering properties of the sample. We do not assume that the channels equilibrate with each other. Our result differs from those given earlier by other authors, except for that of Azbel [J. Phys. C **14**, L225 (1981)], which is confirmed. We point out that a similar result is obtained for the conductance in a single channel at a temperature above zero. As an application, we obtain the dependence on channel number N of the contributions to the conductance of a small ring, periodic in the Aharonov-Bohm flux through it. Terms whose period is h/e as well as those with period $h/2e$ vary with N as $1/N$.

I. INTRODUCTION

The conductance G due to elastic scattering of an obstacle, characterized by transmission and reflection coefficients T and R, is given by[1,2]

$$G=(e^2/\pi\hbar)T/R \ . \tag{1.1}$$

At zero temperature T and R are evaluated at the Fermi energy. This quantum-mechanical result expresses a transport coefficient in terms of static scattering properties, rather than those of the usual temporal correlation functions. Equation (1.1) applies to samples of arbitrary shape and structural complexity. The carriers, however, can enter or leave the sample only through leads with a single quantum channel, i.e., a wire with two states at the Fermi energy.

Equation (1.1) has been used to obtain the scaling of the resistance of a one-dimensional wire as a function of its length.[2] This has been used to develop the modern scaling theory for one-dimensional conduction.[3,4] Small structures of normal metal, with an opening, are sensitive to an Aharonov-Bohm flux. Reference 5 analyzed cylinders and rings through the correlation-function approach, whereas Eq. (1.1) was used in Refs. 6 and 7 to treat strictly one-dimensional rings.

Generalizations of Eq. (1.1), treating samples connected to leads with many channels, have been presented in Ref. 3 and Refs. 8—10. References 3 and 9 add the conductances of channels in parallel, whereas no simple result is found in Ref. 8 and the need to solve a large system of equations is stressed. The result of Azbel[10] differs from these and will be reestablished here.

It is desirable to have a many-channel generalization of Eq. (1.1) for various reasons. Such a result might enable one to generalize the scaling theory[3] from one dimension to higher dimensions or, at least, to finite-thickness wires.[8—10] A multichannel formula can be applied in numerical studies of the scaling of the conductance of higher-dimensional systems,[10—15] testing the scaling theory of localization[16] in such systems. An understanding of the conductance of many-channel systems will also be crucial, for example, to the elucidation of quantum interference effects[5—7] in real conductors, with height and width equal to a number of atomic layers.

A brief discussion of the single-channel result, Eq. (1.1), and its generalization to nonzero temperature, is contained in Sec. II. Many-channel generalizations of Eq. (1.1) are presented in Secs. III and IV. An application to Aharonov-Bohm[5—7] resistance oscillations in many-channel rings is given in Sec. V. Further possible generalizations are mentioned in Sec. VI.

II. THE SINGLE-CHANNEL CASE

In this section we present a brief derivation of the single-channel conductance formula Eq. (1.1) and its generalization to nonzero temperature. Extended discussions can be found in Refs. 1—3, 8, and 17—19. To derive Eq. (1.1), we consider the situation depicted in Fig. 1. The

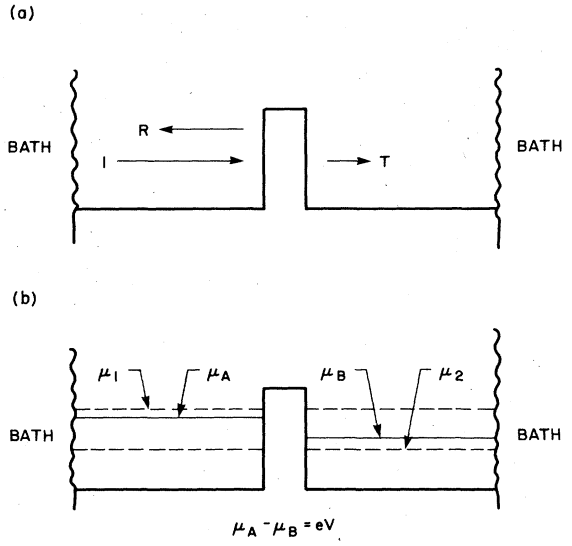

FIG. 1. (a) The geometry of Ref. 1. The obstacle is connected to two incoherent reservoirs by ideal 1D conductors. A stream of particles with unit density hits the barrier from the left, a fraction R is reflected, and a fraction T transmitted. (b) The chemical potentials in the single-channel case. The LHS reservoir emits electrons up to a quasi-Fermi-energy μ_1, and the RHS reservoir emits electrons up to a quasi-Fermi-energy μ_2. μ_A and μ_B are the chemical potentials in the perfectly conducting leads to the left and the right of the barrier.

sample is connected to perfect but identical one-dimensional conductors which in turn connect to electron reservoirs. The reservoir to the left injects carriers into the perfect wire up to a quasi-Fermi-energy μ_1 and the reservoir to the right emits carriers up to a quasi-Fermi-energy μ_2 [see Fig. 1(b)]. (The assignment of a chemical potential to a particular class of electrons can be considered simply as a mathematical alternative to the use of carrier densities.) The reservoirs are incoherent; the waves emerging from separate reservoirs do not have any phase relationship. The current, for two spin directions, emitted by the left reservoir in the energy range between μ_2 and μ_1 is

$$I = ev(\partial n/\partial E)(\mu_1 - \mu_2) . \tag{2.1}$$

Here v is the Fermi velocity, and $\partial n/\partial E$ is the density of states for two spin directions and for carriers with positive velocity. In one dimension $\partial n/\partial k = 1/\pi$ and $\partial n/\partial E = 1/\pi\hbar v$. Thus, the total current emitted by the left reservoir due to the difference in the quasi-Fermi-levels is $I = (e/\pi\hbar)(\mu_1 - \mu_2)$. These carriers have a probability T for traversal of the sample and a probability R of being reflected. Therefore, the net current flow is given by

$$I = (e/\pi\hbar)T(\mu_1 - \mu_2) . \tag{2.2}$$

The difference between the quasi-Fermi-energies μ_1 and μ_2 is chosen to be small enough so that the energy dependence of T (and R) within this range can be neglected. Next, we have to determine the voltage across the sample. This potential difference is determined by the piled up charges to the left and right of the sample, and the screening of these charges. We must invoke either self-consistent screening[1,17] or the Einstein relation,[2] starting from the differences in carrier density, between the two ideal conductors. The carrier densities can be characterized by the chemical potentials μ_A and μ_B [see Fig. 1(b)]. Their respective levels, between μ_1 and μ_2, are determined such that the number of occupied states (electrons) above μ_A (μ_B) is equal to the number of empty states (holes) below μ_A (μ_B). Below the energy μ_2 all states are fully occupied and we need to consider the energy range from μ_2 to μ_1 only. The total numbers of states in this range is $2(\partial n/\partial E)(\mu_1 - \mu_2)$. The factor 2 arises because we have a state with positive velocity and a state with negative velocity at each energy, and $\partial n/\partial E$ is the density of states for carriers moving to the right (or the left) only. Consider now the perfect wire to the right. Since carriers have a transmission probability T, the number of occupied states is $T(\partial n/\partial E)(\mu_1 - \mu_B)$ and the number of unoccupied states is $(2-T)(\partial n/\partial E)(\mu_B - \mu_2)$. Thus, the chemical potential μ_B to the right of the sample is determined by

$$T(\partial n/\partial E)(\mu_1 - \mu_B) = (2-T)(\partial n/\partial E)(\mu_B - \mu_2) . \tag{2.3}$$

To the left of the barrier we have both incident carriers and reflected carriers. The number of occupied states is $(1+R)(\partial n/\partial E)(\mu_1 - \mu_A)$ and the number of unoccupied states is $[2-(1+R)](\partial n/\partial E)(\mu_A - \mu_2)$. Therefore, the chemical potential μ_A to the left of the sample is determined by

$$(1+R)(\partial n/\partial E)(\mu_1 - \mu_A) = (1-R)(\partial n/\partial E)(\mu_A - \mu_2) . \tag{2.4}$$

Charge neutrality does not allow different densities to the left and right of the sample over distances large compared to a screening length.[1,2,17] This requires that the separation between the chemical potential μ_A or μ_B, respectively, and the band bottom must be the same as in equilibrium. Thus, the conduction-band bottoms of the perfect wires are displaced against each other by a potential difference,

$$eV = \mu_A - \mu_B . \tag{2.5}$$

Therefore, Eqs. (2.3) and (2.4) can be used to determine the voltage across the sample. The result of this simple calculation yields $eV = R(\mu_1 - \mu_2)$, and with Eq. (2.2), we find for the conductance $G = I/V = (e^2/\pi\hbar)T/R$, i.e., Eq. (1.1).

To arrive at Eq. (1.1) we have assumed that the reservoirs feed all states up to their quasi-Fermi-energies equally. This is strictly correct only at zero temperature. At higher temperatures we assume that the reservoirs fill the states according to the Fermi distribution. The left reservoir fills the states with probability

$$f(E-\mu_1) = \{\exp[\beta(E-\mu_1)]+1\}^{-1}$$

and the right reservoir fills the states with probability

$$f(E-\mu_2) = \{\exp[\beta(E-\mu_2)]+1\}^{-1} .$$

The net current flowing from the left reservoir to the right reservoir is now given by

$$I=(e/2\pi\hbar)\left[\int dE\frac{-df}{dE}T(E)\right](\mu_1-\mu_2)\ , \tag{2.6}$$

instead of Eq. (2.2). Here

$$\frac{-df}{dE}\cong[f(E-\mu_1)-f(E-\mu_2)]/(\mu_1-\mu_2)$$

is the derivative of the equilibrium Fermi distribution $f(E-E_F)$. Similarly, to determine the chemical potentials we have to multiply Eqs. (2.3) and (2.4) by $-df/dE$ and integrate them over the energy. This yields a voltage

$$eV=\frac{\int dE(-df/dE)R(E)(\partial n/\partial E)}{\int dE(-df/dE)(\partial n/\partial E)}(\mu_1-\mu_2)\ , \tag{2.7}$$

which explicitly depends on the density of states. With Eqs. (2.6) and (2.7) we find a conductance

$$G=\frac{e^2}{\pi\hbar}\left[\int dE\frac{-df}{dE}T(E)\right]\times\frac{\int dE(-df/dE)v^{-1}(E)}{\int dE(-df/dE)R(E)v^{-1}(E)}\ , \tag{2.8}$$

where we have used $\partial n/\partial E=1/\pi\hbar v(E)$. If we assume that the velocity has a negligible energy dependence within the range where $\partial f/\partial E$ is appreciable, Eq. (2.8) reduces to one already obtained by Engquist and Anderson.[18] At zero temperature we have $-df/dE=\delta(E-E_F)$ and Eq. (2.8) reduces to Eq. (1.1).

Each energy range can be considered to provide a separate channel for transmission; in that sense Eq. (2.8) is a *multichannel* result. Indeed, we find in Sec. IV that the conductance in the presence of channels with the same energy has the same structure as Eq. (2.8). Naively, one might think that the generalization of Eq. (1.1) to finite temperatures might proceed by adding the conductances for each energy in parallel, i.e.,

$$G=(e^2/\pi\hbar)\int dE(-df/dE)T(E)/R(E)\ .$$

As Eq. (2.8) shows, this is somewhat misleading. We intentionally avoid the work "incorrect" here for reasons which will become clearer from the discussion in Sec. VI.

III. THE MANY-CHANNEL CASE, GENERALITIES

We now consider a similar physical situation as before, except that the *ideal* conductors include N independent conducting channels characterized, for example, by different quantum numbers for the limited motion transverse to the wire. For zero temperature all the channels have the same energy (the continuous energy associated with the velocity along the wire adds to the discrete transverse level, to give the Fermi energy E_F). These channels can, however, be scattered into each other by the "barrier" which is now represented by a $2N\times 2N$ scattering matrix. All the scattering in the sample is elastic. Evanescent states with imaginary values of the wave number parallel to the wire can be neglected.[20] They cannot contribute to the current, and can only have an effect on the chemical potential right at the interface between the scatterer and the ideal conductor. An incoming wave (see Fig. 2) from the left ith channel has probabilities $T_{ji}=|t_{ji}|^2$ and $R_{ji}=|r_{ji}|^2$ for transmission into the right-hand side (RHS) jth channel and reflection into the left-hand side (LHS) jth channel, respectively. The analogous matrices for incoming waves from the RHS are denoted by primes. The $2N\times 2N$ matrix $\underline{S}$ given by

$$\underline{S}=\begin{bmatrix} r & t' \\ t & r' \end{bmatrix} \tag{3.1}$$

is unitary due to current conservation because the T_{ij}, R_{ij} matrices transform the lead *currents*. Furthermore, when time-reversal symmetry holds,

$$\underline{S}\,\underline{S}^*=\underline{\mathbb{1}},\quad \underline{S}=\underline{\tilde{S}}\ , \tag{3.2}$$

where the asterisk denotes complex conjugation, the tilde the matrix transpose, and $\underline{\mathbb{1}}$ is the unit matrix.

We generalize Fig. 1(b) to the present case by assuming that all electrons *arriving* at the sample from all the LHS channels are characterized by a single chemical potential μ_1. The electrons arriving in the RHS channels are likewise characterized by a single chemical potential $\mu_2<\mu_1$. The chemical potentials of the ideal conductors themselves are denoted by μ_A and μ_B, respectively. In general we can expect that $\mu_A\neq\mu_1$ and $\mu_B\neq\mu_2$; μ_A and μ_B will be determined in the next section.

We make the following assumptions about the coupling to the reservoirs, generalizing those of Fig. 1(b) for the single-channel case. The reservoirs supply electrons so as to keep all the states in the ideal conductor going away from the bath below an energy μ_1 (μ_2) full. On the other hand, any electron incident on a reservoir, and coming from the system, is absorbed and thermalizes in the reservoir. We also assume that the waves in different channels, incident on the system, are incoherent with each other. Thus, interference effects between different input channels are neglected. These are very plausible physical assumptions but they are certainly not unique. There are alternative assumptions, which give the incident carriers a different distribution among channels, leading to different results. We discuss these alternatives in Sec. VI. Here we

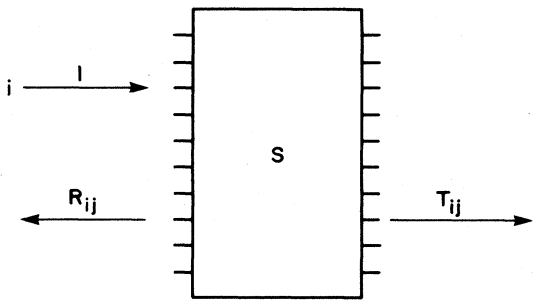

FIG. 2. A multichannel system S. A unit current in channel i is reflected into channel j with probability R_{ji} and transmitted into j with probability T_{ji}. Indices i and j run from 1 to N. Input channels are incoherent with each other.

only emphasize that these distinctions become unimportant in the limit of small transmission.

IV. DERIVATION OF THE MANY-CHANNEL CONDUCTANCE FORMULA

The generalization to many channels of the derivation given in Sec. II is now straightforward. Again we only have to consider the energy range between μ_2 and μ_1. In this range, carriers are injected into the perfect wire only from the LHS reservoir. The current injected into channel j by this reservoir is $ev_j(\partial n_j/\partial E)(\mu_1-\mu_2)$. The density of states (with positive velocity) is given by $\partial n_j/\partial E=1/\pi\hbar v_j$. Therefore, the current fed into the jth channel is $(e/\pi\hbar)(\mu_1-\mu_2)$, i.e., independent of the channel velocity. Thus, the reservoir feeds all channels with the same current. The current from the LHS jth channel transmitted into the RHS ith channel is $(e/\pi\hbar)T_{ij}(\mu_1-\mu_2)$. The total current in the ith channel to the right then becomes

$$(e/\pi\hbar)\left[\sum_{j=1}^{N} T_{ij}\right](\mu_1-\mu_2) .$$

It is convenient to introduce a total transmission probability and reflection probability into the ith channel:

$$T_i=\sum_j T_{ij},\quad R_i=\sum_j R_{ij} . \tag{4.1}$$

Thus the current in the ith RHS channel is

$$I_i=\frac{(\mu_1-\mu_2)e}{\pi\hbar}T_i , \tag{4.2}$$

and the total current is given by

$$I_{\text{tot}}=\sum_i I_i=\frac{(\mu_1-\mu_2)e}{\pi\hbar}\sum_i T_i=\frac{(\mu_1-\mu_2)e}{\pi\hbar}\text{Tr}tt^{\dagger} . \tag{4.3}$$

We can also express the current in terms of the R_{ij}. The current in each channel to the left of the barrier is

$$\begin{aligned} I_i&=\frac{(\mu_1-\mu_2)e}{\pi\hbar}\left[1-\sum_j R_{ij}\right] \\ &=\frac{(\mu_1-\mu_2)e}{\pi\hbar}(1-R_i) , \end{aligned} \tag{4.4}$$

and the total current is

$$I_{\text{tot}}=\sum_i I_i=\frac{(\mu_1-\mu_2)e}{\pi\hbar}\sum_i(1-R_i) . \tag{4.5}$$

Comparing Eqs. (4.3) and (4.5), we find that current conservation implies

$$\sum_i T_i=\sum_i(1-R_i) . \tag{4.6}$$

We also note, for completeness, that the more detailed equalities

$$R_i'+T_i=1,\quad R_i+T_i'=1 , \tag{4.7}$$

are valid only between transmission *to* the right (left) and reflection *from* the right (left). They state that if all incident channels on both sides of the barrier are fully occupied, all outgoing channels will also be fully occupied.

To determine the chemical potentials μ_A and μ_B, we generalize the discussion leading to Eqs. (2.3) and (2.4) by summation over channels. The probability for carriers to be transmitted into the ith channel on the RHS of the sample, if the LHS incident channels are occupied equally, is T_i. This gives rise to $T_i(\partial n_i/\partial E)(\mu_1-\mu_B)$ occupied states. The number of occupied states in all channels to the right is

$$\sum_i T_i(\partial n_i/\partial E)(\mu_1-\mu_B) .$$

The number of unoccupied states is

$$\sum_i(2-T_i)(\partial n_i/\partial E)(\mu_B-\mu_2) .$$

Again μ_B is determined such that the number of occupied states is equal to the number of unoccupied states,

$$\sum_i T_i(\partial n_i/\partial E)(\mu_1-\mu_B)=\sum_i(2-T_i)(\partial n_i/\partial E)(\mu_B-\mu_2) . \tag{4.8}$$

Similarly, the chemical potential to the left is determined by

$$\begin{aligned} &\sum_i(1+R_i)(\partial n_i/\partial E)(\mu_1-\mu_A) \\ &\qquad=\sum_i(1-R_i)(\partial n_i/\partial E)(\mu_A-\mu_2) . \end{aligned} \tag{4.9}$$

Therefore, the voltage across the sample is now given by

$$eV=\mu_A-\mu_B=\frac{\sum_i(1+R_i-T_i)v_i^{-1}}{2\sum v_i^{-1}}(\mu_1-\mu_2) , \tag{4.10}$$

where we have used $\partial n_i/\partial E=1/\pi\hbar v_i$. With the help of Eqs. (4.2) or (4.5) we obtain a conductance

$$G=\frac{e^2}{\pi\hbar}\sum_i T_i\frac{2\sum_i v_i^{-1}}{\sum_i(1+R_i-T_i)v_i^{-1}} . \tag{4.11}$$

This result is identical to that of Azbel.[10] Azbel invoked the same assumptions about the nature of the incident streams and their role in determining the populations of the outgoing streams as we do. We note that this result is different from that of Refs. 3 and 9. For uncoupled channels ($T_{ij}=T_j\delta_{ij}$, $R_{ij}=R_j\delta_{ij}$) Azbel's result [Eq. (4.11)] does *not* reduce to the parallel addition approach, as already indicated in Sec. II in the discussion of energy-dependent transmission coefficients. Equation (4.11) has the same form as the finite-temperature result of Engquist and Anderson,[18] respectively [Eq. (2.8)]. Indeed, we could derive Eq. (2.8) by considering the index i in Eq. (4.11) as continuous, i.e., replacing it by E, and by replacing the sum over i with an integral over $-(df/dE)dE$. Channels are added the same way, whether they are degenerate channels as in Eq. (4.11), i.e., channels belonging to the same energy E_F, or channels belonging to different energies as in Eq. (2.8). A generalization of Eq. (4.11) to finite temperatures will be presented elsewhere.[21]

To derive Eq. (4.11), we have assumed that the perfect wires to the right and left of the obstacle are identical. To be more precise, we have assumed that for each channel to the left characterized by a velocity v_i^l, there exists one channel to the right with a velocity v_i^r which is equal to v_i^l. If we drop this requirement and generalize to the case where there are N channels to the left and N' channels to the right, then a repetition of the steps leading to Eq. (4.11) yields

$$G=\frac{2e^2}{\hbar\pi}\frac{\sum_{i=1}^{N'}T_i}{1+\frac{1}{g_l}\left[\sum_{i=1}^{N}(v_i^l)^{-1}R_i\right]-\frac{1}{g_r}\left[\sum_{i=1}^{N'}(v_i^r)^{-1}T_i\right]}. \tag{4.12}$$

Here we have introduced

$$g_l=\sum_{i=1}^{N}(v_i^l)^{-1}$$

and

$$g_r=\sum_{i=1}^{N'}(v_i^r)^{-1},$$

which are proportional to the density of states to the left and right of the barrier. Equation (4.12) is our most general result. It reduces to Eq. (4.11) in the case where $N=N'$ and $v_i^l=v_i^r$.

Equations (4.11) and (4.12) raise the following interesting question: What happens when one of the channel velocities vanishes? This occurs when the Fermi level E_F coincides with one of the transverse levels in at least one of the leads and the longitudinal part of the energy vanishes. For a system where the Fermi energy E_F can be changed, such as in metal-oxide-semiconductor device, E_F may be made to cross a particular transverse level and to switch on or off its contribution to the conductance. This will lead to sharp changes in the conductance near these crossings. For simplicity, take the case where symmetry exists between the right and left leads. Denote the energy at which the $(N+1)$th channel becomes conducting by E_{N+1}. When the Fermi energy E_F approaches E_{N+1} from above v_{N+1} tends to zero. Once v_{N+1} is smaller than

$$\left[\sum_{i=1}^{N}v_i^{-1}\right]^{-1},$$

the term including v_{N+1}^{-1} in the sum in the denominator and in the density-of-states factor in Eq. (4.11) dominates and the conductance of the $N+1$ channels becomes

$$G_{N+1}(E_{N+1})=\frac{e^2}{\pi\hbar}\sum_{i=1}^{N}T_i. \tag{4.13}$$

Here we have used the fact that as $v_{N+1}\to 0$, $R_{N+1}\to 1$ and $T_{N+1}\to 0$. The diverging one-dimensional density of states of the $(N+1)$th channel causes it to determine μ_A and μ_B and we obtain, in fact, from Eq. (4.10), $eV=\mu_A-\mu_B=\mu_1-\mu_2$. On the other hand, if E_F approaches E_{N+1} from below, we have N conducting channels with nonsingular densities of states and the conductance is simply given by Eq. (4.11). But G_N [Eq. (4.11)], for $E_F=E_{N+1}$, is in general larger than $G_{N+1}(E_{N+1})$. Thus, the conductance exhibits a discontinuous drop at $E_F=E_{N+1}$. Typically, with increasing Fermi energy the conductance increases. An estimate for large N shows that the conductance of $N+1$ channels, after the initial drop, exceeds the conductance of N channels, if the Fermi energy exceeds $E_{N+1}+E_{N+1}/N^2$. This behavior of the conductance as a function of the Fermi energy is *delicately* dependent on the exact details of our model.

In the case where all the transmission probabilities are small, $T_i\ll 1$ and $R_i\approx 1$. Equations (4.11) and (4.12) then yield

$$G=\frac{e^2}{\pi\hbar}\sum_i T_i=\frac{e^2}{\pi\hbar}\mathrm{Tr}tt^{\dagger}, \tag{4.14}$$

in agreement with Refs. 3, 8, and 9. The additional factors in Eqs. (4.11) and (4.12) are only of importance in very small samples. The weak transmission limit [Eq. (4.14)] is relevant in many situations. Anderson *et al.*[3] considered an ensemble of samples characterized by a *fixed conductance* G. As we go from small samples to larger samples, we have to increase the number of channels N to build up the sample in the transverse direction. If we keep the conductance fixed, the length of the sample has to grow accordingly. To keep the conductance constant with growing channel number, the total transmission probabilities have to scale like $1/N$, i.e., $T_i\sim 1/N$. Thus, the conductance of large samples in this ensemble is governed by Eq. (4.14).

In the next section we use a different scaling procedure. We ask, how does the conductance of samples of *fixed length* grow as we increase the number of channels? In this case, the transmission probabilities are of order one and independent of the channel number. This causes G to grow as N. Clearly, for large samples, the conductance is proportional to the cross section of the sample. If the length of the samples is short enough, this suggests that Eqs. (4.11) and (4.12) are relevant even in the limit of large N. However, with increasing N we might obtain channels with very small velocities. The same reasoning that leads to Eq. (4.13) can be applied. If the total density of states is dominated by the density of states of channels with very small velocities, Eq. (4.14) is, even for this scaling procedure, the relevant expression for the conductance in the limit of large N.

V. APPLICATION TO INTERFERENCE PHENOMENA IN SMALL RINGS

The possibility of Aharanov-Bohm—type oscillations in the conductance of small rings or cylinders with a magnetic flux ϕ through its opening has attracted attention.[5–7,22–33] Here, *small* means circumferences less than the inelastic or phase-breaking diffusion length l_T. The *equilibrium* properties of a ring are periodic in ϕ, with a period equal to the single-electron flux quantum, $\phi_0=h/e$.[34–37] The work of Al'tshuler, Aronov, and Spivak,[5] based on an expansion in the disorder in an effec-

tively multichannel system, suggested a periodicity of the *resistance* in the flux ϕ, with a period $\phi_0/2=h/2e$. On the other hand, an exact calculation[6,7] of the conductance of a purely one-dimensional ring, with one-dimensional leads, yields a fundamental periodicity with a period ϕ_0. The amplitude of the fundamental with period ϕ_0 is, in general, larger than that of the first harmonic (period $\phi_0/2$), except for very special cases.[23] It could be argued, however, that this is an artifact of the single-channel case and that when the number of channels becomes large, the term with period $\phi_0/2$ may become dominant. A more specific reason for this expectation which, however, is not borne out by our later considerations, is based on the following argument. The oscillation[24–28] with period $\phi_0/2$ is apparently associated with coherent backscattering[32,33] due to a constructive interference (at $\phi=0$) between two paths going around the ring in opposite directions, as illustrated in Fig. 2.5 of Ref. 33. These two waves return to their original common starting point with a phase difference of $4\pi\phi/\phi_0$, leading to a period $\phi_0/2$, and a minimum transmission at $\phi=0$. On the other hand, the contribution with the longer period ϕ_0 is due to direct interference between waves propagating along the two branches of the ring. Their phase difference, to the extent that it depends on ϕ, is $2\pi\phi/\phi_0$. Since the flux-independent phases of these terms contain random contributions, the contribution to the conductance with period ϕ_0 resulting from their interference can be at any phase of its oscillation at $\phi=0$, with equal probabilities. It is plausible, therefore, that for many channels this contribution will almost cancel out and that the contributions with period $\phi_0/2$ can become dominant. We mention, however, that in the important case where all the T's are very small, the extra path length involved in the extra circuit around the ring for the $\phi_0/2$ periodic terms will diminish their relative importance.

We can use our multichannel conductance formulas to estimate how the contributions periodic in ϕ_0, measured relative to the ϕ-independent contributions, vary with channel number N. We invoke the results of Ref. 5 for the small-ring geometry to determine the relative size of the contributions that are periodic in $\phi_0/2$. Both turn out to scale as $1/N$ so that their ratio remains constant as $N\to\infty$.

We analyze below the contribution to G with period ϕ_0. The contribution is mainly due to interference between partial waves moving in the upper branch of the ring and those in the lower branch. Since all the former pick up the same phase from the flux ϕ, and all the latter again pick up an *identical* phase (but different from the aforementioned) it is possible to estimate the size of the total periodic contribution. Consider Fig. 3 with N channels in both the upper and lower parts of the loop and in the leads. The channels in the upper branch pick up a phase change $(\pi+\epsilon)\phi/\phi_0$ from the vector potential where the constant ϵ allows for asymmetry in the loop. The channels in the lower branch pick up a phase change $(-\pi+\epsilon)\phi/\phi_0$. These two sets of channels come together at the RHS transmission-end splitter (see Fig. 3) and interfere to produce an output there. Note that our picture emphasizes the two contacts with the ring, a point not considered in Ref. 5. We will hereafter assume that the channels in the loop, as they arrive at the terminating contact, or "splitter," are populated relatively uniformly. If this is not the case, our considerations given below *underestimate* the flux sensitivity of the ring. We will also assume that the phases and amplitudes of the wave functions arriving at the LHS input splitter are relatively independent of ϕ. This implies that these waves are largely waves arriving at the splitter without having made a complete circuit and thus have not been affected by the interference effects in the ring.

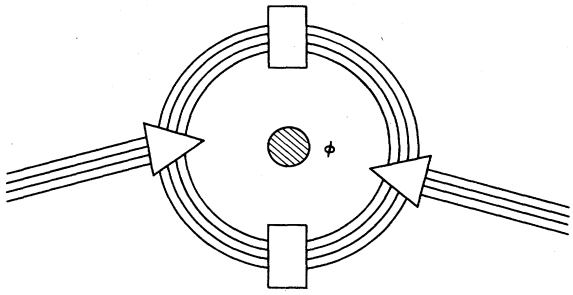

FIG. 3. Schematics of a many-channel ring. N input channels, at the left, are split into N channels in the upper branch and N in the lower branch. These $2N$ channels are then combined into the N output channels, at the right. A flux ϕ is applied in the opening between the upper and lower branches.

Consider now the probability T_{ij} that a wave arriving in channel j, to the left of the ring, is transmitted into the final outgoing channel i. Let the wave amplitudes arriving at the right-end splitter, along the N upper channels, be of the form $\alpha_q e^{i\xi_q} e^{i(\pi+\epsilon)\phi/\phi_0}$: q is a channel index, $1\le q\le N$. We assume that the magnitudes α_q are comparable and that the ξ_q are randomly distributed. Let the wave amplitudes arriving via the lower channels be of the form $\beta_p e^{i\eta_p} e^{-i(\pi-\epsilon)\phi/\phi_0}$. For simplicity, we treat the case where the α_q and the β_p are comparable. We now consider the effect of these arriving waves in a single outgoing channel i. We again assume that the loop channels couple to the outgoing channel in a relatively uniform way and that the additional phase retardations in the splitter do not alter the relatively random distribution of the phases due to the contributions from different channels within the loop branches. The outgoing wave in channel i is, therefore, a sum of terms

$$\psi_i=\sum_q \gamma_q e^{i\theta_q} e^{i(\pi+\epsilon)\phi/\phi_0}+\sum_p \delta_p e^{i\chi_p} e^{-i(\pi-\epsilon)\phi/\phi_0}, \tag{5.1}$$

with the γ's and δ's comparable in magnitude. The transmitted amplitude $|\psi_i|^2$ then becomes

$$\begin{aligned} T_{ij}&=|\psi_i|^2 \\ &=\sum_q \gamma_q^2+\sum_p \delta_p^2+\sum_{q\neq q'} \gamma_q\gamma_{q'} e^{i(\theta_q-\theta_{q'})}+\text{c.c.} \\ &\quad+\sum_{p\neq p'} \delta_p\delta_{p'} e^{i(\chi_p-\chi_{p'})}+\text{c.c.} \\ &\quad+\sum_{q,p} \gamma_q\delta_p e^{i(\theta_q-\chi_p)} e^{2i\pi\phi/\phi_0}+\text{c.c.}\ . \end{aligned} \tag{5.2}$$

Equation (5.1) contains $2N$ incoherent contributions. Thus, we can expect the total magnitude of $|\psi_i|^2$ to be $2N$ times that of a single contribution. It is only the final sum in Eq. (5.2) which depends on ϕ. There are N^2 terms in this final sum, again incoherent with each other. Thus, the net contribution of these terms, varying periodically with period ϕ_0, is proportional to $(N^2)^{1/2}$ or to the total amplitude of $|\psi_i|^2$. This final sum in Eq. (5.2), however, comes with a random phase. Let us rewrite Eq. (5.2) in the form

$$T_{ij}=a_{ij}^{(1)}+a_{ij}^{(2)}\cos\theta_{ij}^{(2)}+a_{ij}^{(3)}\cos\theta_{ij}^{(3)} +a_{ij}^{(4)}\cos\left[2\pi\left[\frac{\phi}{\phi_0}\right]-\theta_{ij}^{(4)}\right] . \tag{5.3}$$

The four terms in Eq. (5.3) represent the four sums in Eq. (5.2). The a coefficients are all positive and comparable. They will simply be denoted by a hereafter. The θ's are random phases.

Equation (4.14) tells us that

$$G=\frac{e^2}{\pi\hbar}\sum_{i,j}T_{ij}$$

if the transmission probabilities are small. Thus, we can simply sum Eq. (5.3) over all incident and transmitted channels. The first three terms in Eq. (5.3) give rise to a flux-insensitive conductance. The sum over the $a_{ij}^{(1)}$ terms gives rise to a conductance proportional to N^2a. The sum of the next two terms represents fluctuations, in the flux-insensitive conductivity, of the order of $(N^2)^{1/2}a$. The sum over the final term in Eq. (5.3) gives a contribution of order $(N^2)^{1/2}a\cos[2\pi(\phi/\phi_0)-\theta]$. Here the phase θ is either 0 or π. The scattering matrix for the ring discussed above must be unitary and time-reversal invariance implies $\underline{S}(\phi)=\underline{\tilde{S}}(-\phi)$, instead of Eq. (3.2). These symmetry properties have the consequence that the conductance calculated with the help of Eq. (4.14) is symmetric if the flux is reversed, i.e., $G(-\phi)=G(\phi)$. The random phases θ_{ij} obey a sum rule. In contrast, the magnetoconductance, based on Eqs. (4.11) and (4.12) which include velocity factors, can indeed be asymmetric as shown in Ref. 38. Equation (4.14), without velocity factors, however, tells us that the conductance has either a local maximum or a local minimum at $\phi=2\pi n\phi_0$. Furthermore, our calculation shows that the terms in G which have the period ϕ_0 are of order $1/N$ of the total conductance. In Ref. 6, it was argued correctly that the contribution to each T_{ij} with period ϕ_0 is of relative order unity. However, since G is a sum of N^2-such random terms, the relative total effect is $1/N$, *not* $1/\sqrt{N}$ as speculated in Ref. 6. Any effects that emphasize the role of particular channels, such as speckle patterns[20,39] or resonances[7,22] in the T_{ij}, may as mentioned before, diminish the effective N. Thus, our $1/N$ estimate may be regarded as an order of magnitude of a *lower* bound on the amplitude of the contribution with period ϕ_0.

We could construct estimates, based on arguments similar to the one presented above, for the relative size of the part of the conductance with period $\phi_0/2$. Rather than doing that, we refer the reader to Ref. 5. For a thin ring the flux-sensitive quantum correction to the flux-insensitive conductance (Eq. 5 in Ref. 5) can be cast into the form

$$\Delta G=e^2/\hbar f(2\phi/\phi_0) . \tag{5.4}$$

Here f is a periodic function of period one which depends only on the ratio of the circumference to the inelastic length, not on the cross section of the ring. If we now assume that the flux-insensitive conductance is proportional to the cross section, i.e., proportional to the number of channels N, then Eq. (5.4) implies that the amplitude of the oscillations with period $\phi_0/2$ is also of order $1/N$. Thus, in a particular N-channel ring the terms with the two periods under consideration are, as far as their N dependence is concerned, of the same order $(1/N)$. Only when an experiment involves an effective averaging over an ensemble of many rings,[30,31,40,41] such as in a long cylinder[24,27] or in a system with many incoherent rings,[28] can we expect a dominance of the $\phi_0/2$ term, due to the averaging out of the ϕ_0 terms that have a different sign in different ensemble members. This is also the case in the calculation of Ref. 5, where ensemble-averaged propagators are used in the calculation, and the ϕ_0-periodic term is suppressed by the choice of procedure.

The ensemble averaging has also been invoked as a formal device, without discussing the physical conditions for its applicability, in the calculation of the participation ratio of closed, small-channel number rings in Refs. 30 and 31. It is not clear why the participation ratio in a closed ring, i.e., in a Hamiltonian system without resistance,[37] should explain resistance oscillations.

VI. ALTERNATIVE RESULTS AND FURTHER GENERALIZATIONS

The situation discussed in connection with Eq. (1.1) has a particular simplicity. The single energy level and single channel result in a well-defined state once the current in the ideal conductors is specified. If we go beyond this, as in the case of Eq. (2.8) where we deal with an energy range, we have to ask how the incident flux is distributed over energy. It is clear that this distribution will have an effect on the net transmission of carriers and therefore on the resistance. One could, in principle, proceed at this point without extraneous assumptions about the energy distribution by treating a complete system with its power source included. This has only been done in the one-dimensional case at zero temperature; in that case, we have a system which exhibits energy storage, but not a resistance.[37] A similar, but more ambitious calculation, which couples the current carrying degrees of freedom to a reservoir providing dissipation, e.g., as in the treatments of macroscopic quantum tunneling,[42] does not yet exist. As a result, we are reduced to plausible assumptions about the distribution of the incident current, which may or may not be satisfied in specific physical arrangements. Our assumption, leading to both Eqs. (2.8) and (4.11), is as follows: The chemical potential for all of the carrier classes incident from the left (whether the classes correspond to different channels or to different energies) is given one value, the chemical potential for all of the car-

rier classes incident from the right is given a different value. It is, in fact, surprising that this assumption has been taken for granted in all the papers that treat the one-dimensional case with energy-dependent transmission,[18,43,44] but has not even been mentioned by most of the multichannel investigators.

In the multichannel case, there are several alternative possible assumptions. We stress that these are alternative assumptions and do not characterize them as correct or incorrect. One alternative pictures an incident-carrier distribution which is controlled by a scattering mechanism within the leads bringing the carriers to the sample. Thus, for example, we might envision the typical solid-state conductivity relaxation-time case with the carrier distribution specified by a Fermi sphere shifted in momentum space. In that case, carrier classes moving along the direction of the arriving current will be farthest from equilibrium, whereas carriers moving almost parallel to the sample interface will be much closer to equilibrium.

Langreth and Abrahams[8] have invoked another alternative set of assumptions, which we describe by adapting and extending a discussion in Ref. 19. Reference 8 assumes (in distinction to our treatment) that the chemical potential for all channels on the left, as determined by the average between both left and right moving carriers, is the same. Similarly the chemical potential for all channels on the right is the same. The chemical potential on the left is, of course, different from that on the right. At first sight, this seems plausible and reasonable. It corresponds to our common-sense notion that there should be a fixed driving force across a sample. Furthermore, the fact that all the channels on the left have the same value of chemical potential means that they are in equilibrium with each other. That may seem appropriate: They are in contact with the same source. On closer inspection, however, the assumptions of Ref. 8 lead to questions. What is the mechanism which keeps the population between different channels perfectly in equilibrium, but permits the left and right moving components in the same channel to be out of equilibrium? Furthermore, in Ref. 8 the way in which the current is distributed among the channels is a function only of the scattering behavior of the conductor and is unrelated to the mechanism in the leads which produce the presumed equilibration between the channels. Satisfactory answers to these questions about Ref. 8 are likely to exist but remain to be supplied.

Still another possibility was discussed in Ref. 19 and in that publication incorrectly identified as identical to the approach of Langreth and Abrahams.[8] This additional possibility considers a long chain of identical obstacles scattering incoherently with each other. In that case, the current flow emerging from one obstacle and incident on the next must mirror the current distribution among channels incident on the original obstacle.

Let us now turn aside from the discussions of alternatives to Eqs. (2.8) and (4.12) and stress the broad applicability of the viewpoint represented by Eq. (1.1). The generalization to many channels discussed in this paper is only one of many ways in which the approach, originally introduced in Ref. 1, can be expanded. It is apparent that the transmission and reflection properties of a sample determine the current flow if we specify the incident flux and also determine the extent to which these moving carriers accumulate on each side of the sample. Self-consistent screening, which will eliminate the long wavelength components of this charge pileup, then determines the potential difference. The scattering in the sample does not have to be elastic.[45] The scattering matrix need not be confined to independent electron effects, as we have assumed, but can include many-body interactions.[46] As already pointed out in this paper, the medium attached to one end of the sample need not be identical to that attached to the other end. The carriers need not be charged; phonon transport, as for example in a Kapitza resistance, should be amenable to a similar view. Finally, we mention that other transport properties, such as the thermal conductivity of the electrons and the associated thermoelectric coefficients,[18,47] can also be expressed using similar ideas.

Our emphasis on the generality of the approach must also be accompanied by a warning about its limitations. In general, if the current distribution in the leads (distribution in space, by channel or by energy) does not match the preferences of the sample, there will be an interface resistance, as discussed in Ref. 19.

Note added in proof. Section V analyzed resistance oscillations for a small ring, with a flux period h/e. These oscillations have now been seen in gold rings, in a striking and clear fashion.[48]

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