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Localization of acoustic waves

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The localization of sound waves moving in a random array of hard scatterers is studied for both two- and three-dimensional systems using a diagrammatic technique. For $d=2$ acoustic waves are found to be localized for all frequencies with the localization length growing exponentially for both high and low frequencies. For $d=3$ there are mobility edges at an intermediate frequency and at a high frequency. Between these mobility edges the sound waves are localized. The connection between this problem and quantum-mechanical electron localization is discussed.

I. INTRODUCTION

In some recent papers John, Sompolinsky, and Stephen[1,2] (JSS) discussed the localization of phonons near two dimensions. They used continuum field theory and renormalization-group techniques to conclude that for $d \leq 2$ all states are localized. In these dimensions they found that the localization length, ξ, increased as the frequency of the phonons was decreased, e.g., for $d=2$, ξ grew exponentially as the frequency was decreased. For $d>2$ JSS found that at sufficiently low frequencies the phonon states were extended but at a higher frequency there was a mobility edge and that all phonon states above this frequency were localized. All of these results show that high-frequency phonons are more easily localized than low-frequency phonons. It should be remarked that since JSS use a continuum field theory there are physical parameters in their calculation that are not specified. Furthermore, these authors do not explicitly calculate coefficients in their work.

The results of JSS should be contrasted with quantum-mechanical electron localization, i.e., Anderson localization.[3-7] There one finds that electrons are most easily localized at low energies since for low energies quantum-mechanical effects are most important.

In this paper the propagation of acoustical (i.e., sound) waves in a fluid with a dilute random distribution of stationary hard-disk ($d=2$) or hard-sphere ($d=3$) scatterers is studied. The problem of wave propagation in a random media is a subject with a long history.[8,9] It is still of considerable interest. Here I extend the usual theory[8] by incorporating recent advances made on the theory of quantum-mechanical electron localization.[3-7] Using a self-consistent diagrammatic theory[6] the differences between electron and "phonon" localization are studied in detail. It is shown that the connection between these two types of localization can be simply understood both physically and mathematically. For low frequencies, defined more precisely below, I recover the form of the results given by JSS and give explicit coefficients for the localization lengths, etc. These coefficients are important in determining the experimental consequences of phonon localization. Further, since the model considered here is well defined for "all" frequencies I can consider higher frequencies than those considered by JSS. For $d=2$ I find that acoustic waves are always localized and that the localization length grows exponentially for *both* high and low frequencies with a minimum at some characteristic frequency. For $d=3$ I find that if there is a mobility edge for intermediate frequencies then for sufficiently high frequencies there is another mobility edge where the waves again become delocalized. The simple physics of these results will be discussed below.

Experimental systems where the model considered here might be realized will be discussed in detail elsewhere.[10] Here we only mention that for $d=3$ sound-wave propagation in fluids with a suspension of particles is a possible candidate. For $d=2$ sound-wave propagation in fluid films with random impurities appears to be a realistic candidate.

The plan of this paper is as follows. In Sec. II the problem of acoustical wave motion in a random array of hard scatterers is reviewed and it is formulated in terms of averaged-over-the-randomness Green's functions and the average of the Green's function squared. Here we also introduce a multiple-scattering formalism to calculate these quantities.[8] In Sec. III we calculate the averaged Green's function to lowest order in the density of scatterers. As is usual for localization due to wave interference the averaged Green's function will not contain any information about localization. In Sec. IV we calculate the Green's function squared—which is related to the energy density of a wave—and discuss localization for $d=2,3$ using a self-consistent diagrammatic theory. In Sec. V we give the results for $d=2,3$ and in Sec. VI we discuss these results and make some concluding remarks.

II. FORMULATION OF THE PROBLEM

In this section I review[8] how the problem of sound-wave motion on a random array of scatterers is transformed into a many-body problem.

A. Basic definitions

The basic problem is to solve the wave equation

$$\nabla^2\phi-\frac{1}{c^2}\partial_t^2\phi=0\ , \tag{2.1a}$$

in the presence of a random array of stationary hard-disk ($d=2$) or hard-sphere ($d=3$) scatterers of radius a. Here ϕ is the wave amplitude, ∇^2 is a d-dimensional Laplacian, and c is the speed of sound in the absence of any scatterers. For typical hydrodynamical systems ϕ is the velocity potential and $\mathbf{V}=\nabla\phi$ is the fluid velocity. Since no fluid can flow into the hard scatterers, the boundary conditions to be imposed on Eq. (2.1a) are that the normal derivative of ϕ vanish at the surface of each scatterer,[11]

$$\partial_n\phi\,|_{\text{surface}}=0\ , \tag{2.1b}$$

where ∂_n denotes a normal derivative. it is important to remark here that Eq. (2.1b) represents the fundamental difference between electron and sound-wave localization. That is, Eq. (2.1a) is identical to the Schrödinger equation except that its time derivative is of second order. This difference is of no consequence once we transform to frequency or energy space. It should be noted that in this paper I sometimes will use the expression frequency of the electron in place of energy so that the connection between sound-wave localization and electron localization can be easily stated. The only real difference between the electron problem and Eqs. (2.1) is the boundary condition given by Eq. (2.1b). If the problem of a quantum-mechanical electron moving in a random array of hard scatterers was considered (i.e., the quantum-mechanical Lorentz gas), then for this case one would require that the wave function itself vanish at the surface of each scatterer. For low frequencies it is well known,[11] and easy to understand physically, that the Neumann boundary condition given by Eq. (2.1b) is not as effective in scattering a wave; the total cross section at frequency E is proportional to[11] E^{d+1} for small E for this case, as is the Dirichlet boundary condition used for the electron problem. For Dirichlet boundary conditions[11] the total cross section is, e.g., a constant for low frequencies for $d=3$. This then already explains the differences between phonon and electron localization for low frequencies discussed in Sec. I. For high frequencies the cross sections for Neumann and Dirichlet boundary conditions become identical so that at these frequencies the phonon and electron problems become identical. Since the high-frequency limit for the electron problem is the classical limit where quantum-mechanical localization effects disappear, the localization behavior discussed in Sec. I for high frequencies can also be easily understood. Physically the delocalization at high energies occurs because the waves are then very localized objects that can easily propagate between the scatterers.

We next define a Green's function for Eqs. (2.1). If we imagine a localized disturbance is created at time $t=0$ at $\mathbf{r}$ then a reasonable initial condition is[1]

$$\begin{aligned}&\phi(\mathbf{r},t\le 0)=0\ ,\\ &\partial_t\phi(\mathbf{r},t=0)=-c^2f(\mathbf{r})\ ,\end{aligned} \tag{2.2a}$$

where $f(\mathbf{r})$ is a given function. In terms of a Green's function Eq. (2.1) is given by

$$\phi(\mathbf{r},t)=\int d\mathbf{r}'G(\mathbf{r},t\,|\,\mathbf{r}')f(\mathbf{r}') \tag{2.2b}$$

with

$$\left[\nabla^2-\frac{1}{c^2}\partial_t^2\right]G(\mathbf{r},t\,|\,\mathbf{r}')=0 \tag{2.2c}$$

and

$$\begin{aligned}&G(\mathbf{r},t\le 0\,|\,\mathbf{r}')=0\ ,\\ &\partial_tG(\mathbf{r},t=0\,|\,\mathbf{r}')=-c^2\delta(\mathbf{r}-\mathbf{r}')\ ,\\ &\partial_nG(\mathbf{r},t\,|\,\mathbf{r}')\,|_{\text{surface}}=0\ .\end{aligned} \tag{2.2d}$$

As mentioned in Sec. I, in order to calculate localization effects one must consider the energy density of the injected disturbance which is proportional to the Green's function squared. We define

$$P(\mathbf{r},t\,|\,\mathbf{r}')\equiv G^2(\mathbf{r},t\,|\,\mathbf{r}')\ , \tag{2.3a}$$

and we will actually be interested in P for a given external frequency ω defined by the Laplace transform,

$$\begin{aligned}P_\omega(\mathbf{r}\,|\,\mathbf{r}')&=\int_0^\infty dt e^{i(\omega+i0)t}G^2(\mathbf{r},t\,|\,\mathbf{r}')\\ &=\int_{-\infty}^{+\infty}\frac{dE}{2\pi}G^+_{E+\omega/2}(\mathbf{r}\,|\,\mathbf{r}')G^-_{E-\omega/2}(\mathbf{r}\,|\,\mathbf{r}')\\ &\equiv\int_{-\infty}^{+\infty}\frac{dE}{2\pi}P_{E,\omega}(\mathbf{r}\,|\,\mathbf{r}')\end{aligned} \tag{2.3b}$$

with

$$G^\pm_E=G_{E\pm i0}\ . \tag{2.3c}$$

Here I have used the convolution property of Laplace transforms and denoted the internal frequency by the symbol E. In JSS they denoted E by ω.

It is the purpose of this paper to calculate the average of $P_{E,\omega}$ defined by

$$\bar{P}_{E,\omega}(\mathbf{r}\,|\,\mathbf{r}')\equiv\lim_{\substack{N,V\to\infty\\ N/V=n}}\frac{1}{V^N}\int d\mathbf{R}^NP_{E,\omega}(\mathbf{r}\,|\,\mathbf{r}')\ . \tag{2.4a}$$

Here $\{\mathbf{R}_i;i=1-N\}$ denotes the location of the ith scatterer in a volume V and the bulk limit is taken with n, the density of scatterers. In defining the average given by Eq. (2.4a) we have for simplicity neglected all correlations between scatterers so that overlapping configurations of scatterers are possible. It will be convenient to work in Fourier space with an external wave vector $\mathbf{k}$,

$$\bar{P}_{\omega,k}(E)=\int(d(\mathbf{r}-\mathbf{r}')\exp[-i\mathbf{k}\cdot(\boldsymbol{r}-\mathbf{r}')]\bar{P}_{E,\omega}(\mathbf{r}\,|\,\mathbf{r}')\ , \tag{2.4b}$$

where I have used the knowledge that space is homogeneous on the average so that $\bar{P}(\mathbf{r}\,|\,\mathbf{r}')$ is a function of $|\,\mathbf{r}-\mathbf{r}'\,|$ only. Below we show that $\bar{P}_{\omega,k}(E)$ is proportional to a diffusive hydrodynamic pole for $k,\omega\to 0$,

$$\bar{P}_{\omega,k}(E)\sim\frac{1}{-i\omega+D(E,\omega)k^2}\ . \tag{2.4c}$$

Here $D(E,\omega)$ is an internal frequency, E, and external frequency, ω, dependent diffusion coefficient that

represents, physically, the diffusion of energy in this amorphous system. Whether or not a wave at frequency E is localized is determined by the behavior of $D(E,\omega\to 0)$. For example, in $d=2$ we will find that for all E,

$$D(E,\omega\to 0)=-i\omega\xi^2(E)+O(\omega^2)\ .$$

This implies that $\bar{P}_E(\mathbf{r},t)$ does not decay in time and that it is localized in a spatial region of extent $\xi(E)$.

B. Transformation to multiple-scattering formalism

In order to transform the complicated boundary condition problem given by Eqs. (2.2) and (2.4) into a standard many-body problem we use the usual[8] multiple-scattering formalism.[7–9,12,13] It is first convenient to introduce an abstract quantum-mechanical-like notation which is representation free. We define bra and ket vectors and abstract operators, denoted by circumflexes, by, for example,

$$G_E^+(\mathbf{r}\,|\,\mathbf{r}')\equiv\langle\mathbf{r}\,|\,\hat{G}^+\,|\,\mathbf{r}'\rangle=\langle\mathbf{r}'\,|\,\hat{G}^+\,|\,\mathbf{r}\rangle\ , \tag{2.5a}$$

where the second equality is due to the reciprocal relation for Green's functions. In a wave-number representation Eq. (2.5a) reads

$$\begin{aligned}G_E^+(\mathbf{p},\mathbf{p}')&=\int\frac{d\mathbf{r}d\mathbf{r}'}{(2\pi)^d}e^{-i\mathbf{p}\cdot\mathbf{r}+i\mathbf{p}'\cdot\mathbf{r}'}G_E^{(+)}(\mathbf{r}\,|\,\mathbf{r}')\\&=\langle\mathbf{p}\,|\,\hat{G}_E^{\pm}\,|\,\mathbf{p}'\rangle\ .\end{aligned} \tag{2.5b}$$

With this notation Eq. (2.4) is given by

$$\begin{aligned}\bar{P}_{\omega,k}(E)&=\int\frac{d\mathbf{p}d\mathbf{p}_1}{(2\pi)^d}\frac{(2\pi)^d}{V}\\&\quad\times\overline{\langle\mathbf{p}_+\,|\,\hat{G}_{E+}^{\pm}\,|\,\mathbf{p}_{1+}\rangle\langle\mathbf{p}_{1-}\,|\,\hat{G}_{E-}^{\mp}\,|\,\mathbf{p}_-\rangle}\\&\equiv\int\frac{d\mathbf{p}d\mathbf{p}_1}{(2\pi)^d}\Phi_{\mathbf{p}\mathbf{p}_1}(\mathbf{k},\omega\,|\,E)\ ,\end{aligned} \tag{2.5c}$$

where

$$\mathbf{p}_\pm\equiv\mathbf{p}\pm\frac{\mathbf{k}}{2} \tag{2.5d}$$

and

$$E_\pm=E\pm\frac{\omega}{2}\ .$$

In giving the second equality in Eq. (2.5c) we have defined the function Φ and have used that from the averaging there will be an additional factor of the volume since on the average space is homogeneous.

We next define a $\hat{T}_1$ operator whose matrix elements are the transition matrix for a scatterer at $\mathbf{R}_1\equiv 1$,[7,13]

$$\hat{G}_E^{\pm}(1)\equiv\hat{G}_{E,0}^{\pm}+\hat{G}_{E,0}^{\pm}\hat{T}_1^{\pm}(E)\hat{G}_{E,0}^{\pm}\ . \tag{2.6a}$$

Here $\hat{G}(1)$ is the Green's operator when there is only one scatterer present at $\mathbf{R}_1$ and $\hat{G}_0$ is the Green's operator in the absence of scatterers,

$$\langle\mathbf{r}\,|\,\hat{G}_{E,0}^{\pm}\,|\,\mathbf{r}_1\rangle=[(E/c\pm i0)^2+\nabla^2]^{-1}\delta(\mathbf{r}-\mathbf{r}_1)\ . \tag{2.6b}$$

With this, we can in the usual way[7–9,12,13] express the N scatterer ($N\to\infty$) Green's operator as

$$\begin{aligned}\hat{G}_E^{\pm}&=\hat{G}_{E,0}^{\pm}+\sum_{i=1}\hat{G}_{E,0}^{\pm}\hat{T}_i^{\pm}(E)\hat{G}_{E,0}^{\pm}(E)\\&\quad+\sum_{i\neq j}\hat{G}_{E,0}^{\pm}\hat{T}_i^{\pm}(E)\hat{G}_{E,0}^{\pm}(E)\hat{T}_j^{\pm}(E)\hat{G}_{E,0}^{\pm}+\cdots\ ,\end{aligned} \tag{2.6c}$$

i.e., an infinite series of binary interactions where no two consecutive $\hat{T}$ operators can have the same scatterer label. The advantages of this technique are that it directly incorporates the complicated boundary conditions into the equations of motion and transforms the problem into a standard many-body or multiple-scattering problem.[4–9,12–14]

The $\hat{T}$ operators can be straightforwardly calculated by solving Eq. (2.6a) and using techniques already given in the literature.[8,13,15–17] I obtain

$$\langle\mathbf{p}\,|\,\hat{T}_1^{\pm}(E)\,|\,\mathbf{p}_1\rangle=e^{-i\mathbf{R}_1\cdot(\mathbf{p}-\mathbf{p}_1)}\langle\mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}_1\rangle\ . \tag{2.7a}$$

For $d=2$,

$$\begin{aligned}\langle\mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}_1\rangle&=-\frac{(E/c\pm i0)^2-p^2}{2\pi}\frac{a}{|\mathbf{p}-\mathbf{p}_1|}J_1(a\,|\,\mathbf{p}-\mathbf{p}_1|)\\&\quad+\frac{1}{2\pi}\sum_{m=0}^{\infty}\epsilon_m\cos(m\theta)\frac{p_1}{\zeta^{\pm}}\frac{J_m'(p_1a)}{H_m^{(\pm)'}(\zeta^{\pm}a)}[a\zeta^{\pm}J_m(pa)H_{m+1}^{\pm}(\zeta^{\pm}a)-paJ_{m+1}(pa)H_m^{\pm}(\zeta^{\pm}a)]\ ,\end{aligned} \tag{2.7b}$$

and for $d=3$,

$$\begin{aligned}\langle\mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}_1\rangle&=-\frac{(E/c\pm i0)^2-p^2}{2\pi^2}a^2\frac{j_1(a\,|\,\mathbf{p}-\mathbf{p}_1)}{|\mathbf{p}-\mathbf{p}_1|}\\&\quad+\frac{a}{2\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\frac{p_1}{\zeta^{\pm}}\frac{j_l'(p_1a)}{h_l^{(\pm)'}(\zeta^{\pm}a)}[a\zeta^{\pm}j_l(pa)h_{l+1}^{\pm}(\zeta^{\pm}a)-paj_{l+1}(pa)h_l^{\pm}(\zeta^{\pm}a)]\ .\end{aligned} \tag{2.7c}$$

Here $\epsilon_0=1$, $\epsilon_2=\epsilon_3=\cdots=2$, $\zeta^{\pm}\equiv|E/c\pm i0|$, θ is the angle between $\mathbf{p}$ and $\mathbf{p}_1$, J_m (j_m) is the ordinary (spherical) Bessel function of order m, $H_m^{\pm}=H_m^{(1,2)}$ ($h_m^{\pm}=h_m^{(1,2)}$) are the ordinary (spherical) Hankel functions of order m, prime

denotes derivative with respect to the argument, and P_l is the Legendre function of order l.

For later use we give here some useful limits of the T matrices given by Eqs. (2.7b) and (2.7c). For low frequencies and wave numbers, $Ea/c<1$, $pa<1$, and $p_1a<1$, Eq. (2.7b) can be approximated[11] by ($d=2$ here)

$$\langle \mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}_1\rangle \simeq \frac{(pa)^2+(p_1a)^2-(Ea/c)^2-2(p_1a)(pa)\cos\theta}{4\pi} \mp \frac{i}{16}[(p_1a)^2(pa)^2+2(p_1a)(pa)(Ea/c)^2\cos\theta]+\cdots , \tag{2.8a}$$

and Eq. (2.7c) can be approximated by ($d=3$ here)

$$\langle \mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}_1\rangle \simeq \frac{a}{6\pi^2}[(pa)^2+(p_1a)^2-(Ea/c)^2-\tfrac{3}{2}(p_1a)(pa)\cos\theta] \mp \frac{ia}{18\pi^2}(Ea/c)[(pa)^2(p_1a)^2+\tfrac{3}{4}(p_1a)(pa)(Ea/c)^2\cos\theta]+\cdots . \tag{2.8b}$$

Also of later use is Eq. (2.7b) in the limit[11] ($d=2$ here)

$$\langle \mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}_1\rangle\,|_{p=p_1=E/c} = \mp\frac{i}{\pi^2}\sum_{m=0}^{\infty}\epsilon_m \frac{J_m'(Ea/c)}{H_m^{(\pm)'}(Ea/c)}\cos(m\theta) , \tag{2.8c}$$

and

$$\mathrm{Im}\langle \mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}\rangle\,|_{p=E/c} = \mp\frac{1}{\pi^2}\sum_{m=0}^{\infty}\epsilon_m \frac{[J_m'(Ea/c)]^2}{[J_m'(Ea/c)]^2+[Y_m'(Ea/c)]^2} \tag{2.8d}$$

with Y_m the ordinary Neumann function of order M.

For $Ea/c \gtrsim 1$, Eq. (2.8d) is given by[18,19]

$$\mathrm{Im}\langle \mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}\rangle\,|_{p=E/c} = \mp\frac{1}{\pi^2}(Ea/c)\left[1-\frac{0.4321}{(Ea/c)^{2/3}}-\frac{0.2137}{(Ea/c)^{4/3}}+\frac{0.055\,73}{(Ea/c)^2} -\frac{0.000\,555}{(Ea/c)^{8/3}}+\frac{0.023\,25}{(Ea/c)^{10/3}}+\cdots\right] . \tag{2.8e}$$

For $d=3$, the analogous results are

$$\langle \mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}_1\rangle\,|_{p=p_1=E/c} = \mp\frac{i}{2\pi^2}\frac{c}{E}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\frac{j_l'(Ea/c)}{h_l^{(\pm)'}(Ea/c)} , \tag{2.8f}$$

and

$$\mathrm{Im}\langle \mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}\rangle\,|_{p=E/c} = \mp\frac{c}{2\pi^2E}\sum_{l=0}^{\infty}(2l+1)\frac{[j_l'(Ea/c)]^2}{[j_l'(Ea/c)]^2+[y_l'(Ea/c)]^2} . \tag{2.8g}$$

For $Ea/c \gtrsim 1$, Eq. (2.8g) is given by[18,19]

$$\mathrm{Im}\langle \mathbf{p}\,|\,\hat{T}^{\pm}(E)\,|\,\mathbf{p}\rangle\,|_{p=E/c} = \mp\frac{a^2E}{4\pi^2c}\left[1-\frac{0.8642}{(Ea/c)^{2/3}}-\frac{0.4163}{(Ea/c)^{4/3}}+\frac{0.7532}{(Ea/c)^2} -\frac{0.029\,85}{(Ea/c)^{8/3}}+\frac{0.058\,62}{(Ea/c)^{10/3}}+\cdots\right] . \tag{2.8h}$$

C. Diagrammatic rules

We next give the diagram technique for calculating $\bar{G}$ and Φ [cf. Eq. (2.5c)].[8] Since space is homogeneous, on the average, $\bar{G}$ is diagonal in the wave-vector representation and can be written

$$\langle \mathbf{p}\,|\,\hat{G}_E^{\pm}\,|\,\mathbf{p}_1\rangle = \delta(\mathbf{p}-\mathbf{p}_1)G(p,E\pm i0) = \delta(\mathbf{p}-\mathbf{p}_1)[(E/c\pm i0)^2-p^2-\Sigma_p^{\pm}(E)]^{-1} , \tag{2.9a}$$

where we have defined a self-energy by

$$\Sigma_p^{\pm}(E)=\Gamma_p(E)\mp i\gamma_p(E) , \tag{2.9b}$$

which consists of a real part Γ_p and an imaginary part $\gamma_p(E)$. Σ is given by the sum of all irreducible diagrams and the first few of them are given in Fig. 1. The diagram rules for the calculation of $\Sigma_p^{\pm}(E)$ are as follows.[7,17]

(1) With each directed line segment $\mathbf{p}_1$, we associate a free-particle Green's function

$$G_0(p_1,E\pm i0)=[(E/c\pm i0)^2-p_1^2]^{-1} .$$

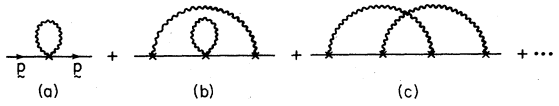

FIG. 1. First few diagrams that contribute to the self-energy Σ_p.

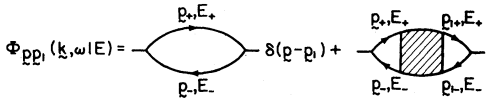

FIG. 2. Diagrammatic representation of Eq. (2.10a).

(2) An x denotes a $\hat{T}$ operator and

$\mathbf{p}\quad\mathbf{p}_1$

denotes the matrix element $\langle \mathbf{p} \mid \hat{T}_i^{\pm}(E) \mid \mathbf{p}_1 \rangle$. (3) When we average over all configurations of the scatterers, the $\hat{T}$ operators are connected by the exponential factors in Eq. (2.7a). We present the connection by a wavy line. (4) With each wavy line we associate a factor of $n(2\pi)^d$ times a δ function of the sum of the momenta entering and leaving the wavy line. The δ function appears as a result of the $\mathbf{R}_i$ integrations of the exponential factors in the T matrices. (5) Diagrams with the element $\frown$ are not allowed due to the restrictions on the sums in Eq. (2.6c).

Having formulated the diagram rules for the calculation of $\bar{G}$ as an expansion in powers of n, we now turn our attention to deriving an equation for Φ. We can represent the function $\Phi_{\mathbf{pp}_1}(\mathbf{k},\omega \mid E)$ in terms of the "bubble" diagrams given in Fig. 2.[6,7] Here the directed line segments denote exact averaged Green's functions. Analytically one has

$$\Phi_{\mathbf{pp}_1}(k,\omega \mid E)=\bar{G}(p_+,E_+ +i0)\bar{G}(p_-,E_- -i0)[\delta(\mathbf{p}-\mathbf{p}_1)+\Gamma_{\mathbf{pp}_1}(\mathbf{k},\omega \mid E)\bar{G}(p_{1+},E_+ +i0)\bar{G}(p_{1-},E_- -i0)] \ . \quad (2.10a)$$

The complete four-point vertex function $\Gamma_{\mathbf{pp}_1}$ in Eq. (2.10a) can be expressed as the sum of all reducible and irreducible diagrams connecting the upper and lower parts of the diagram. The reducible diagrams can be split into two disjoint parts by drawing a vertical line through the diagram. We can define an irreducible four-point vertex function by $U_{\mathbf{pp}_1}$ that contains only irreducible diagrams by[6,7]

$$\Gamma_{\mathbf{pp}_1}(\mathbf{k},\omega \mid E)=U_{\mathbf{pp}_1}(\mathbf{k},\omega \mid E)+\int d\mathbf{p}' U_{\mathbf{pp}'}(\mathbf{k},\omega \mid E)\bar{G}(p',E_+ +i0)\bar{G}(p',E_- -i0)\Gamma_{\mathbf{p}'\mathbf{p}_1}(\mathbf{k},\omega \mid E) \ . \quad (2.10b)$$

With this Eq. (2.10a) integrated over $\mathbf{p}_1$ can be written

$$\Phi_{\mathbf{p}}(\mathbf{k},\omega \mid E)\equiv\int d\mathbf{p}\Phi_{\mathbf{pp}_1}(\mathbf{k},\omega \mid E)=\bar{G}(p_+,E_+ +i0)\bar{G}(p_-,E_- -i0)\left[1+\int d\mathbf{p}_2 U_{\mathbf{pp}_2}(\mathbf{k},\omega \mid E)\Phi_{\mathbf{p}_2}(\mathbf{k},\omega \mid E)\right] . \quad (2.10c)$$

Equation (2.5c) is now given by

$$\bar{P}_{\omega,k}(E)=\int \frac{d\mathbf{p}}{(2\pi)^d}\Phi_{\mathbf{p}}(\mathbf{k},\omega \mid E) \ . \quad (2.10d)$$

The first few diagrammatic contributions to $U_{\mathbf{pp}_2}$ are given in Fig. 3.

Finally, Eq. (2.10c) can be written in the form of a generalized Boltzmann equation as, from Eq. (2.9a),[6,7]

$$[-2E\omega/c^2+2\mathbf{k}\cdot\mathbf{p}+\Sigma_{p_+}^{+}(E_+)-\Sigma_{p_-}^{-}(E_-)]\Phi_{\mathbf{p}}(\mathbf{k},\omega \mid E)$$
$$=\Delta\bar{G}(p_+,p_-,E_+,E_-)+\Delta\bar{G}(p_+,p_-,E_+,E_-)\int d\mathbf{p}_2 U_{\mathbf{pp}_2}(\mathbf{k},\omega \mid E)\Phi_{p_2}(\mathbf{k},\omega \mid E) \ , \quad (2.11a)$$

with

$$\Delta G(p_+,p_-,E_+,E_-)=\bar{G}(p_+,E_+ +i0)-\bar{G}(p_-,E_- -i0) \ . \quad (2.11b)$$

In the calculation given below we take $E>0$. The result for $E<0$ can be obtained from these results.

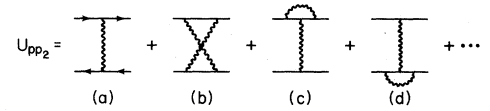

FIG. 3. First few diagrams that contribute to the irreducible four-point vertex function $U_{\mathbf{pp}_2}$.

III. THE AVERAGED GREEN'S FUNCTION

In this section we calculate the averaged Green's functions to lowest order in the density of scatterers. The dimensionless parameter that is assumed to be small is the reduced density, na^d.[20] In this section we also give a useful representation for the self-energies for both low, $Ea/c \leq 1$, and high, $Ea/c > 1$, frequencies.

The leading approximation to the self-energy for $na^d < 1$ is given in Fig. 1(a), and its analytic value is[8]

$$\Sigma_p^{\pm}(E) = n(2\pi)^d \langle \mathbf{p} \mid \hat{T}^{\pm}(E) \mid \mathbf{p} \rangle \equiv \Gamma_p(E) \mp i\gamma_p(E) . \quad (3.1)$$

For future reference we also note that for small k and ω, and small n, the $\Delta\bar{G}$ functions in Eqs. (2.11) are approximately[6,7]

$$\Delta\bar{G}(p_+,p_-,E_+,E_-) \simeq -2i\pi\delta(p^2 - E^2/c^2) , \quad (3.2)$$

so that we will need[6,7]

$$\begin{aligned}\Sigma_p^{\pm}(E) \mid_{p=E/c} &\equiv \Sigma^{\pm}(E) \\ &\equiv \Gamma(E) \mp i\gamma(E) \\ &= n(2\pi)^d \langle \mathbf{p} \mid \hat{T}^{\pm}(E) \mid \mathbf{p} \rangle \mid_{p=E/c} . \end{aligned} \quad (3.3)$$

The important part of the self-energy for localization theory is the imaginary part. Equations (2.8) give $\gamma(E)$ for $d = 2,3$ for essentially all frequencies.

IV. CALCULATION OF $\bar{P}_{\omega,\mathbf{k}}(E)$

In this section $\bar{P}_{\omega,\mathbf{k}}(E)$ is calculated. We first solve Eq. (2.11a) to lowest order in the density of scatterers. The resulting equation has the structure of a Boltzmann equation. We then derive Eq. (2.4c) with D given by its Boltzmann equation value which we denote by D_B. We next examine density corrections to this Boltzmann result. In particular, the maximally crossed diagrams important for electron localization[3-7,17,21] are taken into account. Using the techniques developed by Vollhardt and Wölfle[5,6] for the electron problem, we discuss the localization of acoustic waves for $d = 2,3$. Since the technique is essentially identical to that given in the literature[5-7] for the electron problem I will be very brief.

A. Boltzmann equation results

To lowest order in the density the irreducible four-point vertex function, U_{pp_2}, is approximated by its Boltzmann value $U^B_{pp_2}$, given by the diagram in Fig. 3(a),

$$\begin{aligned} U^B_{pp_2}(\mathbf{k},\omega \mid E) = n(2\pi)^d &\langle \mathbf{p}_+ \mid \hat{T}^+(E_+) \mid \mathbf{p}_{2+} \rangle \\ &\times \langle \mathbf{p}_{2-} \mid \hat{T}^-(E_-) \mid \mathbf{p}_- \rangle . \end{aligned} \quad (4.1a)$$

Also, to lowest order in the density we can neglect the k and ω dependence in the Σ's and U functions[22] in Eqs. (2.11), and we can also use Eq. (3.2). With these approximations the Boltzmann approximation for $\Phi(\equiv\Phi^B)$ is given by the solution of the equation,

$$\left[-2i\frac{\omega E}{c^2} + 2i\mathbf{k}\cdot\mathbf{p} + 2\gamma_p(E)\right]\Phi^B_{\mathbf{p}}(\mathbf{k},\omega \mid E) = 2\pi\delta(p^2 - E^2/c^2)\left[1 + n(2\pi)^d \int d\mathbf{p}_2 \mid \langle \mathbf{p} \mid \hat{T}(E)\mathbf{p}_2 \rangle \mid^2 \Phi^B_{\mathbf{p}_2}(\mathbf{k},\omega \mid E)\right] . \quad (4.1b)$$

The solution to Eq. (4.1b) for small $\mathbf{k}$ and ω can be readily constructed by noting that it is an integral equation only in the unit vector $\hat{p} = \mathbf{p}/\mid \mathbf{p} \mid$. Defining the self-adjoint collision operator,

$$\begin{aligned} n\Lambda(\hat{p}) = n\pi(2\pi)^d &(E/c)^{d-2} \\ &\times \int d\hat{\mathbf{p}}_2 \sigma(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_2)[P(\hat{\mathbf{p}}\hat{\mathbf{p}}_2) - 1] , \end{aligned} \quad (4.2a)$$

where

$$\sigma(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_2) = \mid \langle \mathbf{p} \mid \hat{T}(E) \mid \mathbf{p}_2 \rangle \mid^2 \mid_{p=p_2=E/c} \quad (4.2b)$$

and $P(\hat{\mathbf{p}}\hat{\mathbf{p}}_2)$ is a permutation operator that changes $\hat{\mathbf{p}}$ to $\hat{\mathbf{p}}_2$ when it acts on an arbitrary function $f(\hat{\mathbf{p}})$. With this and defining

$$\Phi^B_{\mathbf{p}}(\mathbf{k},\omega \mid E) \equiv \delta(p^2 - E^2/c^2)\phi_{\hat{\mathbf{p}}}(\mathbf{k},\omega \mid E) , \quad (4.2c)$$

Eq. (4.1b) gives

$$\left[-2i\frac{\omega E}{c^2} + 2i\mathbf{k}\cdot\mathbf{p} - n\Lambda(\hat{\mathbf{p}})\right]\phi_{\hat{\mathbf{p}}}(\mathbf{k},\omega \mid E) = 2\pi . \quad (4.2d)$$

In giving Eq. (4.2a) we have used the optical theorem relating the imaginary part of the self-energy, γ, and the angular integral of Eq. (4.2b). It is this property that leads to a collision operator that has an eigenfunction, a constant, with a zero eigenvalue which in turn leads to a conservation equation for Φ^B and the diffusive pole result given by Eq. (2.4c). The generalization of this property for the exact equation given by Eq. (2.11) is the Ward identity,[6]

$$\begin{aligned} \int d\mathbf{p}\, \Delta G(p_+,p_-,E_+E_-) &U_{pp_2}(\mathbf{k},\omega \mid E) \\ &= \Sigma^+_{p_{2+}}(E_+) - \Sigma^-_{p_{2-}}(E_-) . \end{aligned} \quad (4.3)$$

For small k and ω, Eq. (4.2d) can be readily solved and one obtains

$$\phi_{\hat{\mathbf{p}}}(k,\omega \mid E) \simeq \frac{c^2\pi}{E[-i\omega + D_B(E)k^2]} , \quad (4.4a)$$

with $D_B(E)$ the Boltzmann diffusion coefficient.

For d dimensions,

$$D_B(E)=-\frac{2E}{d^2}[\langle\hat{p}_x\,|\,n\Lambda(\hat{\mathbf{p}})\,|\,\hat{p}_x\rangle]^{-1}\,. \tag{4.4b}$$

For $d=2$ for small E, $Ea/c<1$, Eqs. (2.8) and (4.4b) yield

$$D_B(E)=\frac{2}{5}\frac{c}{\pi^2 na}\left[\frac{c}{Ea}\right]^3, \tag{4.4c}$$

and for large E, $Ea/c>1$, Eqs. (2.8) and (4.4b) give

$$D_B(E)=\frac{c}{4na}\,. \tag{4.4d}$$

For $d=3$ for low frequencies, Eqs. (2.8) and (4.4b) yield

$$D_B(E)=\frac{3c}{11\pi na^2}\left[\frac{c}{Ea}\right]^4, \tag{4.4e}$$

and for high frequencies Eqs. (2.8) and (4.4b) give

$$D_B(E)=\frac{c}{6\pi na^2}\,. \tag{4.4f}$$

Finally, Eqs. (4.2c), (4.4a), and (2.10d) yield

$$\bar{P}^{\,B}_{\omega,k}(E)=\frac{(d-1)\pi^2 c}{(2\pi)^d}(E/c)^{d-3}\frac{1}{[-i\omega+D_B(E)k^2]}\,. \tag{4.5}$$

The results given above are valid only when $kl(E)<1$, where $l(E)$ is a hydrodynamic, E-dependent, mean free path given approximately by $D_B(E)\simeq cl(E)$.

B. Self-consistent diagrammatic theory

We next consider the density corrections to the Boltzmann diffusion coefficient given by Eqs. (4.4). In particular, we first take into account the set of maximally crossed diagrams[3–7,17,21] important for electron localization. It is first convenient, following Vollhardt and Wölfle,[5–7] to approximately solve Eq. (2.11a) and obtain an equation for $D(E,\omega)$ directly in terms of the irreducible four-point vertex function. Using Eq. (4.3) and techniques already given in the literature,[5–7] one obtains

$$\bar{P}_{\omega,k}(E)=\frac{(d-1)\pi^2 c}{(2\pi)^d}\left[\frac{E}{c}\right]^{d-3}\frac{1}{-i\omega+D(E,\omega)k^2}\,, \tag{4.6a}$$

with

$$\frac{1}{D(E,\omega)}=\frac{1}{D_B(E)}+\frac{\alpha_d}{c\pi^2}\left[\frac{c}{E}\right]^4 \times\int d\mathbf{p}\,d\mathbf{p}_2 p_x p_{2x}\Delta G(p,E)\Delta G(p_2,E) \times\delta U_{\mathrm{pp}_2}(0,\omega\,|\,E)\,, \tag{4.6b}$$

where

$$\alpha_2=E/c,\quad \alpha_3=\tfrac{9}{8} \tag{4.6c}$$

and

$$\Delta G(p,E)=\Delta G(p,p,E,E)\,, \tag{4.6d}$$

and

$$\delta U_{\mathrm{pp}_2}(0,\omega\,|\,E)=U_{\mathrm{pp}_2}(0,\omega\,|\,E)-U^B_{\mathrm{pp}_2}(0,\omega\,|\,E)\,. \tag{4.6e}$$

In giving Eq. (4.6b) I have neglected all the ω and k dependences that are nonsingular for $k,\omega\to 0$, and I have used that $\delta U_{\mathrm{pp}_2}(\mathbf{k},\omega\,|\,E)$ is a nonsingular function of k.[23]

If one next examines the intensity corrections to D given by Eq. (4.6b) then one finds that if a power-series expansion is assumed then the individual terms are divergent, for $\omega\to 0$, after some order in the density that depends on the dimension of system.[7,17] Amongst the most divergent terms are the set of maximally crossed diagrams given in Fig. 4, which are divergent for $|\mathbf{p}+\mathbf{p}_2|\to 0$. It is this class of diagrams which are assumed to cause localization. These diagrams can be easily resummed by rotating the bottom line in Fig. 4 180° and relating the resulting infinite series to the Boltzmann diffusive pole given above.[3–7,17,21] For small ω and $q=|\mathbf{p}+\mathbf{p}_2|$, one then obtains that $\delta U^M_{\mathrm{pp}_2}$ (M denotes maximally crossed diagrams) is proportional to a hydrodynamic diffusive pole,

$$\delta U^M_{\mathrm{pp}_2}(0,\omega\,|\,E)\simeq\frac{\gamma^2(E)c}{(d-1)\pi^2}(c/E)^{d-1} \times\frac{1}{-i\omega+D_B(E)(\mathbf{p}+\mathbf{p}_2)^2} \times[1+O(|\mathbf{p}+\mathbf{p}_2|^2,\omega)] \tag{4.7}$$

for $d=2,3$, where $\gamma(E)=\gamma_{p=E/c}(E)$.

Inserting Eq. (4.7) into Eq. (4.6b) and evaluating the result to lowest order in n yields

$$\frac{1}{D(E,\omega)}=\frac{1}{D_B(E)}+\frac{d\gamma(E)}{2(d-1)\pi^2}(c/E)^d \times\int_{q<q_0}d\mathbf{q}\frac{1}{-i\omega+D_B(E)q^2}\,. \tag{4.8a}$$

Here q_0 is a hydrodynamic cutoff on the order of $l^{-1}(E)$, where $l(E)$ is the hydrodynamic mean free path. The cutoff q_0 is needed because in giving Eq. (4.7) we have assumed that $|\mathbf{p}+\mathbf{p}_2|/q_0<1$. For $d\le 2$ the localization effects calculated here are essentially independent of q_0 while for $d>2$ the results are cutoff dependent.

It is important to note that the second term on the right-hand side of Eq. (4.8a) is still divergent (even after the resummation given in Fig. 4) for $d\le 2$. The self-consistent theory of localization, for all d, consists of replacing the low-density hydrodynamic pole in Eq. (4.7) by an exact hydrodynamic pole,[5–7] with $D(E,\omega)$ replacing $D_B(E)$, so that Eq. (4.8a) is replaced by

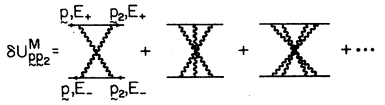

FIG. 4. The maximally crossed diagrams.

$$\frac{1}{D(E,\omega)}=\frac{1}{D_B(E)}+\frac{d\gamma(E)}{2(d-1)\pi^2}(c/E)^d \times\int_{q<q_0}d\mathbf{q}\frac{1}{-i\omega+D(E,\omega)q^2}\ . \tag{4.8b}$$

An algebraic equation for $D(E,\omega)$ is then obtained. Equation (4.8b) can also be written

$$D(E,\omega)=D_B(E)-\frac{d\gamma(E)D_B(E)}{2(d-1)\pi^2}(c/E)^d \times\int_{q<q_0}\frac{d\mathbf{q}}{q^2-i\omega/D(E,\omega)}\ . \tag{4.8c}$$

V. RESULTS OF SELF-CONSISTENT THEORY

In this section Eq. (4.8c) is solved for $d=2,3$.

A. $d=2$

For $d=2$ Eq. (4.8c) is given by

$$D(E,\omega)=D_B(E)-\frac{\gamma(E)D_B(E)}{\pi}\left[\frac{c}{E}\right]^2\ln\left[\frac{q_0^2D(E,\omega)}{-i\omega}\right]\ . \tag{5.1a}$$

It is easy to show that the solution to Eq. (5.1a) for $\omega\to 0$ is of the form

$$D(E,\omega\to 0)=-i\omega\xi^2(E)+O(\omega^2)\ . \tag{5.1b}$$

Here $\xi(E)$ is the frequency-dependent localization length that can be determined by inserting Eq. (5.1b) into Eq. (5.1a). I obtain

$$\xi(E)=\frac{1}{q_0}\exp\left[\frac{\pi}{q\gamma(E)}\left[\frac{E}{c}\right]^2\right]\ . \tag{5.2}$$

Defining $q_0=l^{-1}(E)$ with the hydrodynamic mean free path given by $D_B(E)\equiv cl(E)$ and using Eqs. (4.4) and (2.8), we can explicitly determine $\xi(E)$ for essentially all frequencies E. For low frequencies, $Ea/c<1$, these equations yield

$$\xi(E)\simeq\frac{2}{5\pi^2na}\left[\frac{c}{Ea}\right]^3\exp\left[\frac{2}{3\pi na^2}\left[\frac{c}{Ea}\right]^2\right]\ , \tag{5.3a}$$

and for high frequencies these equations give

$$\xi(E)\simeq\frac{1}{4na}\exp\left[\frac{\pi}{8na^2}\left[\frac{Ea}{c}\right]\frac{1}{f_2(Ea/c)}\right]\ , \tag{5.3b}$$

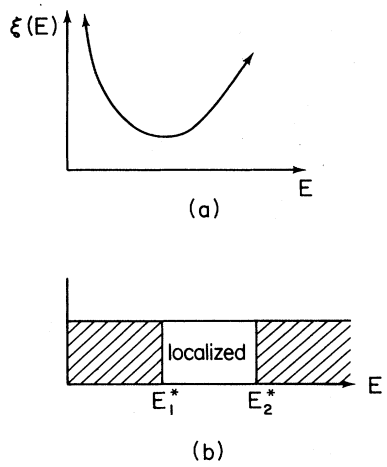

FIG. 5. (a) Localization length, $\xi(E)$, as a function of energy for $d=2$. The minimum occurs near $Ea/c\sim 0.9$. (b) Regions of extended states, lined boxes, and localized states, the open box in $d=3$.

with

$$f_2(x)=1-\frac{0.4321}{x^{2/3}}-\frac{0.2137}{x^{4/3}}+\frac{0.055\,73}{x^2}-\frac{0.000\,555}{x^{8/3}}+\frac{0.023\,25}{x^{10/13}}+\cdots\ . \tag{5.3c}$$

In Fig. 5(a) we graph $\xi(E)$ as a function of Ea/c. The minimum occurs near $Ea/c\sim 0.9$.

It should be noted that Eq. (4.4d) for $D_B(E)\equiv cl(E)$ is an accurate representation only for $Ea/c\gg 1$. For a larger range of energies, $Ea/c\geq 1$, one expects that an additional factor such as $f_2(Ea/c)$ multiplies the factor $4na$ in Eq. (5.3b). Similarly, in Eq. (5.5c) for $d=3$, one anticipates the f_3 in Eq. (5.5c) should be replaced by a term such as $[f_3(Ea/c)]^2$. These factors have not been explicitly indicated in these equations since the value of the cutoff, q_0, is not precisely given in the theory presented here.

B. $d=3$

For $d=3$ Eq. (4.8c) is

$$D(E,\omega)=D_B(E)\left[1-\frac{3\gamma(E)}{\pi}\left[\frac{c}{E}\right]^3\int_0^{q_0}dq\frac{q^2}{q^2-i\omega/D(E,\omega)}\right]$$

$$=D_B(E)\left[1-\frac{3\gamma(E)}{\pi}\left[\frac{c}{E}\right]^3q_0-\frac{3\gamma(E)}{\pi}\left[\frac{c}{E}\right]^3\frac{i\omega}{D(E,\omega)}\int_0^{q_0}\frac{dq}{q^2-i\omega/D(E,\omega)}\right]\ . \tag{5.4}$$

As already discussed, we expect both a lower mobility edge, at E_1^*, and a higher mobility edge, at E_2^*, for $d=3$. This of course assumes that the density of scatterers is not too small. For a fixed density of scatterers the acoustic wave is extended for $E<E_1^*$ and $E>E_2^*$, and $D(E,0)=D(E)$ is given by

$$D(E)=D_B(E)\left[1-\frac{3\gamma(E)}{\pi}\left[\frac{c}{E}\right]^3 q_0\right] . \tag{5.5a}$$

Defining q_0 by $D_B(E)=cl(E)=cq_0^{-1}$ and using Eqs. (5.5a) and (2.8), $D(E)$ for $E<E_1^*$ and $Ea/c<1$ is given by

$$D(E)=D_B(E)\left[1-\frac{77\pi}{9}(na^3)^2\left[\frac{Ea}{c}\right]^6\right] . \tag{5.5b}$$

For $E>E_2^*$ and $Ea/c>1$, Eqs. (5.5a) and (2.8) yield

$$D(E)=D_B(E)\left[1-36\pi(na^3)^2\left[\frac{c}{Ea}\right]^2 f_3(Ea/c)\right] , \tag{5.5c}$$

with

$$f_3(x)=1-\frac{0.8642}{x^{2/3}}-\frac{0.4163}{x^{4/3}}+\frac{0.7352}{x^2} -\frac{0.029\,85}{x^{8/3}}+\frac{0.058\,62}{x^{10/13}}+\cdots . \tag{5.5d}$$

For a fixed density, E_1^* and E_2^* are the solutions of the equation[24]

$$\frac{3\gamma(E)}{\pi}\left[\frac{c}{E}\right]^3\frac{c}{D_B(E)}=1 , \tag{5.6a}$$

and in the localized region the localization length is again defined by

$$D(E,\omega\to 0)=-i\omega\xi^2(E)+O(\omega^2) . \tag{5.6b}$$

Inserting this into Eq. (5.4) yields an expression for $\xi(E)$ given by

$$\xi(E)=\frac{\pi D_B(E)}{2c}\left[1-\frac{D_B(E)\pi}{3c\gamma(E)}\left[\frac{E}{c}\right]^3\right]^{-1} . \tag{5.6c}$$

For $E\simeq E_1^*$ one obtains

$$\xi(E)\sim(E-E_1^*)^{-1} , \tag{5.7a}$$

and for $E\simeq E_2^*$ one obtains

$$\xi(E)\sim(E_2^*-E)^{-1} , \tag{5.7b}$$

i.e., the critical exponents are unity. In Fig. 5(b) we present the localized and extended states for $d=3$.

VI. DISCUSSION

This paper is concluded with some remarks.

(1) The main results of this paper are illustrated in Fig. 5. For two dimensional fluids we found that sound waves are always localized with the localization length growing exponentially for both high and low frequencies. Furthermore, we have given explicitly the coefficient in the important exponential factor in Eq. (5.2) for essentially all frequencies. In form our low-frequency results are in agreement with JSS.[1]

For $d=3$ we found that if the density scatterers is sufficiently large then there are two mobility edges. Between these mobility edges the sound waves are localized and the critical exponent for the localization length near the mobility edges is unity. In form, the low-frequency results are in agreement with JSS.

It is important to remark that the high-frequency limit considered here is an allowable physical limit for the model defined by Eqs. (2.1). Since we have not specified the size of the scatterers and since Eq. (2.1a) is wave equation that should be valid in a liquid down to length scales on the order of a molecular diameter, σ, there exist a range of frequencies such that $Ea/c>1>E\sigma/c$. Finally, the tendency for delocalization at higher frequencies is due to the fact that the waves then have a small spatial extent and can easily propagate between the scatterers.

(2) Although the quantitative features of our results for high frequencies depend on the assumption of hard scatterers, it is reasonable to expect that they are qualitatively correct for a much larger class of scatterers.

(3) For low frequencies the model considered here, Eqs. (2.1), is almost identical to the model

$$\partial_t^2\phi(\mathbf{r},t)-[V_0+V(\mathbf{x})]\nabla^2\phi(\mathbf{r},t)=0 , \tag{6.1a}$$

where $V(x)$ is a Gaussian random function with correlations

$$\langle V(\mathbf{x})\rangle=0 , \tag{6.1b}$$

$$\langle V(\mathbf{x})V(\mathbf{x}')\rangle=\mu_d\delta(\mathbf{x}-\mathbf{x}') , \tag{6.1c}$$

where d is a constant. The only physical difference between Eqs. (6.1) and the low-frequency limit of our model is that the white-noise assumption given by Eq. (6.1c) is equivalent to assuming isotropic scattering. From Eqs. (2.8a) and (2.8b) it is clear that this is not entirely correct even for low frequencies. However, the localization properties of Eqs. (6.1) are identical to those obtained in this paper for low frequencies. For the electron localization problem the Gaussian white-noise model is physically identical to the low-energy limit of the hard scatterer model since the low-energy scattering is isotropic for Dirichlet boundary conditions.

(4) As was already discussed in Sec. II, the difference between sound-wave localization and electron localization in this class of models is due to the difference between Neumann and Dirichlet boundary conditions.[11] For low frequencies—or energies—the scattering due to these boundary conditions is fundamentally different. For high energies the scattering becomes identical. Physically this implies that for high frequencies the diffusion of the energy density of a sound wave in this amorphous model is mathematically similar to diffusion in a classical Lorentz gas.

(5) As for the electron problem,[25] renormalization-group techniques in $d=2+\epsilon$ dimensions can be used directly together with diagrammatic technique given here. One then finds for $d=3$ ($\epsilon=1$) the results given by Eq. (5.7).

(6) It is interesting to note that the presence of scatter-

ers renormalize the effective speed of sound in the fluid. For example, for low frequencies the effective speed of sound, c_{eff}, is given by, from Eqs. (2.8), (2.9), and (3.1), for $d=2$,

$$c_{\text{eff}}^2 = \frac{c^2}{1+\pi n a^2} , \tag{6.2a}$$

and for $d=3$,

$$c_{\text{eff}}^2 = c^2 \frac{1+(2\pi n a^3)/3}{1+(4\pi n a^3)/3} . \tag{6.2b}$$

As expected the effective speed of sound is decreased from its value in the absence of scatterers. This is an example of the acoustic extinction theorem.[8]

(7) In this paper we have neglected the effects due to intrinsic fluid dissipation, e.g., viscosity, as well as nonlinearities. In terms of the electron problem the analogous neglected physical processes are inelastic scattering mechanisms, which corresponds to dissipation, and electron-electron interactions, which corresponds to the nonlinearities. We have also neglected classical percolation effects. All of these must be considered before a reliable experimental prediction can be made.

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