Josephson-junction arrays in transverse magnetic fields: Ground states and critical currents

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(Received 23 July 1984)

Josephson-junction arrays in transverse magnetic fields are most simply described by uniformly frustrated *XY* models. Frustration in these models is parametrized by f, the ratio of the magnetic flux through one cell of the array to the magnetic flux quantum. We propose a quasi-one-dimensional structure for the ground states of these models, valid for $\frac{1}{3} \leq f \leq \frac{1}{2}$. The energies and zero-temperature critical currents for this structure can be calculated exactly. For rational $f = p/q$, energies and critical currents are functions of q only; for irrational f, they are independent of f. Numerical calculations of the ground-state form agree with this quasi-one-dimensional conjecture for several f in the range $\frac{1}{3} \leq f \leq \frac{1}{2}$.

I. INTRODUCTION

Rarely in physics does the rationality of a dimensionless external parameter play any role in determining the behavior of a system. Josephson-junction arrays in transverse magnetic fields offer an example of just such a system, the dimensionless parameter whose rationality plays a key role being the magnetic flux piercing a unit cell of the array in units of the magnetic flux quantum. These arrays are a realization of uniformly frustrated lattice spin models, and present a host of new problems in statistical physics. In this study we will focus upon the $T=0$ properties of these models in the classical limit, discussing in particular a conjectural form for the ground state and its consequences for ground-state energies and $T=0$ critical currents.

Consider a two-dimensional square lattice of superconducting islands in the presence of a magnetic field perpendicular to the plane of the lattice (see Fig. 1). Assume that each island is proximity coupled to its four nearest neighbors. The Hamiltonian of this system will be

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j - A_{ij}) , \tag{1.1}$$

where θ_i denotes the phase of the superconducting order parameter at the ith island or site, and A_{ij} is proportional to the line integral of the vector potential $\mathbf{A}$ between the ith and the jth sites,

$$A_{ij} = \frac{2e}{\hbar c} \int_i^j \mathbf{A} \cdot dl . \tag{1.2}$$

This Hamiltonian is a sum of the Josephson coupling energies between the neighboring islands; $\theta_i - \theta_j - A_{ij}$ is the gauge-invariant phase difference between the superconducting order parameter on the ith and on the jth site. In general, J will be a function of the magnetic field and of the temperature. We require that

$$\sum_P A_{ij} = 2\pi f , \tag{1.3}$$

i.e., that the directed sum of the A_{ij} about each plaquette be a constant, which is proportional to the magnetic flux piercing the plaquette in units of the magnetic flux quantum,

$$2\pi f = 2\pi H a^2 / \Phi_0 , \tag{1.4}$$

where a is the lattice constant of the array. It is easy to show that the properties of these Hamiltonians are invariant under $f \to f+1$ and under $f \to -f$; we need therefore only consider f in the range $0 \leq f \leq \frac{1}{2}$.

Throughout the following we will assume that the A_{ij} are quenched in by an external magnetic field. This corresponds to taking the limit of infinite London penetration depth for the array. In this limit the magnetic field must penetrate the array homogeneously, in contrast to the case of type-II superconductivity in bulk samples, in which the magnetic field penetrates entirely within Abrikosov flux tubes.[1] In the real systems that have been studied experimentally, the penetration depth is the same order of magnitude as the size of the system; so that taking the limit $\lambda_T \to \infty$ is a reasonable approximation.[2,3]

These models are related via a Villain transformation to a class of lattice plasma models of the form

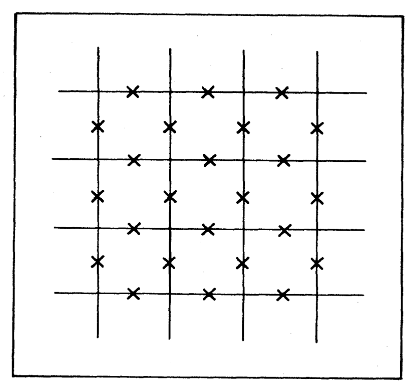


FIG. 1. Schematic representation of a square lattice of Josephson junctions (indicated by crosses). The magnetic field is perpendicular to the plane of paper.

$$H=-\pi\tilde{J}\sum_{r,r'}[m(r)+f]G(r-r')[m(r')+f] , \quad (1.5)$$

with $G(r-r')\rightarrow\ln|r-r'|$ as $|r-r'|\rightarrow\infty$ and the charges $m(r)$ restricted to integral values.[4] The average value of the plasma charges over the entire lattice is constrained to equal f.

The continuum version of these arrays is a two-dimensional superconducting film in a transverse magnetic field. The ground state of this system is a triangular lattice of logarithmically interacting vortices in the phase field of the superconducting order parameter.[5] It is topologically necessary for these vortices to possess small core regions in which the film is actually in the normal state. The positions of these normal cores are gauge invariant. These vortices should not be confused with the charges of (1.5), which are defined within a transformed model.

Because most of the surface area of the array is taken up by normal regions, there are no vortex cores.[6] The only physically meaningful quantities are the gauge-invariant phase differences between the various sites. It is thus possible, and even convenient, to eschew the discussion of vortices and concentrate instead upon these phase differences. This will be our strategy.

At the value of $f=0$ this model is the classical XY model on a square lattice, a model whose properties are quite well understood.[7] At $f=\frac{1}{2}$ the model is equivalent to the fully-frustrated XY model, which was introduced by Villain and has been studied by Teitel and Jayaprakash and by this author.[8–10] The antiferromagnetic XY model on a triangular lattice is a variant of this fully-frustrated XY model; this system has been studied numerically by a number of investigators.[11]

One approach to the model for general f is to linearize in the superconducting order parameter, allowing its amplitude and phase to vary from site to site. In this approximation, which is a modification of Abrikosov's mean-field theory of type-II superconductivity, the array problem is reduced to the determination of the spectrum of a tight-binding Bloch electron in a magnetic field. This problem has been the subject of much recent work, notably that of Hofstadter.[12] This question reduces to the analysis of the properties of Harper's equation for a wave function $\psi(m)$ defined on the integers,

$$\psi(m+1)+\psi(m-1)+[2\cos(2\pi mf-\nu)-\epsilon]\psi(m)=0 . \quad (1.6)$$

These properties depend strongly upon the rationality of f. The application of this linearized approximation to the physics of superconducting arrays has been pursued by Rammal *et al.* and by Shih and Stroud.[13,14] However, since the collective effects in the real arrays occur at temperatures well below the temperature of the resistive transition in the superconducting islands, fluctuations in the amplitude of the superconducting order parameter will be well suppressed in this regime of interest.[2,3] We thus believe that the linearized approximation is highly unrealistic in the case of the real arrays. Such an approximation should be more useful in the study of three-dimensional arrays.

Teitel and Jayaprakash have conducted an extensive numerical study of the properties of the model (1.1).[15] They have found the ground state and zero-temperature critical currents at several low-order rational f, and have also offered interesting speculations on the general behavior of the model as a function of the rationality of f. They suggest that the zero-temperature critical currents obey a bound of the form

$$i_c(f=p/q)<\frac{e\pi}{\hbar q}|\epsilon_0(f)| , \quad (1.7)$$

where $|\epsilon_0(f)|$ is the absolute value of the ground-state energy per site.

There has also been considerable experimental interest in these arrays. Tinkham *et al.* and Webb *et al.* have reported resistance variation as a function of H which shows some structure at values of H corresponding to low-order rational values of f.[2,3] Pannetier *et al.* have reported similar results on the related system of an array of thin superconducting wires.[16] Kimhi *et al.* have also reported magnetoresistance studies.[17]

We restrict ourselves to the zero-temperature properties of the array, and to the classical limit. This corresponds to assuming that the capacitance of the junctions is infinite. We will explore the consequences of a simple, physically motivated assumption concerning the nature of the ground states, namely, that they are quasi one dimensional in character.

A ground state, or indeed any state, of the array may be specified by specifying the gauge-invariant phase difference across every junction of the array. Note, however, that not every conceivable set of phase differences corresponds to an actual state of the array, as the sum of the phase differences around each plaquette is constrained by

$$\sum_p(\theta_i-\theta_j-A_{ij})=2\pi f \ (\mathrm{mod}2\pi) . \quad (1.8)$$

The junctions of the array may be uniquely assigned to "staircases" which run perpendicularly to one of the diagonals of the array. These staircases each contain an infinite number of alternately horizontal and vertical bonds (see Fig. 2). The type of quasi-one-dimensional state to which we refer is one in which the gauge-invariant phase differences across all junctions on the same staircase are identical. To characterize the phase differences across all

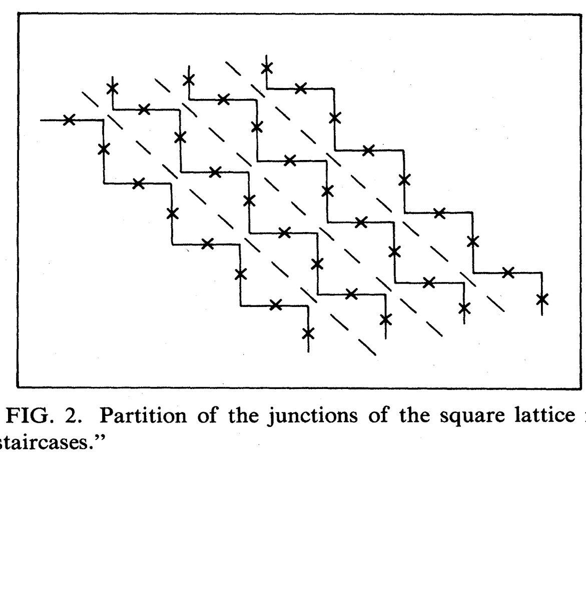


FIG. 2. Partition of the junctions of the square lattice into "staircases."

junctions in the array it is necessary in this type of state to characterize only the phase difference for each staircase. These states are an array version of the type of striped phase common to many commensurate-incommensurate problems. If one assumes that the ground states exhibit this staircase form, then it is possible to determine their structure exactly.

These states are local minima of the Hamiltonian for all values of f, and reduce to the known ground states at $f=0$ and at $f=\frac{1}{2}$. Their properties are strongly dependent upon the rationality of f. If f is rational, $f=p/q$, then the phase differences in these ground states are spatially periodic with a $q\times q$ unit cell. If f is irrational, the phase differences do not possess such a spatial periodicity. The staircase-state energies can be calculated exactly. For $f=p/q$, the energy per site is given by

$$\epsilon(f=p/q)=-(2J/q)\csc(\pi/2q)\ , \tag{1.9a}$$

while for irrational f,

$$\epsilon=-4J/\pi\ , \tag{1.9b}$$

which is the limit of (1.9a) as $q\to\infty$.

This assumption about the form of the ground state can also be exploited to determine zero-temperature critical currents. These critical currents are quite anisotropic, that is, dependent upon direction within the array. In the diagonal direction perpendicular to the quasi-one-dimensional modulation, the $[1\bar{1}]$ lattice direction, the critical currents have a particularly simple form. For rational $f=p/q$,

$$ic^{[1\bar{1}]}(f=p/q,\ T=0)=\frac{1}{q}ic^{[1\bar{1}]}(f=0,\ T=0)\ , \tag{1.10a}$$

while for irrational f,

$$ic^{[1\bar{1}]}(T=0)=0\ , \tag{1.10b}$$

so that at irrational f it is impossible for current to flow without dissipation in the array.

We have verified numerically that the actual ground states are of the staircase form at $f=\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, and $\frac{3}{8}$. There are heuristic reasons for believing that the true ground state may be of the staircase form for numerous f satisfying $\frac{1}{3}\le f\le\frac{1}{2}$; we do not, however, believe that the staircase form gives the true ground state for all such values of f.

Several effects present in real experimental systems have been neglected in this analysis. Chief among these are the finite size of the experimental arrays, their finite London penetration depth, the finite capacitance of the junctions, and the inhomogeneity in the coupling constants and in the sizes of the array cells. It is impossible to fix the magnetic field experimentally to equal any particular rational value. Therefore these effects may be important in allowing the observation of experimental consequences of the proximity of the magnetic field to some low-order rational f.

This paper is organized into four sections and four appendixes. Section I is the Introduction. Section II is a discussion of the ground states in the charge model (1.5). While no calculations are performed, the physical motivation for the postulation of a quasi-one-dimensional form for the ground states is developed in detail. In Sec. III the nature of the ground states of the original phase model (1.1) is explored. An exact quasi-one-dimensional form for the putative ground states is derived, and the ground-state energies (1.9) are calculated for all such states. In Sec. IV we discuss the critical currents following from these states, deriving (1.10) and displaying numerical calculations of critical current anisotropies.

Several more technical discussions are relegated to appendixes. Appendix A examines in detail the relationship between the charge model (1.5) and the original phase model (1.1). Appendix A should be read in parallel with Sec. II. In Appendix B the stability of the states discussed in Sec. III is demonstrated, and certain instabilities relevant to the critical current discussion of Sec. IV are displayed. Appendix C discusses some properties of these quasi-one-dimensional states that may be simply generated from the continued fraction representation for f. In Appendix D the details of the numerical calculations are presented, both of the ground-state verification and of the calculation of the critical current anisotropy.

II. GROUND STATES OF THE CHARGE MODEL

We will find it instructive to consider the nature of the ground states of the charge model introduced in Sec. I above.

This model describes the statistical mechanics of a set of lattice integral charges interacting logarithmically with one another and with a constant uniform background field f. The Hamiltonian is given by (1.5),

$$H=-\pi\tilde{J}\sum_{r,r'}[m(r)+f]G(r-r')[m(r')+f]$$

with

$$G(r-r')\to\ln|r-r'|\ \text{ as }\ |r-r'|\to\infty\ .$$

The positions of the charges $m(r)$ are the centers of the plaquettes of the phase model (1.1). The allowed charge configurations $\{m(r)\}$ are constrained by global charge neutrality. This is shown in Appendix A. This condition can only be satisfied for a finite-volume system if f is rational, although for f irrational it may be true as a limiting statement as the volume of the system becomes arbitrarily large.

As we restrict our attention to the cases where $0\le f\le\frac{1}{2}$, and to a determination of the ground state of (1.5), we will assume that it is only necessary to consider the possibilities $m(r)=0,-1$. Crudely speaking, the ground state of (1.5) will be that state which most successfully minimizes *local charge fluctuations,* the state in which compact regions possess a charge $\sum[m(r)+f]$ that as closely approximates zero as possible.

At the value of $f=\frac{1}{2}$ the ground state of this system is the "checkerboard" state exhibited in Fig. 3. This state is a square lattice of (-1) charges with a lattice constant of $\sqrt{2}\times a$, where a is the lattice constant of the array. The density of these charges is thus $\frac{1}{2}$, in accordance with the charge neutrality condition. This state displays a Z_2 broken symmetry (here denoted a "chiral" symmetry—in the

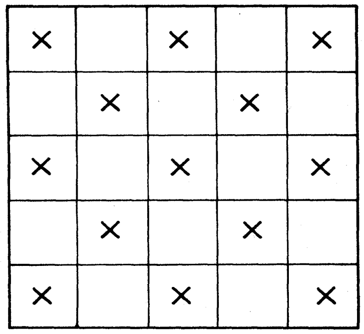


FIG. 3. Section of the $f=\frac{1}{2}$ charge model ground state displaying the $\sqrt{2}\times\sqrt{2}$ "checkerboard" structure of -1 charges.

phase model this symmetry is related to the sense of current circulation about elementary plaquettes) according to which of the two possible $\sqrt{2}\times\sqrt{2}$ superlattices is chosen by the charges.

The $f=\frac{1}{2}$ state has two distinct sorts of thermal excitations.[8,9] Interstitial-vacancy pairs may be excited from the $\sqrt{2}\times\sqrt{2}$ superlattice [see Fig. 4(a)]; these pairs will interact logarithmically at sufficiently low temperatures. These interstitial-vacancy pairs do not disrupt the long-range chiral ordering of the charge superlattice. Domain walls may also be excited; these domain walls separate regions of differing chirality [see Fig. 4(b)]. The excitation of arbitrarily long domain walls will disrupt the long-range chiral ordering.

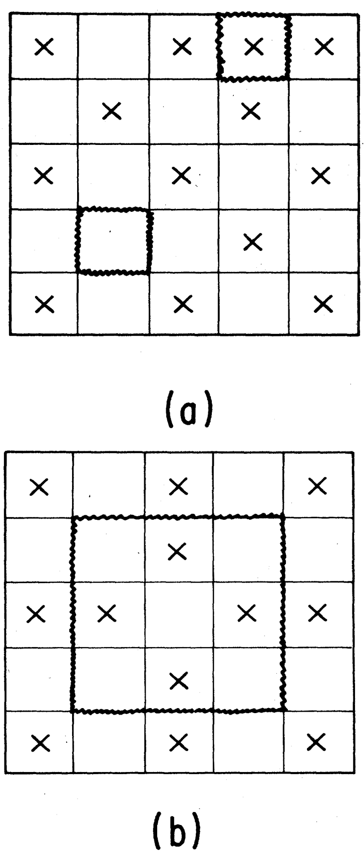


FIG. 4. (a) Interstitial-vacancy pair superimposed on the $f=\frac{1}{2}$ ground state. (b) Domain wall separating regions of different chirality at $f=\frac{1}{2}$.

These domain walls have a significant and peculiar property: corners of domain walls possess a net charge of $\pm\frac{1}{4}$. This is most easily seen by averaging at each vertex where four plaquettes meet the charge of the neighboring four sites with the value of the background field f, as in Fig. 5. At vertices coinciding with the corners of domain walls, there will be either three charges or one charge on the neighboring four plaquettes, depending upon the precise configuration of the domain wall. Thus at these vertices, the average charge will be $\pm\frac{1}{4}$. Along a closed domain wall these fractional charges must sum to an integer. This is possible because closed domain walls always possess an even number of corners. These fractional charges are important in determining the finite-temperature properties of the $f=\frac{1}{2}$ model.[10]

In the neighborhood of $f=\frac{1}{2}$, in particular for $f=\frac{1}{2}-\epsilon$, the ground state should locally appear very similar to the checkerboard $f=\frac{1}{2}$ state. At longer length scales, however, there must be some breakdown of this state due to the fact that the average value of $m(r)$ does not equal $-\frac{1}{2}$, because of the charge neutrality condition. Thus defects must be introduced into the $f=\frac{1}{2}$ state so that its average density of charges falls below $\frac{1}{2}$.

This breakdown of the $f=\frac{1}{2}$ state may occur in two ways (or in some combination of two ways). An appropriate number of the charges in the $\sqrt{2}\times\sqrt{2}$ superlattice could be removed without a destruction of the long-range chiral ordering, these vacancies would presumably form some sort of square or triangular superlattice of their own. If $f=-\langle m(r)\rangle=\frac{1}{2}-\epsilon$, the lattice constant of this superlattice of vacancies (which is a superlattice on top of the original $f=\frac{1}{2}\sqrt{2}\times\sqrt{2}$ superlattice) will be of order $\epsilon^{-1/2}$. One such possible superlattice of vacancies is illustrated in Fig. 6(a) for the case $f=\frac{3}{8}$.

There is a second possibility. The local checkerboard structure may be preserved while insuring that $-\langle m(r)\rangle=\frac{1}{2}-\epsilon$ by introducing a structure of domains of alternating chirality. These structures exploit the fact that domain-wall corners have a net charge of $\pm\frac{1}{4}$ with

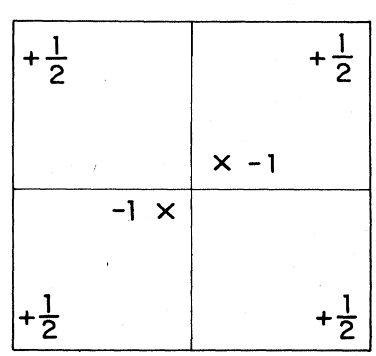


FIG. 5. Average charge of the plaquettes surrounding this vertex is $\frac{1}{4}[4(\frac{1}{2})+2(-1)]=0$.

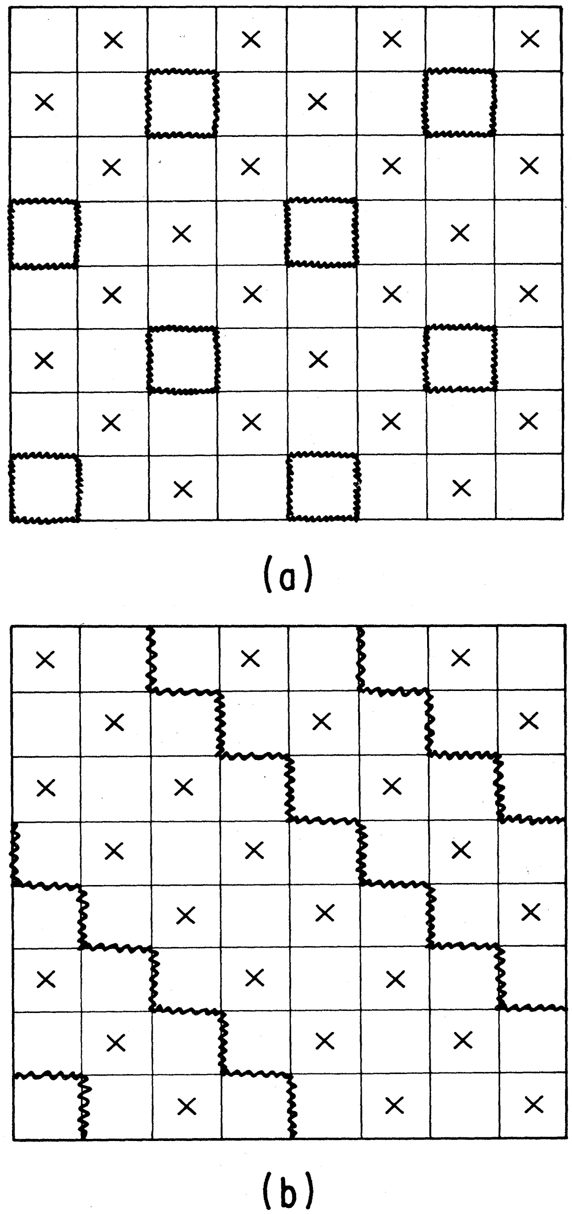


FIG. 6. (a) Conceivable form for the $f=\frac{3}{8}$ ground state: a "superlattice of vacancies." (b) Another conceivable form for the $f=\frac{3}{8}$ ground state: a striped array of domain walls.

respect to the $f=\frac{1}{2}$ state. An appropriately configured domain wall will thus have net charge per unit length. The simplest such domain wall is a diagonal domain wall, inclined at an angle of 45° with respect to the underlying square lattice (such a domain wall is parallel to one of the simplest lattice vectors of the $\sqrt{2}\times\sqrt{2}$ superlattice). Such a domain wall is illustrated in Fig. 7. The $\sqrt{2}\times\sqrt{2}$ superlattices have been displaced by an underlying lattice constant a relative to one another. Restricting ourselves only to a consideration of such diagonal domain walls, there are still many possible ways of arranging such walls relative to one another. The simplest possibility is the arrangement of *all* walls parallel to one another to form a

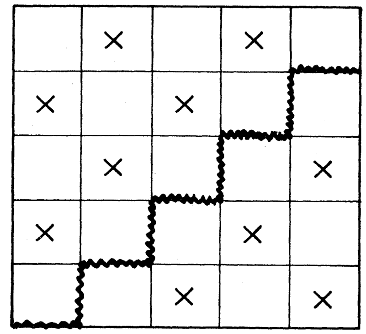


FIG. 7. Diagonal domain wall carrying net charge per unit length.

striped state. This possibility is shown in Fig. 6(b) for the case of $f=\frac{3}{8}$. An advantage of such a striped phase over other configurations is that since the charge is removed in $\frac{1}{4}$ integral units rather than in integral units, local charge fluctuations will be less pronounced.

Although it is possible to construct these diagonally varying quasi-one-dimensional states for any value of f, this procedure can only be interpreted as an insertion of defects into the $f=\frac{1}{2}$ state for $f\geq\frac{1}{3}$. At $f=\frac{1}{3}$, a domain wall has been introduced between every neighboring pair of diagonal lines of charges (see Fig. 8). Indeed, although we find below that the preferred quasi-one-dimensional states are local minima of the Hamiltonian (1.1) for all values of f, they do not appear to be global minima for $f<\frac{1}{3}$. The published numerical work of Teitel and Jayaprakash shows diagonally varying charge model ground states at $f=\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, and $\frac{1}{2}$, all values of $f\geq\frac{1}{3}$, and nondiagonally varying ground states of $f=\frac{1}{7}$, $\frac{1}{6}$, $\frac{1}{5}$, and $\frac{1}{4}$, all values of $f<\frac{1}{3}$.[15,18]

III. GROUND STATES OF THE PHASE MODEL: THE STAIRCASE STATE

We have argued above that the ground states in the charge model (1.5) for $\frac{1}{3}\leq f\leq\frac{1}{2}$ may have a striped,

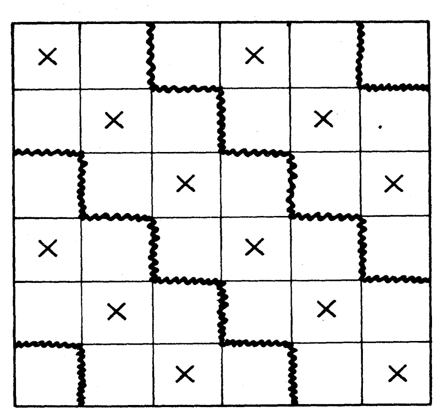


FIG. 8. Striped array of domain walls at $f=\frac{1}{3}$.

quasi-one-dimensional character. In this section we explore the consequences of assuming that the ground states in the phase model (1.1) have this character (in a sense to be defined precisely below).

The condition that a particular set of phases $\{\theta_i\}$ be an extremum of the Hamiltonian (1.1) implies that

$$\sum_{j'} \sin(\theta_i - \theta_{j'} - A_{ij'}) = 0 \;, \tag{3.1}$$

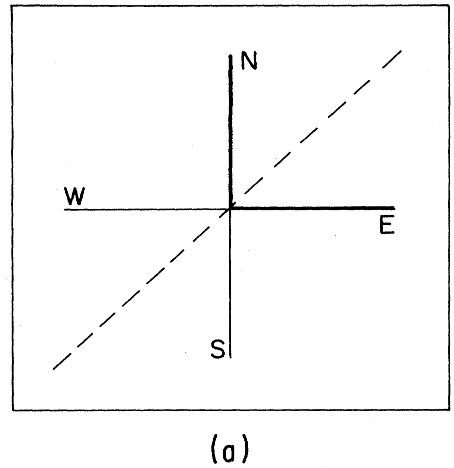
for all i, where the sum is over nearest neighbors to i. The sine of the gauge-invariant phase difference across a Josephson junction is proportional to the supercurrent flowing through the junction. Equation (3.1) therefore states that an extremum of the Hamiltonian is a state in which supercurrent is conserved at every site of the array. This is Kirchhoff's law for supercurrent. The currents flowing in the ground state of $f=\frac{1}{2}$ are shown in Fig. 9. The magnitudes of all of these currents are equal, and correspond to gauge-invariant phase differences of $\pm\pi/4$.[8] The arrows indicate the directions in which the currents flow. It is clear by inspection that current is conserved at each site.

The relationship between the correlation functions in the charge model and in the phase model is complicated, and it is difficult to extract quantitative information from this relationship. However, it is shown in Appendix A that the symmetry properties of the ground states of the two models can be related, so that if the ground state in the charge model is symmetric about some axis, then the energies of the junctions, or bonds, in the phase model ground state must also be symmetric about that axis.

The quasi-one-dimensional states of the charge model discussed above possess a family of symmetry axes. All diagonal axes perpendicular to the domain walls are axes of reflection symmetry of these states, provided that they are axes of reflection symmetry of the underlying lattice.

Imagine that a diagonal axis of symmetry passes through a particular site in the phase model, say in the northeast-southwest direction. Thus, as in Fig. 10(a), we see that the discussion above implies that the energies of the bonds to the north and to the east of this site must be identical to one another, as must be the energies to the south and to the west of this site. There is no relationship imposed by this symmetry between the northern and eastern energies, on the one hand, and the southern and western energies, on the other. The coupling energies are proportional to the cosines of gauge-invariant phase differences across those junctions; the currents flowing are proportional to the sines of the gauge-invariant phase differences. Thus the energies determine the magnitudes but not the signs of the currents.

If the energies of northern and eastern bonds differ from those of the southern and western bonds, then current can only be conserved at this site if it is separately conserved on the northern and eastern bonds considered in isolation and on the southern and western bonds considered in isolation. Thus the current entering the site along the northern bond will be identical to that leaving the site along the eastern bond, and the, in general different, current entering the site along the southern bond



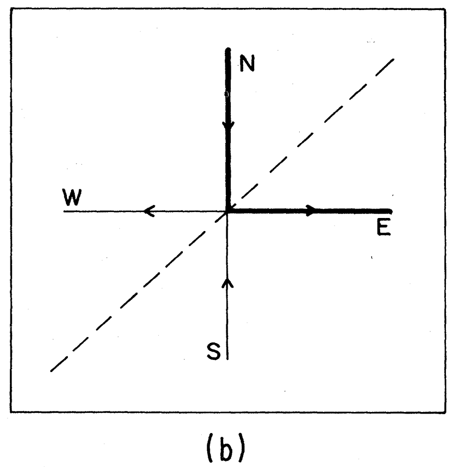


FIG. 10. (a) If the dashed line is an axis of reflection symmetry, then the energies of the N and E bonds must be the same, as must be the energies of S and W bonds. (b) If current is conserved at this site, the bond currents must flow in the sense indicated, though the arrows on the NE pair could be reversed with respect to those on the SW pair.

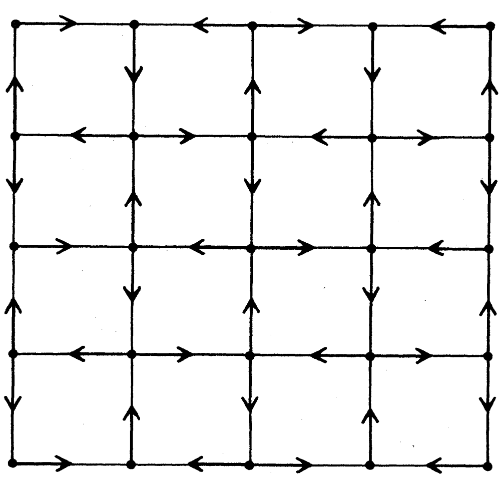


FIG. 9. Orientation of the currents in the $f=\frac{1}{2}$ ground state. The magnitudes of all currents are the same.

will be identical to that leaving the site along the western bond. This is illustrated in Fig. 10(b). In general, this is the only possibility that is consistent with both a diagonal symmetry axis and current conservation, as it can be shown that (except at $f=0,\frac{1}{2}$) there are no configurations of phases $\{\theta_i\}$ such that the energies of all bonds in the lattice are identical.

It is elementary to extend this argument to the entire lattice using the family of diagonal axes of symmetry. The conclusion is that quasi-one-dimensional phase model states that are extrema of the Hamiltonian (1.1) must consist of adjoining staircases of current (see Fig. 11). Each staircase consists of alternate horizontal and vertical steps; staircases do not terminate save at the boundaries of the system. Along each staircase the magnitude and the sign of the current remain constant. We have not yet introduced any restrictions upon the magnitudes or relative signs of the currents flowing upon neighboring staircases.

Such a state will automatically conserve current. Each staircase enters and then leaves a site once. As remarked above, however, the model (1.1) is defined not in terms of currents on bonds but in terms of phases on sites. This introduces a large set of constraints on the possible current configurations, so that it is not immediately obvious that such a staircase state could even exist. A simple calculation shows that staircase states can exist, and determines the conditions satisfied by currents on neighboring staircases.

We can conveniently choose the Landau gauge for the A_{ij}. In this gauge the A_{ij} on all horizontal bonds are zero, while the A_{ij} on the vertical bonds are a function only of horizontal position $A(m)$, so that

$$A(m)=2\pi f m \ , \tag{3.2}$$

satisfying

$$\sum_{P} A_{ij}=2\pi f \ ,$$

where horizontal distance is measured in units of the lattice constant a.

Consider two adjacent rows of the square lattice (see Fig. 12). If these rows are a section of a state of the staircase form, then the currents flowing within one row will be identical to those flowing within the other row, although displaced by one lattice constant. Because we are using the Landau gauge, the gauge-invariant phase differences within the rows are the same as the naive phase differences. Thus if the phases of the sites in the top row are $\tilde{\theta}(m)$, then the phases of the sites in the bottom row must be $\tilde{\theta}(m-1)+\alpha$, where α is constant along the entire bottom row. The staircase form further implies that the currents on the vertical bonds are the same as the currents flowing on the appropriate horizontal bonds. Specifically, this requirement is that

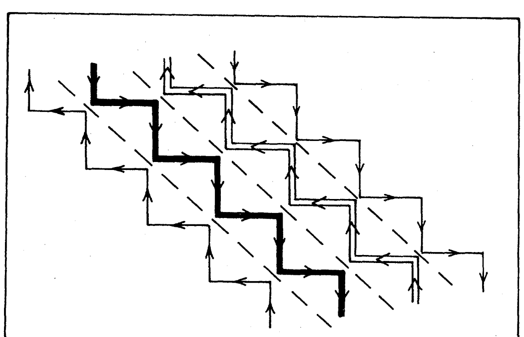


FIG. 11. Staircase form of the current flow. Each staircase extends to the boundaries of the system in both directions.

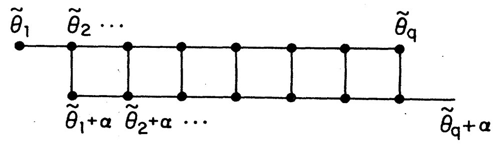


FIG. 12. Phases in two adjacent rows of a staircase state in Landau gauge.

$$\sin[\tilde{\theta}(m-1)-\tilde{\theta}(m)] = \sin\{\tilde{\theta}(m)-[\tilde{\theta}(m-1)+\alpha]-2\pi f m\} \ , \tag{3.3}$$

or that

$$\tilde{\theta}(m-1)-\tilde{\theta}(m)=\tilde{\theta}(m)-[\tilde{\theta}(m-1)+\alpha]-2\pi f m+2\pi k \ , \tag{3.4}$$

where k is an integer. This equation has two solutions of interest,

$$\tilde{\theta}(m)-\tilde{\theta}(m-1)=\pi f m+\alpha/2 \tag{3.5a}$$

or

$$\tilde{\theta}(m)-\tilde{\theta}(m-1)=\pi f m+\alpha/2-\pi \ . \tag{3.5b}$$

Choosing the answer so that

$$-\pi/2 \le \tilde{\theta}(m)-\tilde{\theta}(m-1)<\pi/2 \ ,$$

we obtain

$$\tilde{\theta}(m)-\tilde{\theta}(m-1)=\pi f m+(\alpha/2)-\pi\,\mathrm{nint}[fm+(\alpha/2\pi)] \ , \tag{3.6}$$

where nint is the nearest integer function; nint(x) is the integer part of $x+(\frac{1}{2})$. We will discuss in Sec. IV the justification for requiring that

$$-\pi/2 \le \tilde{\theta}(m)-\tilde{\theta}(m-1)<\pi/2 \ .$$

The construction developed above for two rows can be easily extended to the entire lattice. If the row is indexed by n then

$$\theta(m,n)=\tilde{\theta}(m+n)-n\alpha+n(n+1)\pi f \ , \tag{3.7}$$

where the $\tilde{\theta}(m)$ are given by (3.6).

These phase differences $\tilde{\theta}(m)-\tilde{\theta}(m-1)$ can be easily visualized. If we plot the successive phase differences $\tilde{\theta}(m)-\tilde{\theta}(m-1)$ upon a half-circle with the points at $\pi/2$ and $-\pi/2$ identified with one another, then the phase differences occurring sequentially along this circle as m is increased are separated by πf (see Fig. 13).

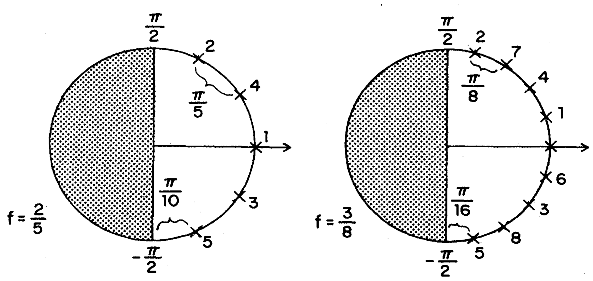


FIG. 13. Polar plot of the staircase phase differences occurring at $f=\frac{2}{5}$ and at $f=\frac{3}{8}$. The integers index the successive staircases.

We still have the freedom to change the parameter α. Thus, if we are interested in possible ground states, we must minimize the energy with respect to α. We must distinguish three cases, according to the nature of f.

A. f rational; $f=p/q$ with $q=2k+1$, $k\in\mathbb{N}$

The staircase form insures that the energy of the entire lattice per spin will be twice the energy of the bonds in a single row per spin. The phase differences $\tilde{\theta}(m)-\tilde{\theta}(m-1)$ are invariant under $m\to m+q$ by (3.6). Furthermore, the phase differences that appear for $1\le m\le q$ are given by $\pi i/(2k+1)+\alpha/2$ for $|i|\le k$, where the requirement that $|\tilde{\theta}(m)-\tilde{\theta}(m-1)|\le\pi/2$ implies that

$$|\alpha/2|\le\pi/[2(2k+1)]=\pi/2q\ .$$

Each of these q phase differences occurs exactly once. Therefore the energy per spin is

$$\begin{aligned}\frac{E(\alpha)}{N}&=-\frac{2J}{q}\sum_{m=1}^{q}\cos[\tilde{\theta}(m)-\tilde{\theta}(m-1)]\\&=-\frac{2J}{q}\sum_{m=1}^{q}\cos\left[\pi fm+\frac{\alpha}{2}-\pi\,\mathrm{nint}\left[fm+\frac{\alpha}{2\pi}\right]\right]\\&=-\frac{2J}{q}\sum_{i=-k}^{k}\cos\left[\frac{\pi i}{2k+1}+\frac{\alpha}{2}\right],\end{aligned}\tag{3.8}$$

so that

$$\frac{E(\alpha)}{N}=-\frac{2J}{q}\left\{\left[\sum_{i=-k}^{k}\cos\left[\frac{\pi i}{2k+1}\right]\right]\cos\frac{\alpha}{2}-\left[\sum_{i=-k}^{k}\sin\left[\frac{\pi i}{2k+1}\right]\right]\sin\frac{\alpha}{2}\right\}.\tag{3.9}$$

Since

$$\sum_{i=-k}^{k}\cos\left[\frac{\pi i}{2k+1}\right]=\csc\left[\frac{\pi}{2(2k+1)}\right],$$

and

$$\sum_{i=-k}^{k}\sin\left[\frac{\pi i}{2k+1}\right]=0\ ,$$

we immediately see that the energy is minimized for $\alpha=0$, and, writing the energy per spin as $\epsilon(f)$, we have

$$\epsilon(f=p/q,\ q=2k+1)=-\frac{2J}{q}\csc\frac{\pi}{2q}\ .\tag{3.10}$$

B. f rational, $f=p/q$ with $q=2k$, $k\in\mathbb{Z}$

In this case the phase differences $\tilde{\theta}(m)-\tilde{\theta}(m-1)$ for $1\le m\le q$ are given by $\pi i/2k+\alpha/2$, where $-k\le i<k$, and $\alpha/2\le\pi/q$ to insure, as in subsection A above, that

$$|\tilde{\theta}(m)-\tilde{\theta}(m-1)|\le\pi/2\ .$$

Thus in this case

$$\begin{aligned}\frac{E(\alpha)}{N}&=-\frac{2J}{q}\sum_{m=1}^{q}\cos[\tilde{\theta}(m)-\tilde{\theta}(m-1)]\\&=-\frac{2J}{q}\sum_{m=1}^{q}\cos\left[\pi fm+\frac{\alpha}{2}-\pi\,\mathrm{nint}\left[fm+\frac{\alpha}{2\pi}\right]\right]\\&=-\frac{2J}{q}\sum_{i=-k}^{k-1}\cos\left[\frac{\pi i}{2k}+\frac{\alpha}{2}\right],\end{aligned}\tag{3.11}$$

and

$$\frac{E(\alpha)}{N}=-\frac{2J}{q}\left\{\left[\sum_{i=-k}^{k-1}\cos\left[\frac{\pi i}{2k}\right]\right]\cos\frac{\alpha}{2}-\left[\sum_{i=-k}^{k-1}\sin\left[\frac{\pi i}{2k}\right]\right]\sin\frac{\alpha}{2}\right\}.\tag{3.12}$$

Since

$$\sum_{i=-k}^{k-1}\cos\left[\frac{\pi i}{2k}\right]=\csc\frac{\pi}{2(2k-1)}\ ,$$

and

$$\sum_{i=-k}^{k-1}\sin\left[\frac{\pi i}{2k}\right]=-1\ ,$$

we see that the energy is minimized for $\alpha/2=\pi/4k=\pi/2q$, so that again

$$\epsilon(f=p/q,\ q=2k)=-\frac{2J}{q}\csc\left[\frac{\pi}{2q}\right].\tag{3.13}$$

C. f irrational, $f\neq p/q$

In this case the phase differences $\tilde{\theta}(m)-\tilde{\theta}(m-1)$ are not periodic. In the limit of an infinitely large system, the phase differences generated by (3.6) will occur with a constant density over the interval

$$-\pi/2<\tilde{\theta}(m)-\tilde{\theta}(m-1)<\pi/2\ ,$$

regardless of the value of α. Thus the energy is given by

$$\epsilon(f\neq p/q)=-J\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta=-J\frac{4}{\pi}\ .\tag{3.14}$$

Note that

$$\lim_{q\to\infty}\left[\frac{2}{q}\csc\frac{\pi}{2q}\right]=\frac{4}{\pi}\ ,$$

so that (3.14) is a limiting case of (3.10) and (3.13).

Figure 13 displays the phase differences occurring for the $f=\frac{2}{5}$ and the $f=\frac{3}{8}$ staircase states. Each point on these diagrams indexes a set of staircases that are separated from one another by q horizontal or vertical lattice spacings. For $f=p/q$, the phase-difference points are separated by π/q; the largest and the most negative phase differences are separated from the points at $\pi/2$ and $-\pi/2$, respectively, by phases of $\pi/2q$. If α is varied from $\alpha_{\min}$, the value that minimizes the energy, all of the points on these phase diagrams rotate rigidly; the distances between the points remains constant. As $q\to\infty$, this pattern approaches a limit in which there is a constant density of points filling the entire half-circle between $-\pi/2$ and $\pi/2$.

The current flowing along a staircase is proportional to the sine of the gauge-invariant phase difference characterizing that staircase. These sines of phase differences are proportional to the projections of the points in the phase diagram onto the y axis of the diagram. The points in the minimum-energy staircase-state phase diagrams are "balanced" about the x axis (see Fig. 13), thus the currents flowing along the different staircases sum to zero, and no macroscopic current flows in these states.

The energy formulas (3.10), (3.13), and (3.14) are remarkable. They state that the staircase-state energy is a function only of the degree of rationality of f. For rational $f=p/q$, not only is the energy a strong function of q, but it is not even a weak function of p. At irrational f, the energy is identically $\epsilon=-J(4/\pi)$, and is not a function of f at all. The staircase-state energy $\epsilon(f)$ is a continuous function of f at all irrational f, and is a discontinuous function of f at all rational f. The function $\epsilon(f)$ is shown as a function of f for $\frac{1}{3}\le f\le\frac{1}{2}$ in Fig. 14.

While the energies of these structures do not depend upon p, the structures clearly do depend on p. Recalling that each point on the phase diagram corresponds to the gauge-invariant phase differences occurring along a particular set of staircases, we see that p determines the ordering of the staircases whose phase differences are determined by q, as in Fig. 13.

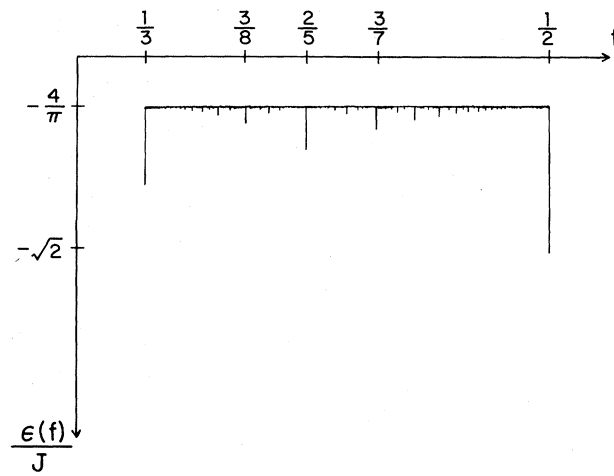


FIG. 14. Staircase-state energy $\epsilon(f)$ plotted against f for $\frac{1}{3}\le f\le\frac{1}{2}$. Certain prominent features are identified.

In Appendix B we show that the staircase states are not only extrema of the Hamiltonian (1.1) but are also local minima of the Hamiltonian for all values of f. We are only, however, proposing the staircase states as candidates for the true ground states for f in the range $\frac{1}{3}\le f\le\frac{1}{2}$.

For rational $f=p/q$, the phase differences on the bonds are spatially periodic with a $q\times q$ unit cell for any value of α. One quantity of interest is the current that flows around any particular plaquette, which we will term the "circulation" of that plaquette, $c(m,n)$. In a staircase state this quantity is proportional to

$$\begin{aligned} c(m,n)&=\tilde{c}(m+n)\\ &=\sin\left[\pi f(m+n)+\frac{\alpha}{2}-\pi\,\mathrm{nint}\left[f(m+n)+\frac{\alpha}{2\pi}\right]\right]-\sin\left[\pi f(m+n-1)+\frac{\alpha}{2}-\pi\,\mathrm{nint}\left[f(m+n-1)+\frac{\alpha}{2\pi}\right]\right], \end{aligned} \tag{3.15}$$

where the (m,n) plaquette is the plaquette bounded by the lattice sites (m,n), $(m,n-1)$, $(m-1,n)$, and $(m-1,n-1)$. Of particular interest is the sign of this function. The quantity $(-1)^{m+n}\mathrm{sgn}[c(m,n)]$ is related to the chirality of a plaquette as discussed in Sec. II. This quantity is constant in the ground state at $f=\frac{1}{2}$, and changes sign on either side of a domain wall in that model.[8–10] The $\mathrm{sgn}[c(m,n)]$ is given by

$$\begin{aligned} \mathrm{sgn}[c(m,n)]&=\mathrm{sgn}\left\{\pi f(m+n)+\frac{\alpha}{2}-\pi\,\mathrm{nint}\left[f(m+n)+\frac{\alpha}{2\pi}\right]-\left[\pi f(m+n-1)+\frac{\alpha}{2}-\pi\,\mathrm{nint}\left[f(m+n-1)+\frac{\alpha}{2\pi}\right]\right]\right\}\\ &=\mathrm{sgn}\left\{\pi f-\pi\left[\mathrm{nint}\left[f(m+n-1)+\frac{\alpha}{2}\right]\right]\right\}, \end{aligned} \tag{3.16}$$

$$\mathrm{sgn}[c(m,n)]=1-2\left[\mathrm{nint}\left[f(m+n)+\frac{\alpha}{2\pi}\right]-\mathrm{nint}\left[f(m+n-1)+\frac{\alpha}{2\pi}\right]\right]. \tag{3.17}$$

Thus the quasi-one-dimensional pattern formed by this function sgn$[c(m,n)]$ is identical to the commensurate and incommensurate structures discussed by Hubbard and by Pokrovsky and Uimin in the context of truly one-dimensional problems.[19] These structures can be simply generated from the continued fraction expansion for f, as is discussed in further detail in Appendix C.

We have also conducted a numerical investigation of the ground states of the Hamiltonian (1.1). The procedure (discussed in detail in Appendix D) was to commence with a finite-size phase system in a random initial state and to allow it to develop via a Monte Carlo algorithm. The characteristic temperature of this algorithm decreased exponentially with increasing timestep number. When sufficiently low temperatures had been reached, the system was quenched to zero temperature. This procedure was guaranteed to generate, at the very least, local minima of the Hamiltonian.

Ground states at $f=\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{1}{6}, \frac{2}{7}, \frac{3}{7}$, and $\frac{3}{8}$ were investigated in this fashion. Numerical studies of this sort are always limited to finite-size systems; it is therefore difficult to investigate by this method the ground states at high-order rational f. It is easy to disprove hypotheses concerning the nature of ground states in this manner; it is necessary only to produce a local minimum state with an energy lower than the energy of the alleged ground state.

Indeed, at the values of $f<\frac{1}{3}$ investigated it was quite easy to numerically generate states with energies lower than that of the staircase state. However, for $f \geq \frac{1}{3}$ no state was ever generated by this method with energy lower than that given by (3.10) or by (3.13).[19] We have, by another method, succeeded in generating states with energies lower than the staircase-state energies at certain higher-order rational values of f close to $\frac{1}{2}$ and $\frac{1}{3}$. These states, which are modified staircase states, will be discussed in a separate study.[20]

At $f=\frac{1}{3}, \frac{2}{5}, \frac{3}{7}$, and $\frac{3}{8}$ it was possible to generate on a fair proportion of the runs the staircase state itself. The staircase states generated at $f=\frac{1}{3}, \frac{2}{5}$, and $\frac{3}{8}$ are displayed in Fig. 15. The $f=\frac{1}{2}$ ground state is included for comparison. Limitations on the computer time available reduced the effectiveness of the computer studies for $q>8$. A serious practical problem of these studies is the fact

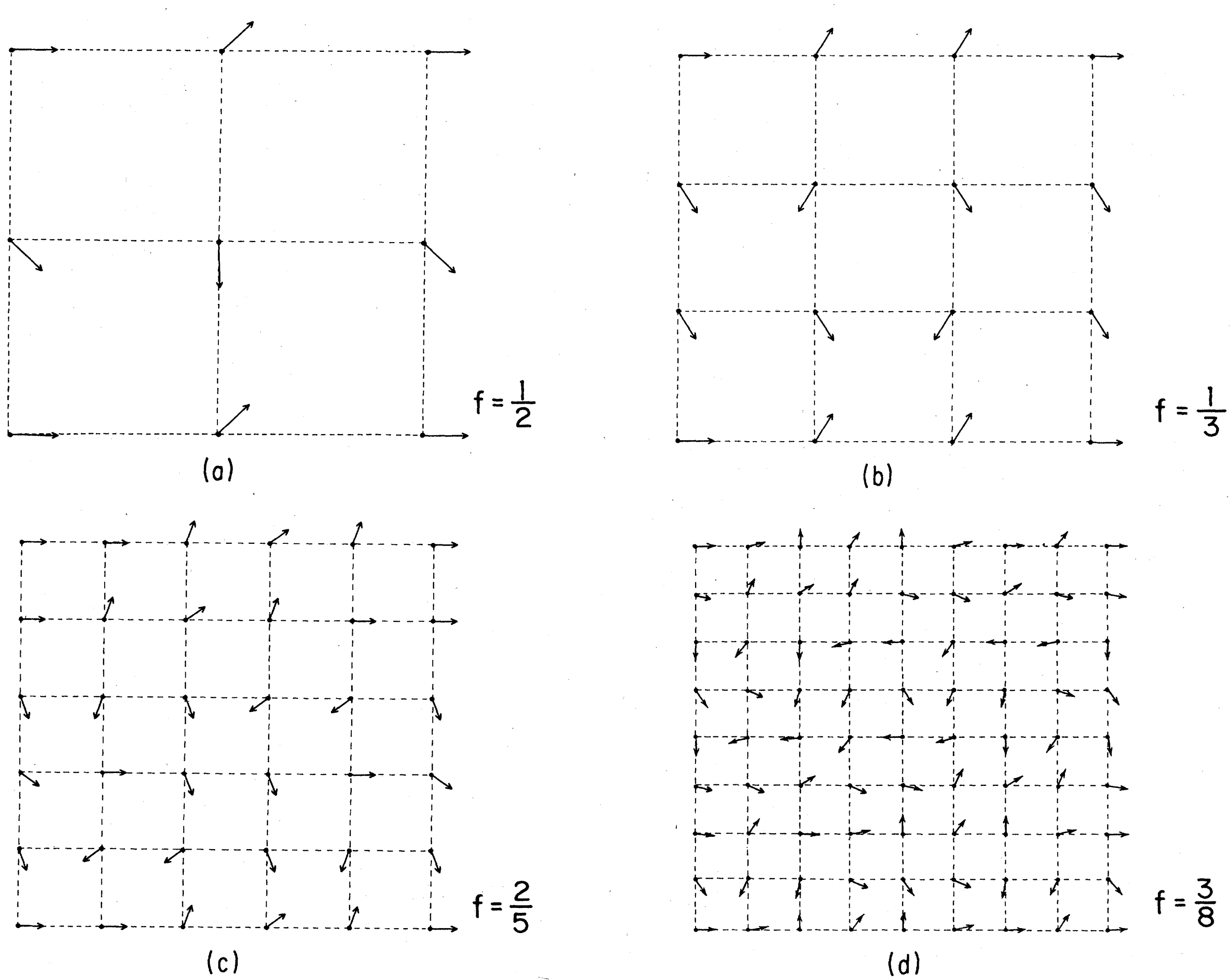


FIG. 15. Staircase-state phases for $f=\frac{1}{2}, \frac{1}{3}, \frac{2}{5}$, and $\frac{3}{8}$ in the Landau gauge (all A_{ij} are zero on horizontal bonds). The staircase form can be seen by studying the phases on successive rows.

that these systems possess a large number of metastable states with energies within a few percent of the actual ground-state energy. This problem becomes severe for larger values of q.

The hypothesis that the ground states are of the staircase form was thus corroborated for several $f \geq \frac{1}{3}$, but for no $f < \frac{1}{3}$. One serious drawback of these simulations was that they were all performed on $q \times q$ cells with periodic boundary conditions. Since the staircase states are periodic with $q \times q$ unit cells, the choice of periodic boundary conditions may have constituted a bias in favor of staircase states. However, it seems likely that ground states at rational f are always periodic with $q \times q$ unit cells, regardless of whether these ground states are staircase states or show more complicated configurations.[15]

IV. ZERO-TEMPERATURE CRITICAL CURRENTS

The zero-temperature critical current of an array is the largest macroscopic current that can flow in a state that is still a metastable state of the array with a Hamiltonian (1.1). If the array is connected in parallel with a resistor, then at currents below this critical current no voltage will develop across the resistor and there will be no dissipation of energy. If a system possesses a zero-temperature critical current, then presumably at sufficiently low temperatures the resistance will be zero, although if current flows some voltage will develop.

We determine $T=0$ critical currents assuming that metastable critical-current carrying states are continuously connected to the ground state. Imagine a two-parameter family of continuous changes of a state, indexed by δ_1 and δ_2, such that at $\delta_1=\delta_2=0$ the state is the ground state. More specifically, we choose the phases in a finite-size system to be continuous functions of δ_1 and δ_2; $\{\theta_i\}=\{\theta_i(\delta_1,\delta_2)\}$, such that $\{\theta_i(0,0)\}$ are the phases in the ground state. We further choose the deformations $\{\theta_i(\delta_1,\delta_2)\}$ so that, with the appropriate boundary conditions, these states are metastable states of the Hamiltonian (1.1). These states will be current-carrying states. By varying δ_1 and δ_2 we can alter the magnitude and the direction of the macroscopic current flowing. In general, there will be some region in the δ_1-δ_2 plane where this is possible, including a compact region about the origin. The points on the boundary of this compact region we identify as the critical current-carrying states. This assumption concerning the nature of the critical current states, and also the assumption that all spatially translated versions of the original ground state $\{\theta_i(0,0)\}$ can be generated by the appropriate choice of (δ_1,δ_2), are implicit in the arguments of Teitel and Jayaprakash leading to the bound discussed in Sec. I, that[15]

$$i_c(f=p/q) < \frac{e\pi}{hq} \mid \epsilon_0(f) \mid . \tag{4.1}$$

We shall examine the critical currents of staircase states subject to the first assumption above; we assume that the values of f are chosen so that these are the true ground states.

The critical currents in different directions can be quite different. We will consider current-carrying deformations of a staircase state oriented in a particular direction, with all of the staircases running in parallel with the $[1\bar{1}]$ diagonal lattice direction. At the end of this section we discuss those effects stemming from the availability of two global choices for staircase direction.

A. The $[1\bar{1}]$ critical current

The staircase state defined by (3.7) is metastable for all values of α provided that all of the phase differences satisfy

$$-\frac{\pi}{2} < \tilde{\theta}(m) - \tilde{\theta}(m-1) < \frac{\pi}{2} \tag{4.2}$$

with the $\tilde{\theta}(m)-\tilde{\theta}(m-1)$ given by (3.6),

$$\tilde{\theta}(m)-\tilde{\theta}(m-1) = \pi f m + \frac{\alpha}{2} - \pi \,\text{nint}\left[fm + \frac{\alpha}{2\pi} \right] .$$

This is proven in Appendix B.

It is easy to understand why the staircase state becomes unstable if, for any m,

$$\mid \tilde{\theta}(m)-\tilde{\theta}(m-1) \mid > \frac{\pi}{2} .$$

The staircase along which the phase difference exceeds $\pi/2$ divides the lattice into two regions. If all phases in one of these regions are rotated by a constant value β, while all of the phases in the other region are held fixed, then only the energies of the bonds making up this staircase change. This energy change per unit length of the staircase is

$$\Delta\epsilon = -\frac{J}{2}(\{\cos[\tilde{\theta}(m)-\tilde{\theta}(m-1)+\beta] + \cos[\tilde{\theta}(m)-\tilde{\theta}(m-1)-\beta]\} - \{2\cos[\tilde{\theta}(m)-\tilde{\theta}(m-1)]\}) \tag{4.3}$$

or

$$\Delta\epsilon = +\frac{J}{2}(\{\cos[\tilde{\theta}(m)-\tilde{\theta}(m-1)]\}[1-\cos\beta]) . \tag{4.4}$$

If $\cos[\tilde{\theta}(m)-\tilde{\theta}(m-1)]<0$, then $\Delta\epsilon<0$. Thus if $\mid \tilde{\theta}(m)-\tilde{\theta}(m-1) \mid > \pi/2$ for any staircase, the state is unstable. In general, if there exist contours which either close upon themselves or else extend through the entire lattice such that the average energy of the bonds pierced by these contours is >0, then the state will be unstable.

Staircase states with $\alpha \neq \alpha_{\min}$ are states carrying current in the $[1\bar{1}]$ direction, since choosing $\alpha \neq \alpha_{\min}$ corresponds to rigidly rotating the phase difference pattern so that it is no longer "balanced" about the x axis (see the discussion in Sec. III above). Here the parameter $(\alpha-\alpha_{\min})$ plays the role of one of the δ parameters mentioned above. The maximum current that may be carried will be carried when $\mid \alpha-\alpha_{\min} \mid = \pi/q$; this is the threshold of instability, where for some m, $\mid \tilde{\theta}(m)-\tilde{\theta}(m-1) \mid = \pi/2$. Only the staircase corresponding to this m carries net current, the currents on all other staircases cancel by symmetry (see Fig. 16). Thus one immediately obtains

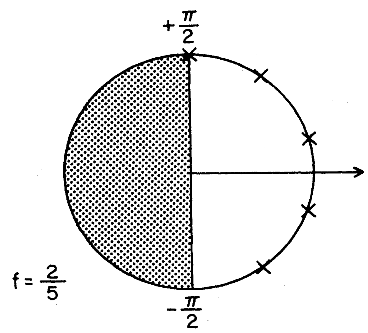


FIG. 16. Phase differences of a staircase state carrying its maximum current in the $[1\bar{1}]$ lattice direction.

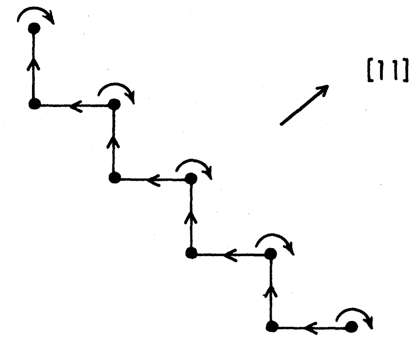


FIG. 17. In the [11] current-carrying states, successive planes bordering the same staircase undergo relative phase rotation.

$$i_c^{[1\bar{1}]}(f=p/q,\ T=0)=\frac{1}{q}i_c^{[1\bar{1}]}(f=0,\ T=0)\ , \tag{4.5a}$$

and for irrational f,

$$i_c^{[1\bar{1}]}(f\neq p/q,\ T=0)=0\ , \tag{4.5b}$$

since the irrational pattern cannot be rotated at all.

If the $T=0$ critical current at a value of f is zero, then presumably at finite temperatures there will be a finite resistance in the array at that value of f. This should not be a surprise. Consider a continuum superconducting film in the presence of a transverse magnetic field. In the absence of pinning forces a supercurrent could be degraded by a uniform motion of the vortex lattice in a direction perpendicular to the supercurrent, so that at finite temperatures this film would also possess a finite resistance. The more interesting result is that for rational f there can be superconductivity in the array, even in the absence of pinning forces. This is a consequence of the existence of periodic ground states at these values of f.

B. The [11] critical current and other directions

The staircase form of the states can also be exploited to determine the form of states that carry current in the [11] direction, the direction perpendicular to the direction of current flow along staircases. Two successive diagonal planes of phases oriented perpendicularly to this direction will form the boundaries of a particular staircase (see Fig. 17). Suppose that the two successive planes of phases are rotated by different angles such that the angle of rotation along a single plane is constant and the difference in the two angles of rotation is γ. If the original gauge-invariant phase difference of the staircase was $\widetilde{\theta}(m)-\widetilde{\theta}(m-1)$, then the net current per unit length of the staircase flowing between these two planes will be proportional to

$$i(m)=\tfrac{1}{2}\{\sin[\widetilde{\theta}(m)-\widetilde{\theta}(m-1)+\gamma] -\sin[\widetilde{\theta}(m)-\widetilde{\theta}(m-1)-\gamma]\}\ , \tag{4.6}$$

or

$$i(m)=\cos[\widetilde{\theta}(m)-\widetilde{\theta}(m-1)]\sin\gamma\ . \tag{4.7}$$

We immediately obtain a bound on $i_c^{[11]}$, as this quantity cannot exceed $\cos[\widetilde{\theta}(m)-\widetilde{\theta}(m-1)]$ in magnitude:

$$i_c^{[11]}(f=p/q,\ T=0)\leq \min_{(m)}\cos[\widetilde{\theta}(m)-\widetilde{\theta}(m-1)] \times i_c^{[11]}(f=0,\ T=0)\ , \tag{4.8}$$

using the results of Sec. III,

$$i_c^{[11]}(f=p/q,\ T=0)\leq \sin\frac{\pi}{2q}i_c^{[11]}(f=0,\ T=0) \tag{4.9a}$$

and for irrational f,

$$i_c^{[11]}(f\neq p/q,\ T=0)=0\ . \tag{4.9b}$$

This net current must be the same between every successive pair of planes so that current is conserved and the state is metastable. Thus the relative rotation γ must be a function of m,

$$\gamma(m)=\sin^{-1}\left[\frac{\chi}{\cos\widetilde{\theta}(m)-\widetilde{\theta}(m-1)}\right]\ , \tag{4.10}$$

where χ is a constant, which determines the magnitude of the current flowing; χ plays the role of the second δ parameter discussed above.

The nature of states carrying currents in directions other than the [11] and $[1\bar{1}]$ directions has been implicitly determined by the above. Nowhere in the above, except in the derivation of (4.9a), was it assumed that the phase differences $\widetilde{\theta}(m)-\widetilde{\theta}(m-1)$ corresponded to the lowest-energy staircase state with $\alpha=\alpha_{\min}$. If $\alpha-\alpha_{\min}$ and χ are both not equal to zero, and are sufficiently small, then the state implicitly determined by (4.10) will be a metastable state carrying current in a direction intermediate between the [11] and $[1\bar{1}]$ directions, provided that f is rational.

We thus conclude that the critical current-carrying states for arbitrarily oriented currents are given by the limit of metastability of states corresponding to points in a compact region about the origin, in a plane whose coordinates are $\alpha-\alpha_{\min}$ and χ. It is simple to determine numerically the boundary of the metastable region in this plane, and thus to determine the critical currents in the various directions. The details of this numerical calculation are presented in Appendix D. A polar plot of these critical currents will be reflection symmetric about the [11] and $[1\bar{1}]$ axes, it is thus only necessary to calculate these currents for one quadrant between the [11] and $[1\bar{1}]$ directions. The results of this calculation are shown (not to scale) in Fig. 18 for $f=\frac{1}{2}$ and $\frac{1}{3}$, and also for $f=0$.

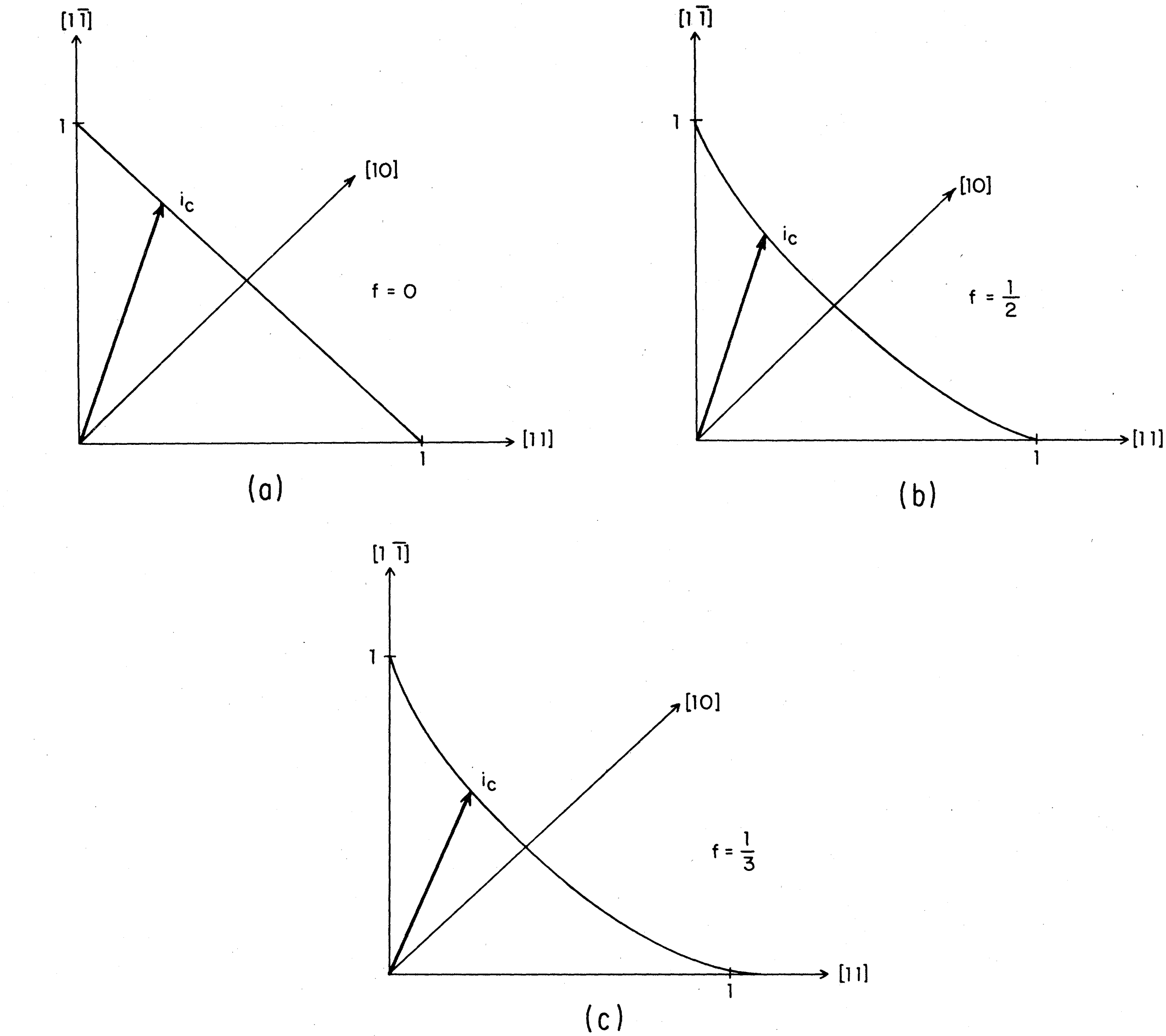


FIG. 18. Polar plot of the critical currents calculated for $f=0$, $\frac{1}{2}$, and $\frac{1}{3}$. The relative scale of the diagrams can be obtained from $i_c^{[1\bar{1}]}(f=p/q)=(1/q)i_c^{[1\bar{1}]}(f=0)$.

We should make several remarks concerning these results.

(1) At $f=\frac{1}{2}$, the two possible diagonal orientations of the staircases lead to states which are, in fact, identical (up to translations). Thus the critical current plot in Fig. 18 is symmetric about the [10] axis.

(2) The critical-current plots are strongly cusped about the [11] and [$1\bar{1}$] axes. Although even at $f=0$ we see that

$$i_c^{[1\bar{1}]}(f=0,\ T=0)=\sqrt{2}\,i_c^{[10]}(f=0,\ T=0)\ , \tag{4.11}$$

this anisotropy is still more pronounced at other values of f. Thus, for instance, at $f=\frac{1}{2}$ we have

$$i_c^{[1\bar{1}]}(f=\tfrac{1}{2},\ T=0)=\tfrac{1}{2}i_c^{[1\bar{1}]}(f=0,\ T=0)\ , \tag{4.12}$$

in accord with (4.5a), while

$$i_c^{[10]}(f=\tfrac{1}{2},\ T=0)=(\sqrt{2}-1)i_c^{[10]}(f=0,\ T=0)\ . \tag{4.13}$$

(3) For $f\neq\frac{1}{2}$ and 0, there is no symmetry reason why $i_c^{[11]}$ calculated in this manner need equal $i_c^{[1\bar{1}]}$, and indeed at $f=\frac{1}{3}$ and $\frac{2}{5}$ they are not equal.

A particular sample that has been cooled at zero applied current will probably possess a complicated quenched domain structure, and its current characteristics will be more isotropic than Fig. 18 would suggest. However, the anisotropy of Fig. 18 suggests that samples cooled in small applied diagonal currents might develop "single-domain" states which would then be more anisotropic in their properties.

ACKNOWLEDGMENTS

It is a great pleasure to acknowledge the many stimulating conversations and considerable encouragement offered by David R. Nelson and by Christopher J. Lobb, who also made critical remarks on the manuscript. Christopher L. Henley advised me on the numerical determination of ground states, and also read the manuscript. Bertrand I. Halperin pointed out to me the insufficiency of the staircase-state hypothesis at high-order rational values of f. Mehran Kardar contributed to my understanding of the contents of Appendix A. C. Jayaprakash shared unpublished results of computer simulations. This work was supported by the National Science Foundation, through the Harvard University Materials Science Laboratory, and through Grant No. DMR-82-07431.

APPENDIX A

In this appendix we will display the approximate relationship between the partition function generated by the Hamiltonian (1.1) and that generated by the charge model Hamiltonian (1.5). We will also show the relationship between the ground-state symmetries of the two models, which was mentioned in Sec. III. Throughout this discussion we will ignore constants that multiply the partition function Z or correlation functions:

$$H=-J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j-A_{ij})\ , \tag{A1}$$

with

$$\sum_p A_{ij}=2\pi f\ , \tag{A2}$$

where the $\sum_p$ indicates the sum over the bonds bordering the plaquette p, so that

$$Z(\beta)=\int\prod_i(d\theta_i)\exp[-\beta H\{\theta_i\}]\ . \tag{A3}$$

Let $\theta_i-\theta_j-A_{ij}\equiv\theta_{ij}$; index the plaquette by p, and write $-\beta J=J'$. Then

$$Z(J')=\int\prod_{\langle ij\rangle}(d\phi_{ij})\prod_p\left[\delta\left[\sum_{i,j\in p}\phi_{ij}-2\pi f\right]\right]\times\exp\left[J'\sum_{\langle ij\rangle}\cos\phi_{ij}\right]\ . \tag{A4}$$

The further transformation of this partition function follows a well-known approximate technique.[4] Writing

$$\delta\left[\sum_{i,j\in p}\phi_{ij}-2\pi f\right]\propto\sum_{k_p=-\infty}\exp\left[ik_p\left[\sum_{i,j\in p}\phi_{ij}-2\pi f\right]\right]\ , \tag{A5}$$

performing the integral over the $\{\phi_{ij}\}$ in the saddle-point approximation, and using the Poisson summation formula, we have

$$Z(J')=\int\prod_p(d\phi_p)\prod_p\left[\sum_{m_p=-\infty}^{\infty}\right]\times\exp\left[\frac{-1}{2J'}\sum_{p,\hat{i}}(\phi_{p+\hat{i}}-\phi_p)^2-2\pi i\sum_p[\phi_p(m_p-f)]\right]\ , \tag{A6}$$

where the $\hat{i}$ are the elementary lattice vectors $\hat{\mathbf{x}},\hat{\mathbf{y}}$.

We now proceed to integrate over the ϕ field, noting that this integration introduces the constraint

$$\sum_p(m_p-f)=0\ . \tag{A7}$$

Thus is obtained

$$Z(J')=\prod_p\left[\sum_{m_p=-\infty}^{\infty}\right]\exp\left[-\pi J'\sum_{p,p'}(m_p-f)G(p,p')\times(m_{p'}-f)\right]\ ,$$

with $G(p,p')$ given for large separations between p,p' by

$$G(p,p')=\ln|p-p'|\ . \tag{A8}$$

The expectation value of a bond energy in the original θ model,

$$\langle J\cos(\theta_i-\theta_j-A_{ij})\rangle_\theta\ ,$$

can be approximately related to a correlation function in the intermediate ϕ model (A6)

$$\langle\cos(\theta_i-\theta_j-A_{ij})\rangle_\theta\propto\langle(\phi_{p+\hat{j}}-\phi_p)^2\rangle_\phi\ , \tag{A9}$$

where the bond ij is the boundary between the plaquette p and the plaquette $p+\hat{j}$, and

$$\langle(\phi_{p+\hat{j}}-\phi_p)^2\rangle_\phi=Z^{-1}\int\prod_p(d\phi_p)\prod\left[\sum_{m_p=-\infty}^{\infty}\right](\phi_{p+\hat{j}}-\phi_p)^2\times\exp-\beta H\{\phi_p,m_p\}\ . \tag{A10}$$

Thus if the ensemble is restricted so that as $\beta\to\infty$ the $\{m_p\}$ are fixed to correspond to a particular ground state, then the expectation value $\langle(\phi_{p+\hat{j}}-\phi_p)^2\rangle$ will possess the same spatial symmetries as that configuration of $\{m_p\}$. This includes the symmetry of reflection about a diagonal axis of which use was made in Sec. III.

APPENDIX B

Herein we will show that the quasi-one-dimensional staircase states, which were proposed in Sec. III, are not only extrema of the Hamiltonian (1.1), as they are by construction, but are also local minima of the energy. We are unable at this time to show under what conditions such a state is a global minimum of the energy.

Recall the Hamiltonian

$$H=-J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j-A_{ij}),\quad \sum_p A_{ij}=2\pi f\ . \tag{B1}$$

Suppose the $\{\theta_i\}$ corresponding to a staircase state are $\{\theta_i\}=\{\phi_i\}$. Then the staircase state energy will be

$$E=-J\sum_{\langle ij\rangle}\cos(\phi_i-\phi_j-A_{ij})\ . \tag{B2}$$

The condition for this state to be a local minimum of the Hamiltonian (B1) is that the matrix of second derivatives

$$M_{ij}\equiv\left[\frac{\partial^2 E}{\partial\phi_i\partial\phi_j}\right], \tag{B3}$$

possess no negative eigenvalues. This matrix will have at least one zero eigenvalue corresponding to the uniform rotation of all phases $\{\phi_i\}\to\{\phi_i+\alpha\}$.

Let us consider in more detail this matrix of second derivatives. The diagonal terms

$$M_{ii}=\frac{\partial^2 E}{\partial\phi_i^2}$$

are simply the opposites of the sums of the energies of all bonds connected to a particular site. The off-diagonal terms

$$M_{ij}=\frac{\partial^2 E}{\partial\phi_i\partial\phi_j},\quad i\neq j$$

are simply the energies of the bonds ij if i and j are nearest neighbors. Otherwise these terms are zero. This matrix is thus very sparse.

The staircase form of the states can be exploited to simplify the analysis of this matrix. In particular, imagine that the staircases are, as in the above, oriented parallel to the $[1\bar{1}]$ lattice direction. The indices i and j used above can be further indexed by the horizontal and vertical coordinates, (m,n), respectively, of the lattice site to which they refer. If then the translation $(m,n)\to(m+p,n-p)$ is performed on the entries of this matrix, the matrix will be unchanged. It follows that any eigenvector can be expressed in the form

$$\lambda(i=(m,n))=C(n+m)e^{ik(n-m)} \tag{B4}$$

or in degenerate linear combinations of such forms.

If we write the cosine of the gauge-invariant phase difference between the nth and $(n+1)$th sites of the $m=0$ column as $\alpha(n)$, then it is easy to see that the energy change associated with a displacement of the phases according to the eigenvector $\lambda(i)$,

$$\delta E(\lambda(i))=\sum_{i,j}\lambda(i)\tfrac{1}{2}M_{ij}\lambda(j)\ ,$$

is proportional to a simple one-dimensional sum

$$\delta E\propto\sum_{i=1}^{M}\{[\alpha(i)+\alpha(i-1)]c^2(i)-\tfrac{1}{2}f(k)\alpha(i)c(i)c(i+1)\}\ . \tag{B5}$$

We perform the sum over M terms, which we imagine to be the size of the system. If f is rational with denominator q, then we suppose that $M=Nq$, where N is a large integer. We will also impose the boundary condition $c(M)=c(0)$; the extent to which this restricts the proof will be discussed below. All phases have been absorbed into $f(k)$, which can now be a function of i; we may thus regard the $c(i)$ as real. $f(k)$ satisfies $-1\leq f(k)\leq 1$. We will choose $f(k)\geq 0$, and adjust the signs of the $c(i)$ accordingly.

We wish to prove that there are no solutions to

$$[\alpha(i)+\alpha(i-1)]c(i)-[\tilde{x}(i)\alpha(i)c(i+1)+\tilde{x}(i-1)\alpha(i-1)c(i-1)]=\lambda c(i)\ , \tag{B6}$$

with $0\leq\tilde{x}(i)\leq 1$, such that $\lambda<0$. As anticipated above, if $\tilde{x}(i)=1$, then $c=$const is a solution with eigenvalue 0.

If $\lambda<0$, then

$$\left[\alpha(i)\tilde{x}(i)\left[\frac{c(i+1)}{c(i)}\right]+\alpha(i-1)\tilde{x}(i-1)\left[\frac{c(i-1)}{c(i)}\right]\right]>\alpha(i)+\alpha(i-1)\ . \tag{B7}$$

Defining

$$\frac{c(i+1)}{c(i)}\equiv\gamma_i\ ,$$

we note that $\prod_{i=1}^{M}\gamma_i=1$.

In the case where all $\alpha(i)>0$, which is the staircase-state case, we see that the most advantageous case for the satisfaction of the inequality is all $\tilde{x}(i)=1$, and all $\gamma(i)>0$. Presuming that this is the case, we have

$$\alpha(i)\gamma(i)+\frac{\alpha(i-1)}{\gamma(i-1)}>\alpha(i)+\alpha(i-1)\ . \tag{B8}$$

If $\alpha(i)/\alpha(i-1)\equiv\delta(i)$, then

$$\delta(i)\gamma(i)+\frac{1}{\gamma(i-1)}>\delta(i)+1\ . \tag{B9}$$

But it is impossible for this inequality to hold for all i. If $\gamma(i)<1$, $\gamma(i-1)>1$ the inequality must fail. Say $\gamma(i)>1$, then it is easy to see that all $\gamma(j)>1$ and we cannot have $\prod_i\gamma(i)=1$. Or suppose $\gamma(i-1)<1$, then all $\gamma(i)$ will be <1, and again it is impossible for $\prod_i\gamma(i)=1$. Thus there are no $\{\gamma(i)\}$ for which the inequality holds, and no negative eigenvalues. It is straightforward to extend this argument to allow for $\gamma(i)\leq 0$. This proves the stability of the staircase states.

The demonstration above requires only that M_{ij} be of the staircase form and that all $\alpha>0$. For rational f, it thus also constitutes a proof of the metastability of the states that carry current in the $[1\bar{1}]$ direction (see Sec. IV), as long as that current is sufficiently small so that all $\alpha(i)>0$. If one or more of the $\alpha(i)<0$, then, as pointed out in Sec. IV, the staircase state becomes unstable. Thus in this case there will be at least one negative eigenvalue to M_{ij}.

The proof above was restricted to finite-size systems; periodic boundary conditions were enforced for the variations allowed. For rational $f=p/q$ we can always choose the size of the system to be much larger than the $q\times q$ unit cell of the staircase-state structure. In this case the requirement of periodic boundary conditions for the allowed eigenvectors is probably not very important, and we may be confident that the same result would hold for any boundary condition that does not allow unbounded variations of the phases at the boundaries of the system. If f is irrational, there are no finite-size systems larger than the unit cell, as there is no unit cell. In this case the situation is more obscure, and the importance of the imposition of periodic boundary conditions is less clear.

The metastability of the current-carrying states does not obviate the fact that the zero-current state has the lowest energy. Any variation of a state that changes the macroscopic current flowing in that state will necessarily involve unbounded variations at the boundaries, and has thus been excluded by the requirement of periodicity.

APPENDIX C

In Sec. III it was shown that the sign of the circulation of the current about successive plaquettes in the direction of the quasi-one-dimensional modulation is given by

$$\mathrm{sgn}c(m,n)=1-2\left[\mathrm{nint}\left[f(m+n)+\frac{\alpha}{2\pi}\right]-\mathrm{nint}\left[f(m+n-1)+\frac{\alpha}{2\pi}\right]\right]. \quad \text{(C1)}$$

The positions of the plaquettes that support negative-current circulation is thus given, as a function of the coordinate $m+n$, by the continued fraction structure of Hubbard and of Pokrovsky and Uimin.[19] In this appendix we will give a description of this structure and apply it to the staircase states. It will first be necessary to review some properties of the continued fraction representation of real numbers.[21]

Any real number f can be represented uniquely by a continued fraction of the form

$$f=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}} \quad \text{(C2)}$$

with $a_0\in\mathbb{Z}$ and $a_i\in\mathbb{N}$ (the positive integers).

If f is rational, then the continued fraction representation terminates at some a_n. If f is irrational, then the continued fraction representation does not terminate.

A continued fraction representation can be truncated at level k simply by taking $a_{k+1}\to\infty$, thus obtaining an approximant to f, which is by construction rational. This procedure defines a sequence of approximants to f, $\{f_k\}$,

$$\{f_k\}:f_k=\frac{p_k}{q_k}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots\, a_{k-2}+\cfrac{1}{a_{k-1}+\cfrac{1}{a_k}}}}}$$

with the following properties.

(1) A particular f_k is the best approximant to f of order q_k (with denominator q_k).

(2) The error of the approximation alternates sign with k. Thus if $f_k-f>0$, then $f_{k+1}-f<0$, and vice versa.

(3) f_{k+1} can be obtained recursively from f_k and f_{k-1} with the formula

$$f_{k+1}=\frac{p_{k+1}}{q_{k+1}}=\frac{a_{k+1}p_k+p_{k-1}}{a_{k+1}q_k+q_{k-1}}, \quad \text{(C3)}$$

with

$$p_{k+1}=a_{k+1}p_k+p_{k-1} \text{ and } q_{k+1}=a_{k+1}q_k+q_{k-1},$$
$$p_{-2}=0,\ p_{-1}=1,\ q_{-2}=1,\ q_{-1}=0.$$

The continued fraction structure is a concrete implementation of this recursion formula. Choose a horizontal sequence of plaquettes from the staircase state discussed in Sec. III. Because of the quasi-one-dimensional form of the state, it will be possible to determine the current circulation of all plaquettes in the lattice from the values of the circulation in this single row. The row presents a one-dimensional structure of plaquettes whose circulation is either positive or negative. This structure is built up out of substructures in a hierarchical fashion, so that the kth sequence is constructed by spatially repeating the $(k-1)$th sequence a_k times and by then appending the $(k-2)$th sequence. An example will make this procedure more clear.

We will display the successive approximants to $f=\frac{5}{12}$ (see Fig. 19). The continued fraction representation for $f=\frac{5}{12}$ is

$$f=0+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}},\quad a_0=0;\ a_1,\ a_2,\ a_3=2$$

thus

$$f_0=\frac{(0)\times1+0}{(0)\times1+1}=\frac{0}{1}=0,$$
$$f_1=\frac{2\times(0)+1}{2\times(1)+0}=\frac{1}{2},$$
$$f_2=\frac{2\times(1)+0}{2\times(2)+1}=\frac{2}{5},$$
$$f_3=\frac{2\times(2)+1}{2\times(5)+2}=\frac{5}{12}.$$

This sequence has the property that a finite series of m plaquettes includes n plaquettes with negative circulation,

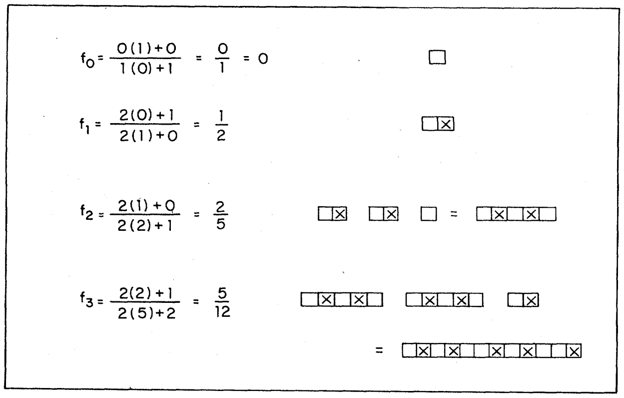


FIG. 19. At the left are displayed the successive approximants to $f=\frac{5}{12}$. At the right we show the structure corresponding to that approximant. Open boxes represents plaquettes with positive circulation, while shaded boxes represent plaquettes with negative circulation.

where either

$$\frac{n}{m} \leq f \leq \frac{n+1}{m} \tag{C4a}$$

or

$$\frac{n-1}{m} \leq f \leq \frac{n}{m} \,. \tag{C4b}$$

In fact, this property can be expressed as the condition that determines the sequence. Note also that the density of negative circulation plaquettes will precisely equal f for f rational, and approach f in the limit of an infinite lattice for f irrational. These negative circulation plaquettes are the lattice version of vortices.

This procedure is well defined for irrational f; since the continued fraction representation for irrational f does not terminate, neither will the recursive procedure.

APPENDIX D

1. Ground states

As discussed in Sec. III, the ground state was shown by explicit numerical simulation to be of the staircase form at $f=p/q=\frac{1}{3}, \frac{2}{5}, \frac{3}{7}$, and $\frac{3}{8}$. The simulation was performed on $q\times q$ samples with periodic boundary conditions. The imposition of periodic boundary conditions restricts the generality of this numerical work; however, Teitel and Jayaprakash have conducted simulations of $q\times q$ and of $nq\times nq$ systems with periodic boundary conditions and have obtained identical results concerning the ground-state form.[15] We have also investigated various values of n at $f=\frac{1}{3}$ and $f=\frac{2}{5}$, and can report no variation in the ground-state form as a function of n. We suspect that the currents in any ground state for rational f are periodic with a $q\times q$ unit cell. This condition is satisfied by a state of the staircase form.

The strategy of the simulation was to allow the state of the $q\times q$ sample to develop via a finite-temperature Monte Carlo algorithm while relaxing the temperature exponentially towards zero. When the system had reached sufficiently low temperatures it was quenched to zero temperature, insuring that the final configuration was a local minimum of the Hamiltonian.

Each run was performed starting from a random initial configuration of the phases of the $q\times q$ sample. A particular site i was chosen at random, and the energetically preferred value for its phase θ_i was calculated from the phases of its neighbors, $\theta_{j'}$. This preferred phase was obtained by minimizing

$$E(\theta_i) = -J\sum_{j'}\cos(\theta_i-\theta_{j'}-A_{ij'}) \,, \tag{D1}$$

yielding

$$\tan\theta_i^{\min} = \frac{-\sum_{j'}\sin(\theta_{j'}-A_{ij'})}{\sum_{j'}\cos(\theta_{j'}-A_{ij'})} \,. \tag{D2}$$

The updated phase was chosen from a Gaussian probability distribution about this preferred value of the phase; the width of this distribution was proportional to the temperature and decayed exponentially with increasing timestep number. The most realistic distribution to have used would have been a Bessel-function distribution

$$\mu(\theta_i) \propto \exp[c(T)\cos(\theta_i-\theta_i^{\min})] \,,$$

but the Gaussian distribution was more numerically convenient. For the last several hundred timesteps of the simulation the phases were set equal to their preferred values; this procedure is the equivalent of quenching the system to zero temperature.

In the larger samples it was necessary to perform runs of up to 50 000 timesteps per site in order to enjoy good convergence to the ground state. These systems possess a large number of metastable states with energies differing by only a few percent from the energy of the ground state. In the larger samples approximately half the runs generated the staircase state; the other half generated metastable states of higher energy.

The generation of any state with a lower energy than that of the staircase state would have immediately disproved the hypothesis that, at these values of f, the ground state has the staircase form. This was not observed in any run. On the other hand, simulations were also performed at $f=\frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, and $\frac{2}{7}$. These simulations easily generated states with energies lower than the staircase energy. While it is impossible to prove beyond a shadow of a doubt that a particular form is the true ground state by this sort of numerical calculation, these results are certainly consistent with the assertion that the staircase form is the true ground state at $f=\frac{1}{3}, \frac{2}{5}, \frac{3}{7}$, and $\frac{3}{8}$.

The situation at higher-order rational f is unclear. Because of the high density of metastable states alluded to above, simulations performed at $f=\frac{4}{9}, \frac{4}{11}$, and $\frac{5}{11}$ were unsuccessful in generating the staircase form, although they were equally unsuccessful in generating states of lower energies than those calculated for the staircase form from (3.10). Here our results differ from those of Teitel and Jayaprakash.[18] Also, as pointed out in Sec. III above, we have successfully generated states with energies lower than the staircase-state energies at some high-order rational values of f.[20]

2. Critical currents of staircase states

While it was a simple matter to determine the critical currents of the staircase states in the $[1\bar{1}]$ direction analytically, in other directions we were obliged to calculate the critical currents by a numerical method.

Given the form of the current-carrying states discussed in Sec. IV, this calculation was not difficult. Recall that the current-carrying states were parametrized by $(\alpha-\alpha_{\min}, \chi)$. The line $\chi=0$ corresponds to states carrying current in the $[1\bar{1}]$ direction, along the staircase, and the line $\alpha-\alpha_{\min}=0$ corresponded to states carrying current in the [11] direction, perpendicular to the staircases.

If $\alpha-\alpha_{\min}$ and $\chi\neq 0$, then the current will flow in various intermediate directions. Because of the reflection symmetry of the staircase states about the [11] and $[1\bar{1}]$ axes, it is necessary only to consider the quadrant $\alpha-\alpha_{\min}$, $\chi\geq 0$. For all values of $\alpha-\alpha_{\min}$ and χ, the state that will be generated following the procedure of Sec. IV will be an extremum of the Hamiltonian (1.1), but for $\alpha-\alpha_{\min}$ and χ sufficiently large, the states will be unstable local saddle points of the Hamiltonian. There will thus be a boundary in the $(\alpha-\alpha_{\min}, \chi)$ plane separating stable from unstable states. The points along this boundary correspond to the critical current-carrying states. This follows from the assumption, discussed in Sec. IV, that the critical current-carrying states are continuously connected to the ground state.

The limit of stability can be determined numerically by the following procedure. The locally extreme configuration can be generated exactly as a function of $\alpha-\alpha_{\min}$ and χ. This configuration is used as a starting configuration for the zero-temperature version of the updating procedure discussed above. Of course, since this configuration is an extremum of (1.1), in principle the phases should not change under this updating procedure. In practice the numerical roundoff error allows a distinction to be drawn between stable and unstable states, as the noise in an unstable state is amplified and leads ultimately to the decay of the initial configuration, while the stable states will not suffer such decay. For states near the threshold of instability, this decay may take several hundred timesteps per spin, if the calculation is performed with standard single-precision accuracy.

The unit cell of these instabilities can be larger than the $q\times q$ unit cell of the underlying structure. Thus, for instance, the instability at $\chi=0$, $\alpha-\alpha_{\min}=\pi/q$ discussed in Sec. IV possesses a $2q\times 2q$ unit cell. The calculations displayed in Sec. IV for $f=\frac{1}{2}$ and $\frac{1}{3}$ were therefore performed for $2q\times 2q$ and for $3q\times 3q$ samples. No difference was observed in the results; it is thus probable that the instability has always a $2q\times 2q$ unit cell.

In these calculations twisted boundary conditions were applied. The amount of the twist was determined by $\alpha-\alpha_{\min}$ and by χ. Thus if the staircase state relaxed into some complicated nonstaircase configuration, the conclusion was that with those exotic boundary conditions the system did not have a staircase-form ground state. The states into which the unstable states relaxed did not carry significant currents, a fact that is at least consistent with the assumption that the critical current-carrying states are continuously connected to the ground state.

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