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Hierarchy of plasmas for fractional quantum Hall states

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Explicit trial wave functions are proposed for the ground state at all odd-denominator rational filling factors of the lowest Landau level of a two-dimensional electron gas. An exact sum rule valid for any isotropic state in the lowest Landau level is used to justify an approximation scheme in which the hierarchy of quantum Hall states is associated with a corresponding hierarchy of classical fluids. Results are given for the energy at many rational filling factors and for the pair-correlation function at $\nu=\frac{2}{7}$.

The occurrence of a fractional quantum Hall effect at filling factor ν demonstrates the existence of especially stable states of the two-dimensional electron gas at that filling factor.[1,2] Experiments[3-5] strongly suggest that such states occur only at rational filling factors (rff) with odd denominators. The states at $\nu=1/m$ where m is an odd integer seem to be well described by the wave functions proposed by Laughlin.[6] For $\nu=1-1/m$ approximate eigenstates may be obtained by particle-hole conjugating Laughlin's states. It has been suggested[7,8] that states at the remaining odd denominator rff occur as a hierarchy in which the elementary excitations of a more primitive Laughlin state themselves condense into a Laughlin state. Here we propose trial many-body wave functions in terms of electron (rather than quasiparticle[8]) coordinates for each state in this hierarchy. We argue, using an exact sum rule, that it is possible to estimate the pair distribution function (pdf) and hence the energy of a hierarchy state by performing a corresponding hierarchy of liquid structure calculations.

Any many-body wave function contained entirely within the lowest Landau level may be written in the form[9]

$$\Psi[z]=P[z]\prod_{k=1}^{n}\exp(-|z_k|^2/4) \ , \tag{1}$$

where $z_k=x_k+iy_k$ is the electron coordinate in complex notation, $P[z]$ a polynomial in the z_k's, and lengths are in units of $a_L\equiv(\hbar c/eH)^{1/2}$. If $\Psi[z]$ describes an isotropic state $P[z]$ must be homogeneous. It follows from the antisymmetry of $\Psi[z]$ that $P[z]$ must be divisible by

$$P_V[z]\equiv\prod_{i<j}(z_j-z_i) \ . \tag{2}$$

Since $P_V[z]$ is a Vandermonde determinant the case $P[z]\equiv P_V[z]$ describes the full Landau level with a uniform density $n_1=(2\pi)^{-1}$. In general the rff of any polynomial $P[z]$ which describes a state of uniform density is given by

$$\nu^{-1}=\lim_{N\to\infty}\frac{n_1}{N}\frac{\left\langle\Psi\left|\sum_k r_k^2\right|\Psi\right\rangle 2\pi}{N\langle\Psi|\Psi\rangle}=2\lim_{N\to\infty}\frac{M(N)}{N^2} \ , \tag{3}$$

where N is the number of electrons and $M(N)$ the homogeneous degree of $P[z]$.

As emphasized by Girvin[10] the particle-hole symmetry within the lowest Landau level, which is exact only in the infinite magnetic field limit, plays an essential role in forming the hierarchy of fractional Hall states. For any polynomial $P_i[z]$ we denote the polynomial part of the particle-hole conjugate of $P_i[z]\exp(-\sum_k|z_k|^2/4)$ by $C(P_i[z])$. Each antisymmetric polynomial is uniquely related to a corresponding symmetric polynomial denoted by $Q_i[z]\equiv P_i[z]/P_V[z]$. We define a conjugate for $Q_i[z]$ as follows:

$$C(Q_i[z])\equiv C(Q_i[z]P_V[z])/P_V[z] \ .$$

With these definitions we can compactly state our recurrence relation for the fractional Hall states:

$$Q_i[z]=C(Q_{i-1}[z])(P_V[z])^{p_i} \ , \tag{4a}$$

or

$$Q_i[z]=C(Q_{i-1}[z])^+(P_V[z])^{p_i} \ , \tag{4b}$$

where p_i must, of course, be even. In Eq. (4a) $Q_i[z]$ corresponds to a state in which quasiholes of the Laughlin state at $\nu^{-1}=1+p_i$ have condensed into the hierarchy state associated with $Q_{i-1}[z]$; in Eq. (4b) it is the quasielectrons which have condensed into this state. (The justification of these statements is discussed below. Note that we use *quasielectron* to describe the positive fractionally charged excitations of Laughlin's states, *quasihole* for negative charges, and *quasiparticle* when referring to the excitations generically.) In Eq. (4b) $Q^+[z]$ is the adjoint of $Q[z]$.[9,11]

It follows from Eq. (3) that the Landau level filling factor associated with $Q_i[z]$ is given by

$$\begin{aligned}\nu_i^{-1}&=1+p_i+\frac{\alpha_{i-1}}{\nu_{i-1}^{-1}-1}\\&=1+p_i+\alpha_{i-1}/p_{i-1}+\alpha_{i-2}/p_{i-2}+\cdots\alpha_0/p_0 \ ,\end{aligned} \tag{5}$$

where α_j is 1 and -1 for $Q_j[z]$ obtained from $Q_{j-1}[z]$ by Eqs. (4a) and (4b), respectively. The notation in Eqs. (4) and (5) has been chosen to match that of Haldane,[7] except that for a state which occurs at level i of the hierarchy his α_j,p_j is our α_{i-j},p_{i-j}. This change is made since the wave functions are constructed by starting with that which describes the condensate of quasiparticles at the innermost level of the hierarchy and proceeding outward. These innermost states are those proposed by Laughlin,[6] i.e., $Q_0[z]=(P_V[z])^{p_0}$. In Table I we list some of the wave functions for states which occur in the first two levels of the hierarchy. (The wave functions at $\nu^{-1}=\frac{7}{2}$ and $\nu^{-1}=\frac{5}{2}$ are identical to the hierarchy states proposed by Laughlin.[11])

We now advance some arguments in support of the contention that the wave functions generated by Eqs. (4) are good approximations to the ground state at the filling factors for which they occur and in support of the statements of interpretation which follow Eq. (4). The lowest energy states are those whose wave functions $\Psi[z]$ vanish most strongly as the $z_{ij}\equiv z_i-z_j$, approaches zero. At larger ν^{-1}

TABLE I. Explicit symmetric polynomials characterizing some hierarchy states. $Q_j^i[z]$ corresponds to a state which occurs at level i of the hierarchy. We list all states with $\nu^{-1}<7$ in which the parent state has $p_i=0$, 2, or 4.

Symmetric polynomial	ν^{-1}		
$Q_1^0=P_V^2$	3	$Q_4^1=C(Q_1^0)P_V^2$	$\frac{7}{2}$
$Q_2^0=P_V^4$	5	$Q_5^1=C(Q_2^0)^+P_V^2$	$\frac{11}{4}$
		$Q_6^1=C(Q_2^0)^+P_V^4$	$\frac{19}{4}$
$Q_1^1=C(Q_1^0)^+P_V^2$	$\frac{5}{2}$	$Q_7^1=C(Q_2^0)$	$\frac{5}{4}$
$Q_2^1=C(Q_1^0)^+P_V^4$	$\frac{9}{2}$	$Q_8^1=C(Q_2^0)^+P_V^2$	$\frac{13}{4}$
$Q_3^1=C(Q_1^0)$	$\frac{3}{2}$		

(lower density compared to n_1) $\Psi[z]$ tends to decrease more rapidly with $|z_{ij}|$ but the polynomial $(P_V[z])^p$ (p even) is most successful in achieving this end for a given contribution to ν^{-1}. Thus, if $P[z]$ is a good approximation to the ground-state polynomial for inverse rff ν^{-1}, $(P_V[z])^2P[z]$ should be a good approximation at $\nu^{-1}+2$. This is just the procedure by which the Laughlin state wave functions[6] are obtained from that of a full Landau level. The second element necessary to justify Eqs. (4) is the observation that if $P[z]$ is the ground state at filling factor ν, $C(P[z])$ is guaranteed by particle-hole symmetry to be the ground state at filling factor $1-\nu$. The physical interpretation provided for Eqs. (4) follows from the facts that $C(Q_i[z])$ corresponds to state $Q_i[z]$ in the holes of a full Landau level and that[6] $H[z](P_V[z])^p$ is the symmetric polynomial corresponding to a hole in the full Landau level if $p=0$ and to fractionally charged holes in Laughlin states for $p=2,4,\ldots$. Here, $H[z]=\prod_k(z_k-z_0)$ for a hole centered at z_0 and $H^+[z]$ creates a quasielectron at z_0.

Given the pdf $g_{i-1}(r)$ for some state, $P_{i-1}[z]$, at level $i-1$ of the hierarchy we determine the pdf at level i, for the case where $P_{i-1}[z]$ is formed in the holes of a Laughlin state [Eq. (4a)] by noting that $|\Psi_i[z]|^2$ can be expressed as

$$|\Psi_i[z]|^2=\exp(-\beta_iU_i[z]) \ , \tag{6a}$$

where, choosing $\beta_i=\nu_i$

$$U_i[z]=-2\nu_i^{-1}p_i\sum_{i<j}\ln|z_i-z_j|-\nu_i^{-1}|C(P_{i-1}[z])|^2+\sum_i\nu_i^{-1}\tfrac{1}{2}|z_i|^2 \ . \tag{6b}$$

As discussed above we wish to approximate the impact of $\nu_i^{-1}|C(P_{i-1}[z])|^2$ in "statistical" averages by an effective classical pairwise interaction. This effective potential $U_{i-1}^{\text{eff}}(R)$ is chosen so that the pdf when $p_i=0$ (i.e., for the particle-hole conjugate of $\Psi_{i-1}[z]$; $\nu_i=1-\nu_{i-1}$)

$$(1-\nu_{i-1})^2g_{i-1}^c(r)=(1-2\nu_i)(1-e^{-r^2/2})+\nu_i^2g_{i-1}(r) \tag{7}$$

is reproduced exactly when calculated in a modified hypernetted-chain (HNC) approximation.[12,13] We take this potential to be the same as that between distributed two-dimensional charges with areal density $\sigma_{i-1}(r)$, i.e.,

$$U_{i-1}^{\text{eff}}(|\mathbf{R}-\mathbf{R}'|)=\int d\mathbf{r}\int d\mathbf{r}'\sigma_{i-1}(|\mathbf{r}-\mathbf{R}|)\sigma_{i-1}(|\mathbf{r}'-\mathbf{R}'|)(-2\ln|\mathbf{r}-\mathbf{r}'|) \ , \tag{8a}$$

where $\sigma_{i-1}(r)$ goes to zero for $r\gg a_L$ and

$$\int d\mathbf{r}\,\sigma_{i-1}(\mathbf{r})=(1-\nu_{i-1})^{-1} \ . \tag{8b}$$

Then the last term in Eq. (6b) describes the interaction of these charges with a uniform positive background of fictional charge density $(2\pi)^{-1}$. This guarantees that the classical potential of Eqs. (6) at $p_i=0$ correctly describes a state of electron with uniform (neutralizing) fictional charge density $(2\pi)^{-1}$ which corresponds to electron density $(2\pi)^{-1}(1-\nu_{i-1})$. The Fourier transform,

$$\sigma(k)=\int dr\exp(i\mathbf{k}\cdot\mathbf{r})\sigma(r) \ ,$$

corresponding to the particle-hole conjugate of the $\nu=\frac{1}{3}$ Laughlin state, is illustrated in Fig. 1. For $p_i=2,4,\ldots$ the total interparticle potential is, from Eq. (6b),

$$-2\nu_i^{-1}p_i\ln|z_{ij}|-\nu_i^{-1}(1-\nu_{i-1})U_{i-1}^{\text{eff}}(R) \ , \tag{9}$$

which, using Eqs. (8b) and (5), approaches $-2\nu_i^{-2}\ln|z_i-z_j|$ for $r\gg a_L$. Thus, all of the hierarchy states are represented by two-dimensional fluids with interparticle interactions which are the same as those of distributed charged particles, with $-2\ln r$ interactions and total charge ν_i^{-1}. The plasma analogy which holds for the original Laughlin states[6] $\{P[z]=(P_V[z])^{p+1}\}$ is thus extended to all states of the hierarchy. One consequence of this extension of the plasma analogy, which provides strong support for our approximation and suggests that the plasma analogy may have a fundamental origin, is the following. The sum rule

$$\int dr\,r^3[1-g(r)]=2\nu^{-1} \ , \tag{10}$$

which holds for any homogeneous isotropic state in the lowest Landau level[14] is obeyed exactly in our approach. In the case of the plasma calculations, Eq. (10) follows from the perfect screening condition[15] which, depending only on the long-range part of the interaction, is the same for point charges as for those with a finite size.

For the hierarchy states where the condensate is formed in the quasielectrons of a Laughlin state [Eq. (4b)] we do not have a simple expression like Eq. (6b). In this case we approximate the pair-correlation function of the daughter state with the aid of the following observations. For the parent Laughlin state, $P[z]=(P_V[z])^{1+p_i}$, $g(r)$ vanishes as $(r^2)^{1+p_i}$ corresponding to each pair of electrons having a relative angular momentum $1+p_i$. When a condensate occurs in the quasiholes of this state, $g(r)$ still vanishes as $(r^2)^{1+p_i}$ and we can regard the electron pairs as having some probability for relative angular momentum $1+p_i$ and some probability for a greater relative angular momentum. The pdf has an extra contribution reflecting this, $g(r)\to g_L^{p_i}(r)+\delta g_{i-1}^{p_i}(r)$. When the same condensate is formed in the quasielectrons of $(P_V[z])^{1+p_i}$ [Eq. (6b)] the electron pairs develop a probability for having a reduced relative angular momentum. Except for the sign of the quasiparticles the situation is more similar to that in which the condensate is formed in the holes of a parent with p_i reduced by two. This suggests that when $Q_i[z]$ is determined by Eq. (4b)

$$g_i(r)\simeq g_L^{p_i}(r)-\delta g_{i-1}^{p_i-2}(r) \ ; \tag{11}$$

i.e., the change in the pdf when a given condensate occurs in the quasielectrons of $P[z]=(P_V[z])^{p_i+1}$ is the same

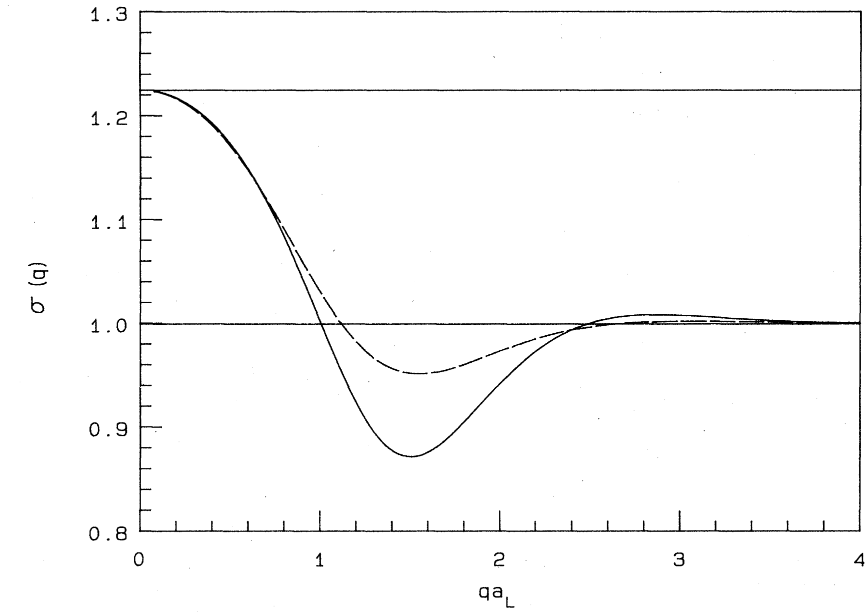

FIG. 1. Fourier transform of the charge distribution which yields the effective potential defined by the hierarchy state which occurs at $\nu=\frac{2}{3}$ $\{Q_3^1[z]=C(P_V^2[z])\}$. The dashed lines shows the result when bridge corrections are dropped.

(apart from sign) as the change which occurs when the same condensate occurs in the quasiholes of $P[z]=(P_V[z])^{p_l-1}$. With this approximation Eq. (10), which is associated with the longer-range correlations, is satisfied. In addition the property[16] that $e^{r^2/4}g(r)$ has no odd terms in a power series expansion in r^2, which severely restricts the form of $g(r)$ at small r, is preserved. An additional piece of evidence for the accuracy of Eq. (11) comes from the numerical calculations of Yoshioka for small systems (see Ref. 17). For $\nu=\frac{2}{5}\{Q[z]=C(P_V^2[z])^+P_V^2[z]\}$ Eq. (11) combined with Eq. (7) yields

$$g_{2/5}(r)\simeq\frac{3}{4}g_{1/3}(r)+\frac{1}{4}(1-e^{-r^2/2}) \ , \tag{12}$$

which, since $g_{1/3}(r)\propto r^6$ at small r, this gives the coefficients of r^2 and $(r^2)^2$ in the expansion of $g_{2/5}(r)$ as $c_1=\frac{1}{8}$ and $c_2=\frac{1}{32}$ which agrees with his results to within their accuracy.

Using the procedures outlined above we have determined pair-correlation functions for many hierarchy states. In Fig. 2 we compare the pair correlation function of the $\nu=\frac{2}{7}$

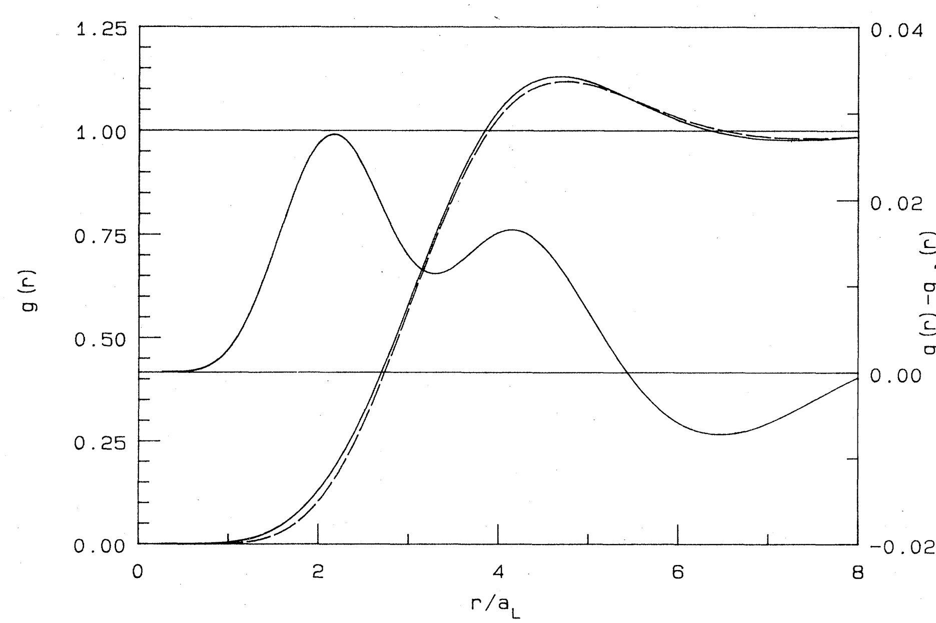

FIG. 2. Comparison of the pair-correlation function $g(r)$ for the hierarchy state which occurs at $\nu=\frac{2}{7}$ $(Q_4^1[z]=P_V^2[z]Q_3^1[z])$ (solid line) and the corresponding plasma pair-correlation function, $g_{\rm pl}(r)$ (dashed line).

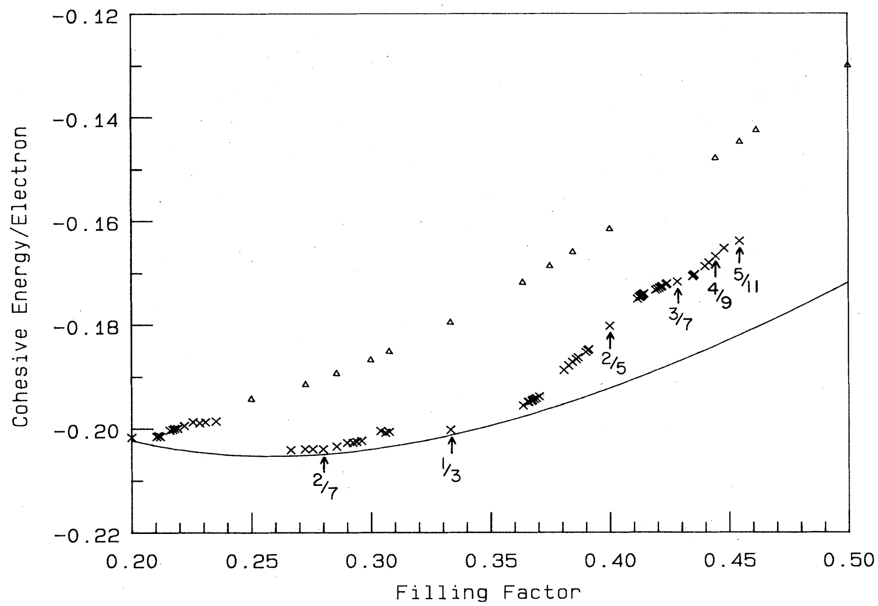

FIG. 3. Cohesive energy per electron $(E/N-\nu\sqrt{\pi/8})$. The triangles are for hexagonal lattice CDW states calculated in the Hartree-Fock approximation (Ref. 18), the crosses were obtained from Eq. (15) using hierarchy state pair-correlation functions, and the solid line is the fit to the reference plasma energy given in Ref. 19.

hierarchy state $\{Q[z]=C(P_{\bar{V}}^2[z])P_{\bar{V}}^2[z]\}$ with that of the corresponding plasma (i.e., point "fictional" charge). The scale of the filling-factor-dependent anomalies, which are responsible for the fractional quantum Hall effect, is that of the differences between these curves. For a e^2/r electron-electron interaction the energy per electron is

$$\frac{E_l}{N}=\frac{e^2}{a_L}\int dr\,[g_l(r)-1] \quad . \tag{13}$$

In Fig. 3 we compare the energies for all the hierarchy states we have calculated with the corresponding plasma energies and with the energies of charge-density-wave (CDW) states.[18] For ν^{-1} odd the difference between the plasma energies and the energies calculated for the hierarchy states are a measure of the accuracy of the modified HNC calculations. For example, the energy per electron we obtain at $\nu^{-1}=3$ in units of e^2/a_L is -0.4092 compared with the more accurate value -0.4100 ± 0.0001 obtained from Monte Carlo calculations.[19] (When bridge corrections are neglected the HNC calculation gives -0.4055.) Note that the hierarchy states remain lower in energy than the CDW states for all values of ν in the range illustrated. The main features of Fig. 3 are consistent with the qualitative discussion of Halperin[8] based on considerations related to the quasiparticles of the Laughlin states. On a more quantitative level, however, there are differences. For example, we find the difference between the hierarchy state energy and the reference plasma energy tends to be larger when a condensate occurs in the quasielectrons rather than the quasiholes. A consequence of this in Fig. 3 is that the anomalies in filling factor dependence of the hierarchy state energy are stronger for the sequence $\nu=\frac{2}{5},\frac{3}{7},\frac{4}{9},\ldots$ than for $\nu=\frac{2}{7}$. The appearance of this same sequence of fractions in analyzing the experimental results has recently been emphasized by Chang *et al.*[3]

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