Resistance fluctuation in a one-dimensional conductor with static disorder

N. Kumar*
Department of Physics and Atmospheric Sciences, Drexel University, Philadelphia, Pennsylvania 19104
(Received 14 September 1984)

The non-self-averaging resistance of a one-dimensional conductor with static disorder is reexamined by the method of invariant imbedding, leading to a Fokker-Planck equation for its probability distribution $W_\rho(\rho,l)$, with varying sample length l. An exact two-point recursion relation for the moments $\langle\rho^n\rangle$ is given along with a closed-form solution for $W_\rho(\rho,l)$ for the case of Gaussian white-noise disorder. The latter confirms $\ln\rho$ as the correct scale variable. The treatment admits generalization to the case of N channels and to general disorder.

The indefiniteness of electrical resistance of a strictly one-dimensional (1D) metal with static disorder has been a subject of much discussion in the recent past.[1–5] It is now generally accepted that the zero-temperature dc resistance of such a random system of noninteracting electrons is a nonadditive as well as non-self-averaging quantity in that the different sections of the macroscopic sample, however long, may not mimic different instances of the sample in the usual ensemble sense. This is due ultimately to the coherent interference effects of scattering from the serial static disorder of the 1D system. The statistical fluctuation of random resistance grows faster than the average value, thus violating the "central limit." This manifests itself in the nonequivalence of different modes of averaging and in the essential sensitivity to the order of certain limits as seen in numerical simulations and analytical calculations.[6] This apparently ill-posed problem was treated properly by Anderson, Thouless, Abrahams, and Fisher[2] (ATAF) employing a Landauer-type[7] scale transformation. They showed $\ln\rho$ to be the correct scaling variable of physical significance for large sample lengths L (measured in terms of localization length L_c). Also, the averaged resistance $\langle\rho\rangle$ grew exponentially as found earlier by Landauer.[7] Moreover in the short-sample limit $L \ll L_c$ one recovered the linear length dependence for the average resistance $\langle\rho\rangle$. In a mathematically rigorous but somewhat hard to comprehend set of papers, Abrikosov and Rhyzkin[3] (AR) and Abrikosov[4] calculated the full probability distribution of resistance $W_\rho(\rho,l)$ that generally reaffirmed the ATAF result that $\ln\rho$ obeys the central limit, but did not agree with the ATAF scaling result in the limit $l=L/L_c \ll 1$. The only explicit approximation made in their treatment was the Born approximation for backscattering. Part of the motivation for our work comes from this disagreement. In this Brief Report we present a novel treatment of this transport problem based on "invariant imbedding"[8,9] that seems natural to any transport problem with serial randomness. Our results for $W_\rho(\rho,l)$ agree qualitatively with those of ATAF and AR in the asymptotic limit $l \gg 1$ but reproduce the ATAF result for $l \ll 1$. Indeed, we obtain an exact Fokker-Planck equation for $W_\rho(\rho,l)$ that is remarkably close to that of AR obtained otherwise, except for the location of a regular singular point. The latter causes the discrepancy for small lengths. We also obtain an exact two-point recursion relation for the moments $\langle\rho^n\rangle$.

Our starting point is the well-known Landauer[7] expression for the resistance (L) given by

$$\rho(L)=\frac{R^*(L)R(L)}{1-R^*(L)R(L)}\ , \tag{1}$$

where $\rho(L)$ is measured in natural units (of $\hbar/e^2$) and $R(L)$ is the complex amplitude reflection coefficient for the one-dimensional conductor of length L. The one-electron motion is described by the Schrödinger equation

$$\frac{\partial^2\psi}{\partial x^2}+k^2(x)\psi=0,\quad k^2(x)=\frac{2m}{\hbar^2}[E-V(x)] \tag{2a}$$

conditioned by matching to the scattering states at the boundaries $x=0,L$. Here $k(x)$ is the local wave number, assumed to be random inasmuch as the potential $V(x)$ is. L is the sample length. In the invariant imbedding[8,9] approach, however, one addresses directly the "emergent" quantity, namely, the complex amplitude reflection coefficient $R(L)$. The latter is known to obey the Riccati equation[9]

$$\frac{\partial R(L)}{\partial L}=f_1(L)+2if_0(L)R(L)-f_1(L)R^2(L)\ ,$$

with

$$f_1(x)=2\left(\frac{\partial k}{\partial x}\right)/k(x)\ \text{ and } f_0(x)=k(x)\ . \tag{2b}$$

Now, we will consider for simplicity the situation where $k^2(x)>0$. In the description of wave propagation in random media this implies that the "refractive" index is random but stays real. Thus any exponential spatial attenuation (localization) is due entirely to interference of random phases and not due to an imaginary "refractive" index that would correspond to barrier penetration. This would be the case for a 1D disordered metal with the electron Fermi energy of interest exceeding the random potential fluctuations. It is then convenient to write $R=R_1+iR_2$, with R_1,R_2 real, and separate Eq. (2) for the complex R as

$$\frac{\partial R_1(L)}{\partial L}=f_1(L)-2f_0(L)R_2(L)-f_1(L)(R_1^2-R_2^2)\ , \tag{3}$$

$$\frac{\partial R_2(L)}{\partial L}=2f_0(L)R_1(L)-2f_1(L)R_1(L)R_2(L)\ . \tag{4}$$

These coupled stochastic differential equations are difficult to treat. What we need, however, is the probability density $W_R(R_1,R_2;L)$ for which we now derive a Fokker-Planck equation. To this end we introduce a spread of "phase points" of density $Q(R_1,R_2,L)$ in the (R_1,R_2) "phase space" evolving in L according to Eqs. (3) and (4), subject to the definite, "initial" condition that $R(L)=0$ at $L=0$. For $R(L)=0$ the angle $\theta=\tan^{-1}(R_2/R_1)$ is undefined and may be taken to correspond to a circular ensemble uniform in θ space. The phase fluid will now evolve according to the stochastic Liouville equation[10]

$$\frac{\partial Q}{\partial L}=-\frac{\partial}{\partial R_1}\left[Q\frac{\partial R_1}{\partial L}\right]-\frac{\partial}{\partial R_2}\left[Q\frac{\partial R_2}{\partial L}\right] , \quad (5)$$

where the "velocities" dR/dL and dR_2/dL are given by Eqs. (3) and (4). We have then the well-known result that[10]

$$W_R(R_1,R_2;L)=\langle Q(R_1,R_2;L)\rangle_f , \quad (6)$$

where $\langle\ \rangle_f$ denotes averaging with respect to the basic random variables f_0 and f_1. As our interest lies in resistance, or equivalently in the reflection coefficient $r=R_1^2+R_2^2$, it is convenient to introduce the angular (θ) average

$$W_r(r,L)=\pi\langle W_R(R_1,R_2;L)\rangle_\theta , \quad (7a)$$

with normalization

$$\int_0^1 W_r(r,L)dr=1 . \quad (7b)$$

Assumption of circular ensemble makes $\langle Q(R_1,R_2;L)\rangle_f$ a function of $r\equiv R_1^2+R_2^2$ alone. This enables us to perform the θ average in the following. From Eqs. (5)–(7), we get

$$\frac{\partial W_r}{\partial L}=-\pi\left\langle\frac{\partial}{\partial R_1}\langle f_1Q\rangle_f\right\rangle_\theta+2\pi\left\langle R_1R_2\frac{\partial}{\partial R_2}\langle f_1Q\rangle_f\right\rangle_\theta +4\pi\langle R_1\langle f_1Q\rangle_f\rangle_\theta+\pi\left\langle(R_1^2-R_2^2)\frac{\partial}{\partial R_1}\langle f_1Q\rangle_f\right\rangle_\theta , \quad (8)$$

where terms involving f_0 cancel out and only f_1, essentially the potential gradient (force), persists. Now, we must evaluate $\langle f_1Q\rangle_f$ occurring on the right-hand side (RHS) of Eq. (8). For this purpose we take the derivative f_1 as our basic random variable and approximate it mathematically as Gaussian and δ correlated in space, i.e.,

$$\langle f_1(L)f_1(L')\rangle=\frac{1}{\xi}\delta(L-L') . \quad (9)$$

Here $1/\xi$ measures the strength of scattering and essentially $\xi\simeq L_c$. In point of fact even for a realistic bounded random potential, the derivative can fluctuate very violently, even become unbounded and the above approximation makes sense as a model. This enables us to close the hierarchy of equations for the averages in virtue of the known Novikov[11] identity for the functional Q of a Gaussian random variable f_1, namely, that

$$\langle f_1(L')Q\rangle_f=\int_0^L\langle f_1(L')f_1(L'')\rangle_f\left\langle\frac{\delta Q}{\delta f(L'')}\right\rangle_f dL'' , \quad (10)$$

with $0\le L'\le L$. Here $\langle\delta Q/\delta f_1\rangle_f$ can be obtained readily from Eq. (5). Thus, closely following an earlier procedure[12] we obtain, after some tedious but straightforward algebra, the evolution equation

$$\frac{\partial W_r}{\partial l}=r(1-r)^2\frac{\partial^2 W_r}{\partial r^2}+(1-r)(1-5r)\frac{\partial W_r}{\partial r} +2(2r-1)W_r , \quad (11)$$

with $l=L/\xi$. Here the "initial" condition is

$$W_r(r,l)\to\delta(r) \text{ as } l\to 0 .$$

The non-self-adjoint equation (11) can be solved formally as a biorthogonal series in terms of the eigenfunctions of the associated hypergeometric equation whose domain of physical interest is bounded by the regular singularities at $r=0$ and $r=1$. Since, however, we are interested in the resistance (ρ) fluctuation, it is apt to change over to $\rho=r/(1-r)$ and the associated probability density $W_\rho(\rho,l)$. We then have

$$\frac{\partial W_\rho}{\partial l}=\rho(1+\rho)\frac{\partial^2 W_\rho}{\partial\rho^2}+(2\rho+1)\frac{\partial W_\rho}{\partial\rho} . \quad (12)$$

This is our central equation. This self-adjoint equation has the remarkable property of yielding an exact two-point recursion relation for the resistance moments $\langle\rho^n\rangle\equiv\rho_n$. Multiplying both sides by ρ^n and integrating by parts on the RHS, we get

$$\frac{\partial\rho_n}{\partial l}=n(n+1)\rho_n+n^2\rho_{n-1} . \quad (13)$$

The boundary terms appearing on integration by parts vanish because of the conditions of normalization and of bounded moments for a given l. In particular we have

$$\rho_1(l)=\tfrac{1}{2}(e^{2l}-1) , \quad (14a)$$

$$\rho_2(l)=\tfrac{1}{12}(2e^{6l}-6e^{2l}+4) . \quad (14b)$$

Thus the variance grows faster than the mean. Indeed one can solve formally for the Laplace transform $\tilde\rho_n(s)$ by taking the transform on both sides of Eq. (13) and iterating the recursion relation. We have

$$\tilde\rho_n(s)\equiv\int_0^\infty e^{-sl}\rho_n(l)dl=\prod_{m=1}^{n}\frac{m^2}{s[s-m(m+1)]} . \quad (15)$$

It is important to note that the transform exists for Res sufficiently large and positive for a given n because the moments grow exponentially with l as $e^{\alpha_n l}$ with $\alpha_n\to\infty$ (monotonically) as $n\to\infty$. This makes the "moment problem," namely, the task of reconstituting the probability $W_\rho(\rho,l)$ from the moments ρ_n, somewhat difficult. An asymptotic closed-form solution for $W_\rho(\rho,l)$ can, however, be obtained directly as follows. We introduce new variables

$$\Omega=[\rho(1+\rho)]^{1/4}W_\rho,\quad \zeta=\frac{1}{J}\int_0^\rho\left[\frac{1}{\rho(1+\rho)}\right]^{1/2}d\rho ,$$
$$J=\frac{1}{\pi}\int_0^{\rho_0}\left[\frac{1}{\rho(1+\rho)}\right]^{1/2}d\rho , \quad (16)$$

where we will take the limit $\rho_0\to\infty$ at the end. [The upper limit ρ_0 is introduced to cut off the weak divergence of J as defined in Eq. (16).] Then for $\rho\gg1$ we get the Schrödinger-like equation

$$\frac{d^2\Omega}{d\zeta^2}-\frac{1}{4}J^2\Omega=J^2\frac{\partial\Omega}{\partial l} . \quad (17)$$

Now we must seek a solution of Eq. (17) subject to the asymptotic condition that it reproduces the correct $\langle\rho\rangle$ given by (14a) for large l (corresponding to the $\rho\gg1$ limit). This gives at once

$$W_\rho(\rho,l)=\frac{e^{-l/4}}{\sqrt{4\pi l}}\rho^{-1/2}\exp\left[-\left(\frac{1}{4l}\right)\ln\rho^2\right] . \quad (18)$$

This completes the mathematical analysis. Apart from numerical details, the expression coincides with that obtained by Abrikosov[4] and confirms that $\ln\rho$ is the meaningful scaling variable. On the other hand the present treatment gives the scaling result [Eq. (14a)] consistent with ATAF in the $l << 1$ limit.

Indeed our central equation (12) for $W_\rho(\rho,l)$ differs from that of Abrikosov in respect of the location of the singular point, viz., his is at $\rho=1$ and ours at $\rho=-1$ (outside the physical domain). Now $\rho=1$ corresponds to a reflection coefficient $r=\frac{1}{2}$ and we do not envisage anything singular about this value.

We would like to conclude with two general remarks. While the present treatment is explicitly for the Gaussian white-noise case, we get essentially the same asymptotic behavior for the general, non-Markovian randomness as can be seen by the straightforward application of the Khasminskii theorem[13] to Eq. (5), at least for weak disorder. Also, generalization of the method of invariant imbedding to the N-channel case already exists[9] and should apply here unchanged. We believe that the present approach based on invariant imbedding is physically transparent and deserves further attention in the context of quantum diffusion in disordered systems.

I would like to thank Drexel University for hospitality and financial support during the course of this work.

---

*On leave of absence from the Indian Institute of Science, Bangalore, India 560012.

[1]For a review, see P. Erdös and R. C. Herndon, Adv. Phys. **31**, 63 (1982).

[2]P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980).

[3]A. A. Abrikosov and I. A. Ryzhkin, Adv. Phys. **27**, 147 (1978).

[4]A. A. Abrikosov, Solid State Commun. **37**, 997 (1981).

[5]N. Kumar, Curr. Sci. **53**, 358 (1984). Some of the results reported here have also been obtained independently and almost simultaneously by U. Frisch and J. L. Gautero, J. Math. Phys. **25**, 1378 (1984).

[6]C. M. Soukoulsi and E. N. Economou, Solid State Commun. **37**, 409 (1984).

[7]R. Landauer, Philos. Mag. **21**, 863 (1970).

[8]S. Chandrasekhar, *Raditative Transfer* (Dover, New York, 1960), p. 161.

[9]For a readable account see R. Bellman and G. M. Wing, *An Introduction to Invariant Imbedding* (Wiley, New York, 1976).

[10]N. G. van Kampen, Phys. Rep. **24C**, 172 (1976).

[11]E. A. Novikov, Zh. Eksp. Teor. Fiz. **47**, 1919 (1964) [Sov. Phys. JETP **20**, 1290 (1965)].

[12]A. M. Jayannavar and N. Kumar, Phys. Rev. Lett. **48**, 553 (1982).

[13]R. Z. Khasminskii, Theory Probab. Its Appl. (U.S.S.R.) **11**, 390 (1966).