Exact results for the fractional quantum Hall effect with general interactions

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Laughlin's state ψ_m is shown to be the exact nondegenerate ground state for repulsive interactions of vanishing range. In this limit, his quasihole states are not exact, and some previously proposed states are ruled out. An experimental prediction is made concerning the competition of ψ_m with charge-density waves. Several exact properties are shared by systems with interactions of shorter range than r^{-2}, for which the center-of-mass motion separates from the other degrees of freedom. These include new invariant subspaces, an operator that creates exact eigenstates, and a subset property of the energy eigenvalues.

I. INTRODUCTION

Experiments on the two-dimensional electron system in GaAs-$Ga_xAl_{1-x}As$ heterojunctions at large magnetic fields have found a quantized Hall conductivity at some simple rational multiples of e^2/h.[1] It has been suggested that the mutual repulsion of the electrons results in a number of unique properties,[2,3] including a discontinuity in the chemical potential at certain fractional fillings of a Landau level.

Laughlin[4] has proposed a correlated wave function ψ_m to explain the quantization of the Hall conductivity at fractions $1/m$ and $1-1/m$ for odd m. The states ψ_m are close to the ground state for small numbers of particles N, but they are not exact.[4] Tao has suggested, however, that the states ψ_m become poor for large numbers of particles.[5] In addition to Laughlin's ψ_m, there have been a growing number of other proposals for ground states.[2,6–8] It has been unclear how accurate any of these states are, or in general how to improve them.

To clarify this situation, we present several exact results. Laughlin's states ψ_m are shown to be exact for any number of particles in the limit in which the particles have a repulsive interaction of vanishing range. Some other proposed states are ruled out in this same limit. A number of exact properties are shown to be shared by all interactions of shorter range than r^{-2} (those that do not require a compensating background), and by the log interaction.

The paper is organized as follows: Sec. II discusses exact results for general interactions that are shorter range than r^{-2}. Section III discusses interactions of vanishing range, including the exactness of ψ_m, a constraint on states at other filling factors, the competition of ψ_m with charge-density waves, a hierarchy in the ordering of gaps, an exact (but trivial) solution for attractive interactions, and Laughlin's quasihole states. Section IV lists conclusions.

We consider the model Hamiltonian

$$H=\frac{1}{2m^*}\sum_j\left[\mathbf{p}_j-\frac{e\mathbf{A}_j}{c}\right]^2+\sum_{j<k}V_2(|\mathbf{r}_j-\mathbf{r}_k|)+\sum_j V_1(|\mathbf{r}_j|), \quad (1)$$

where we have chosen the symmetric gauge $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}$. The boundary conditions are the "open" boundary conditions of a plane used in Ref. 4. There is no need for hard walls, since the system is confined to a region of space by angular momentum conservation. Since we are interested in the limit of high magnetic field, we make the usual approximation of projecting H onto the lowest (spin-polarized) Landau level. As a result, the first term in H is simply a constant. The problem is generally modeled by the two-body Coulomb interaction between electrons, $V_2(\mathbf{r})=e^2/(\epsilon r)$, with the corresponding one-body background potential V_1 due to a uniform density of positive charge. It is expected, however, that many qualitative features of the problem will remain for other choices of V_2 and V_1, so long as they describe a repulsive interaction between particles. Laughlin has in fact found similar results for repulsive Coulomb, Gaussian, and logarithmic interactions between particles.[4] In addition, interactions other than Coulomb may be physically realizable; a nearby conducting plane, for example, would result in a r^{-3} dipole-dipole interaction at large r. The silicon-doped $Al_xGa_{1-x}As$ layer, which is approximately 300 Å from the two-dimensional (2D) electron gas, conducts very poorly in the samples of Ref. 1, so that it acts as a ground plane only on long time scales. A discussion of time scales is contained in Ref. 9.

A complete set of single-particle states for the lowest Landau level is given by

$$|m\rangle=(2^{m+1}\pi m!)^{-1/2}z^m\exp(-|z|^2/4),$$

where the angular momentum m is a non-negative integer and z is the coordinate (x,y) represented as a complex

number $z=(x-iy)/a_0$. The Landau length a_0 is equal to $(\hbar c/eB)^{1/2}$. A basis for the antisymmetric N-particle states can be written in several ways:

$$\psi=|m_1,m_2,\ldots,m_N\rangle \tag{2a}$$

$$=g(z_1,\ldots,z_N)\exp\left[-\sum_j |z_j|^2/4\right] \tag{2b}$$

$$=\left[\prod_{j<k}(z_j-z_k)\right]f(z_1,\ldots,z_N)\times\exp\left[-\sum_j |z_j|^2/4\right] . \tag{2c}$$

Equation (2a) is a Slater determinant of single-particle states, g is a completely antisymmetric polynomial, and f is a completely symmetric polynomial. The form (2c) follows because the quantity in small parentheses is a factor of any antisymmetric polynomial.[10(a)]

The total angular momentum M is a good quantum number, where $M=m_1+m_2+\cdots+m_N$; M is also equal to the degree of the polynomial g. We use the conventional definition of the filling factor $\nu=N(N-1)/2M$.[4,10(b)] Since ν is the dimensionless particle density $\nu=N/(\Omega/2\pi)$, where Ω is the dimensionless volume, the definition implies that volume changes are proportional to the change in M. For fixed N, it is convenient to consider the ground-state energy of a system either as a function of volume (equivalently M), or as a function of pressure P by defining the free energy $G=E+P\Omega$. With this definition, applying pressure corresponds to adding an external potential of the form $V=\frac{1}{2}kr^2$.

Laughlin has suggested that the state ψ_m in which

$$g=\prod_{j<k}(z_j-z_k)^m$$

is a good approximation for the ground state for positive odd integers m, and that additional approximate eigenstates can be obtained by operating with a quasihole operator

$$\hat{A}=\prod_j(z_j-z_0)$$

(and an analogous quasiparticle operator) which create quasiparticles of charge $\pm 1/m$. Equation (2c) implies that the Hilbert space $\mathscr{H}(N,M)$ for N particles and M units of angular momentum is isomorphic to the space $S^{M'}$ of symmetric polynomials in N variables of degree $M'=M-N(N-1)/2$. A standard basis for the space of symmetric polynomials f in N variables can be written in terms of the elementary symmetric functions $s_1,\ldots,s_N$,[10(a)] where

$$\begin{aligned} s_1&=z_1+z_2+\cdots+z_N ,\\ s_k&=\sum_{j_1<j_2<\cdots<j_k} z_{j_1}z_{j_2}\cdots z_{j_k} ,\\ s_N&=z_1z_2\cdots z_N . \end{aligned} \tag{3}$$

It is a theorem in algebra that any completely symmetric polynomial can be written as a unique polynomial of the factors $s_1,\ldots,s_N$.[10(a)]

II. GENERAL INTERACTIONS WITH NO BACKGROUND POTENTIAL

Because angular momentum is conserved, an eigenfunction of the Hamiltonian (1) corresponds to a homogeneous polynomial f, with all terms of degree $M'=M-N(N-1)/2$. Without numerically diagonalizing the Hamiltonian, one can make essentially no further statements about which linear combinations of basis functions correspond to eigenstates for general interactions V_2 and backgrounds V_1. If, however, the background potential V_1 vanishes, additional properties of the eigenstates appear because the center-of-mass motion separates from the other degrees of freedom.

A system with long-range interactions may fail to have a thermodynamic limit unless a background potential is included. If the particles give rise to a potential $\phi(r)$, then the background potential $\phi_b(r)$ is defined as the potential that would result from a uniform density of particles each with a potential $-\phi(r)$. If $\phi(r)$ is shorter range than r^{-2} as $r\to\infty$ (and sufficiently well behaved elsewhere), the background potential will simply be a constant, except close to the edges. For these systems, it is convenient to extend the background density to infinity so that ϕ_b is everywhere a constant, which can be chosen to be zero. In the remainder of this paper, unless specifically mentioned, we will assume a system with no background potential.

The Hamiltonian can be written

$$H=\frac{1}{2M^*}\left[\mathbf{P}-\frac{e\mathbf{A}}{c}\right]^2+\tilde{H}(\tilde{z}_1,\ldots,\tilde{z}_N) . \tag{4}$$

The first term is the kinetic energy associated with the center-of-mass coordinate $R=N^{-1}\sum_j z_j$. The remaining term $\tilde{H}$ is a function of the coordinates relative to the center of mass, $\tilde{z}_j\equiv z_j-R$ (only $N-1$ of the coordinates $\tilde{z}_j$ are independent). Equation (4) implies that all eigenfunctions can be written in the form

$$\psi=R^m e^{-|R|^2/4}\tilde{\psi}(\tilde{z}_1,\ldots,\tilde{z}_N) . \tag{5}$$

The interaction V_2 operates only on the $\tilde{\psi}$ piece of ψ.

It is convenient to replace the elementary symmetric functions s_k by a (nonstandard) set t_k, where

$$\begin{aligned} t_1&=s_1=NR ,\\ t_k&=s_k\quad(z_j\to\tilde{z}_j)\quad\text{for } 2\le k\le N . \end{aligned} \tag{6}$$

The notation means that z_j is replaced by $\tilde{z}_j$ wherever it appears. One can prove that as for the s_j, any symmetric polynomial f can be written as a unique polynomial function of the factors $t_1,\ldots,t_N$.

We define $S_k^{M'}$ $(k=0,1,\ldots,M')$ to be the subspace of $S^{M'}$ spanned by all polynomial functions of $t_1,\ldots,t_N$ that contain precisely k factors of t_1. Some polynomials have the translation invariance (TI) property that they are unchanged when all of the variables are shifted, $z_j\to z_j+c$, where c is an arbitrary complex number. [Example: $(z_1-z_2)^3$ has the property, $z_1z_2(z_1-z_2)$ does not.]

One can then prove the following assertions: (a) $S_0^{M'}$ is the TI subspace. Note that although the polynomials in this subspace are translation invariant, the corresponding

wave functions are not, because they contain an additional exponential factor. (b) The dimension of $S^{M'}$ is $n(N,M')$, the number of distinct ways that N nondecreasing, nonnegative integers can be chosen that add up to M'. The dimension of $S_0^{M'}$ is $n(N,M')-n(N,M'-1)$. (c) Two vectors in different subspaces are orthogonal (the inner product is defined as the usual inner product $\langle\psi_1|\psi_2\rangle$ of the wave functions that correspond to the polynomials). (d) Each subspace $S_k^{M'}$ is left invariant by the interaction potential V_2. (e) If $|\psi\rangle$ is an eigenvector, then $R|\psi\rangle$ is also an eigenvector with the same eigenvalue.

These statements (with minor additions) imply several exact properties of the eigenstates. One has the *subset property* that the list of energy eigenvalues for angular momentum M contains the eigenvalue of any state with lower angular momentum. The subset property arises because one can define many of the eigenstates with angular momentum $M+1$ inductively from those with angular momentum M. Let $|j,k,M\rangle$ be an eigenvector from subspace k of $\mathscr{H}(N,M)$ with eigenvalue $\epsilon(j,k,M)$. Every eigenstate with angular momentum M can be mapped onto an eigenstate with angular momentum $M+1$ by multiplying by R: $R|j,k,M\rangle\sim|j,k+1,M+1\rangle$. (Multiplication by R results in a state with a smaller filling factor.) The new state is in the $k+1$ subspace, and has the same eigenvalue, $\epsilon(j,k+1,M+1)=\epsilon(j,k,M)$. The only states with angular momentum $M+1$ that cannot be constructed inductively from those with angular momentum M are those with $k=0$ (those vectors in the TI subspace). See Fig. 1. It has not been possible to determine *a priori* which vectors in the TI subspace are eigenvectors without resorting to numerical diagonalization. When this is done, the eigenvectors and eigenvalues are observed to change as one changes the interaction potential V_2.

Because one need diagonalize only the TI subspace rather than the entire (N,M) Hilbert space, the range of feasible numerical study is expanded. (We have not exploited this property in the limited numerics presented here.) The invariance property of the TI subspace also constrains the choice of physically relevant operators (Laughlin's quasihole operator $\hat{A}$, for example, does not respect this invariance).

Multiplication of a state by R^n increases its volume by n units, but does not change any of its internal properties, including any k-point function that gives the probability of finding k particles in given positions relative to each other. A state that includes a factor R^n acts somewhat like a marble rattling around in a box that is too large for it.

If the external potential is nonzero, most of the properties discussed above do not hold. One exception is the quadratic external potential $V_1=(k/2)\sum_j|z_j|^2$, which simply adds a constant to the Hamiltonian in a given (N,M) sector. The same results are obtained as in the $V_1=0$ case, except that $k(M+N)$ must be subtracted from all eigenvalues before the subset property holds. Note that if the particles have log interactions, the potential arising from a uniform background charge is precisely quadratic. Particles with a $1/r$ interaction have a background potential that is nearly quadratic away from edges.

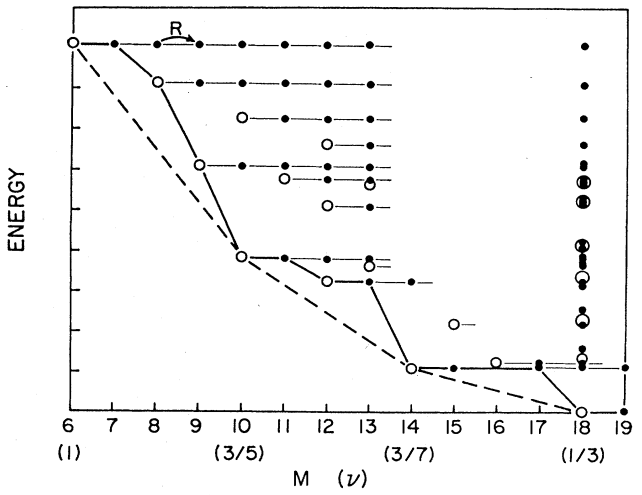

FIG. 1. Numerically obtained eigenvalues are plotted above the angular momentum M for four particles; for clarity, only the ground and first excited states are included for $M=14$ through 17. For this figure and for Table I, particles interact via a repulsive interaction V_2 of vanishing range, $V_2(r)=\nabla^2\delta^2(\mathbf{r})$; there is no background potential V_1. Open circles indicate TI states and closed circles non-TI states. A solid line connects the ground state for every M, and a dashed line connects those ground states that are stable under pressure. The following properties of this figure are shared by all systems with $V_1=0$: (1) Non-TI ground states have zero bulk modulus and cannot be stable under pressure. (2) The eigenvalues for angular momentum M are a subset of those for any $M'>M$. (3) Multiplying an eigenstate by the center-of-mass operator R results in an exact eigenstate with the same eigenvalue and one more unit of angular momentum. All eigenstates that cannot be created by applying R to a state of lower angular momentum are in the TI subspace.

III. SHORT-RANGE INTERACTIONS

Additional properties emerge in the limit that the range of the interaction potential goes to zero. It is convenient to expand a short-range potential $V_2(|\mathbf{r}|)$ in powers of its range b,

$$V_2(\mathbf{r})=\sum_{j=0} c_j b^{2j}\nabla^{2j}\delta^2(\mathbf{r})\ . \tag{7}$$

V_2 is assumed repulsive (non-negative), so that $\langle V_2\rangle\geq 0$. As the range b of the interaction goes to zero, only the leading nonvanishing term in Eq. (7) contributes to the energy. Note that $\langle c_0\delta^2(\mathbf{r})\rangle$ vanishes identically in any (spin-polarized) antisymmetric wave function, so that the pure delta-function term never contributes. All matrix elements of V_2 are $O(b^2)$ or higher. The $O(b^2)$ term, $\nabla^2\delta^2(r)$, is the interaction used in Fig. 1. For m odd and $\nu\leq 1/m$, the energy of the exact ground state vanishes to order b^{2m-2}. This can be seen by constructing a trial function $\psi=\psi_m f(z_1,\ldots,z_N)$, where f is a symmetric polynomial. (Note that for $m>1$, most wave functions cannot be written in the form $\psi_m f$.) Integration by parts gives zero for $\langle V_2\rangle$ unless $2m$ factors of ∇ operate on $\psi_m^*\psi_m$. For $\nu=1/m$, the only state that can be written as $\psi=\psi_m f$, and hence the only state for which $\langle V_2\rangle$ van-

ishes to order b^{2m-2}, is $\psi=\psi_m$. ψ_m then has lower energy as $b\to 0$ [$O(b^{2m})$] than any other state with the same $M=N(N-1)/2\nu$. As a consequence, for all N, Laughlin's state ψ_m is the exact nondegenerate ground state in the limit that the potential range goes to zero.[11] We emphasize that for short-range interactions and open boundary conditions, the exact ground state is *nondegenerate* once the trivial degeneracy that corresponds to moving the center of mass is factored out. With periodic boundary conditions, the ground state for $\nu=1/m$ has been shown to be m-fold degenerate;[12] this degeneracy can, however, be interpreted as a displacement of the center of mass. It also follows that in the limit $b\to 0$, the ground state for $\nu\leq 1/m$ must contain the factor ψ_m. This places a nontrivial constraint on the ground state for $\nu\leq\frac{1}{3}$. For example, for $\nu<\frac{1}{3}$, the new states proposed by Hu,[8] the unperturbed state of Tao and Thouless,[6] and some (but not all) of those proposed by Halperin[2] have the property that their energy divided by the correct ground-state energy becomes infinite as $b\to 0$.

Since the ground-state energy is $O(b^{2m-4})$ as $\nu\to(1/m)^+$ and $O(b^{2m})$ as $\nu\to(1/m)^-$, the chemical potential is discontinuous at $\nu=1/m$ for sufficiently small b. We believe, but have not yet proven, that for large N this discontinuity is of order N^0 (does not vanish as $N\to\infty$). This belief is supported by an analytical argument and by numerical data. Using the $N\to\infty$ particle-hole identity $E(1-\nu)=E(\nu)+(1-2\nu)E(1)$, a discontinuity at $\nu=1/m$ implies one at $\nu=1-1/m$. Table I lists filling factors that are stable under pressure (those at which the chemical potential is discontinuous) for small systems with short-range interactions. Because ψ_m is the only state at filling factor $1/m$ with an energy of order b^{2m}, it also follows that there is a gap in the neutral excitation spectrum (excitations at fixed M) at $\nu=1/m$.

At a filling factor $\nu=1/m$, the Hilbert space has grown just large enough to contain a single state (ψ_m) for which the leading term in the short distance two-point function is proportional to r^{2m}. For $\nu>1/m$, the two-point function is qualitatively different and must have a leading term of order r^{2m-4} or less. An interaction of vanishing range is sensitive only to the $r\to 0$ behavior of the two-point function, and renders the state ψ_m exact.

There may be filling factors other than those at $1/m$ (and $1-1/m$) at which μ is discontinuous (there is a gap). For small b, however, the filling factors $1/m$ are special and separate the spectrum into a hierarchy. As ν decreases, the discontinuity at $1/m$ is the last one of strength b^{2m-4}; those that follow are all weaker, of order b^{2m} or higher.

The exact states ψ_m provide a rigorous starting point from which to do perturbation theory in the range of the interaction. As $\nu\to 0$ for interactions of vanishing range, the states ψ_m remain exact ground states and are never preempted by a charge-density wave (CDW). This is in contrast to the long-range $1/r$ interaction for which a CDW has a lower energy when $\nu<\nu_c$; Lam and Givin[13] estimate $\nu_c\approx\frac{1}{7}$.[14,15] We propose the following picture: At $\nu=1/m$, as the range of the interaction increases, the ground state ψ evolves continuously from ψ_m, maintaining a gap. This behavior persists until there is a transition to a CDW, which occurs first for the smallest filling factors. By performing experiments on different time scales[9] or on samples with and without nearby conducting planes, it may be possible to test the prediction that a CDW first appears at lower filling factors for a shorter-range (r^{-3}) interaction than for a longer-range (r^{-1}) interaction. To substantially stabilize the states ψ_m, a conducting plane should not be much farther from the 2D electrons gas than the order of the electron-electron separation.

TABLE I. For systems of N particles, numerically obtained filling factors $\nu=N(N-1)/2M$ are listed which are stable at some pressure. For $N\geq 6$, some filling factors listed may be rendered metastable by states at lower density than the smallest investigated. Note that filling factors with even denominators are obtained, and that some filling factors are impossible for integer M (such as $\nu=\frac{2}{3}$ for $N=6$).

N	ν					
3	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	
4	1	$\frac{3}{5}$	$\frac{3}{7}$	$\frac{1}{3}$		
5	1	$\frac{2}{3}$	$\frac{5}{9}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$
6	1	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{1}{2}$		
7	1	$\frac{7}{9}$	$\frac{7}{11}$			
8	1	$\frac{14}{17}$	$\frac{7}{10}$	$\frac{2}{3}$		
10	1	$\frac{45}{52}$				

One might hope that a short-range repulsive interaction would result in a ground state with a simple analytical form at general filling factors, just as it does at $\nu=1/m$ and $\nu=1-1/m$. We have been unable to demonstrate that this is true. One can show, however, that multiplying ψ_m by $f_1=S(z_1)$, $f_2=S(z_1-z_2)^2$, and $f_3=S(2z_1-z_2-z_3)^3$, where S is the symmetrization operator, results in exact ground states with 1, 2, or 3 units of angular momentum greater than ψ_m. This applies for all repulsive interactions with $V_1=0$. It is also true that if the force between particles is *attractive,* the exact ground state at any M is $R^n\psi_1$, with the energy independent of n and $n=M-N(N-1)/2$. This statement is illustrated by inverting the y axis in Fig. 1. With the exception of f_1, f_2, and f_3 above, it is not generally true, even for an interaction of vanishing range, that if $f\psi_{m_1}$ is an eigenstate, then $f\psi_{m_2}$ will also be an eigenstate when $m_2\neq m_1$.

We note that if $V_1=0$, the Hamiltonian [Eq. (1)] implies a strictly linear Hall conductivity $\sigma_{xy}=\nu e^2/h$ for all ν. There are no plateaus, even at those ν for which μ is discontinuous. Also, σ_{xx} is zero for all ν. This follows because in a reference frame moving at speed $\mathbf{u}=c(\mathbf{E}_H\times\mathbf{B})/B^2$, the Hall field $\mathbf{E}_H$ is transformed away, and the exact operator identity for the center-of-mass velocity holds,

$$\frac{d}{dt}\mathbf{v}_{\text{c.m.}}=(e\mathbf{B}/m^*c)\times\mathbf{v}_{\text{c.m.}}$$

(there is no restriction to the lowest Landau level). An external pinning potential must be present for plateaus to appear in σ_{xy}. Similar observations were made by Halpe-

rin in Ref. 2.

Finally, we consider Laughlin's quasihole operator $\hat{A}$. $\hat{A}$ creates a superposition of states with the angular momentum M increased by from 0 to N units. A particular angular momentum component of $\hat{A}\psi_m$ cannot be an exact eigenstate, even as $b\to 0$, because it is a mixture of TI and non-TI vectors. The state $\tilde{A}\psi_m$ [where $\tilde{A}\equiv\prod_j(\tilde{z}_j-z_0)$] is a pure TI state with improved overlap against the ground state for small N, but it is still not exact as $b\to 0$.

Although Laughlin's quasihole state $\hat{A}\psi_m$ is not exact, it is a good approximate ground state for small numbers of particles with short-range interactions. For $N=3$, 4, and 5, a quasihole centered at the origin has a good overlap ($\eta\approx 0.9$) against the exact ground state with $\Delta M=N$ units of angular momentum more than the state ψ_1. (The quasihole state has a definite angular momentum if it is at the origin.) The ground state in the sector $\Delta M=N$ is stable under pressure. However, for $N=6$ and 7 the ground state in that sector is only metastable under pressure. This means that as the pressure is reduced, the system jumps past the one-quasihole sector without sticking there. For $N=8$ and 9, the one-quasihole sector is locally unstable under pressure. Finally, for $N=10$, 11, and 12 ($N=12$ is the largest system we simulated), a level crossing occurs so that Laughlin's quasihole state is very nearly an excited state, rather than the ground state in its sector. (The overlap with the ground state is $\eta=0.91$ for $N=9$ and $\eta=0.005$ for $N=10$.) A plot of the single-particle density of states suggests that for $N\geq 10$, the system may be phase separating into a $\nu=1$ phase near the center and a $\nu<1$ phase near the edge, although the system is too small to make a definite statement. A phenomenological theory that explains the anomolous quantum Hall effect in terms of phase separation rather than quasiparticles is contained in Ref. 9.

IV. CONCLUSION

We have shown that for repulsive interactions of vanishing range, Laughlin's state ψ_m is the exact, nondegenerate ground state at filling factor $\nu=1/m$. For filling factors less than $1/m$, the exact ground state ψ must contain ψ_m as a factor. This places a nontrivial constraint on ψ for filling factors less than $\frac{1}{3}$, and rules out some proposed ground states. In contrast to ψ_m, Laughlin's quasihole states are not exact for interactions of vanishing range.

Although a Coulomb interaction is thought to result in a charge-density-wave ground state for filling factors smaller than ν_c (with one estimate of ν_c the order of $\frac{1}{7}$), short-range interactions result in a ground state that is ψ_m for arbitrarily small filling factors. A nearby conducting plane should stabilize ψ_m compared to charge density waves, reducing ν_c.

Several exact properties apply to systems with a vanishing background potential, which corresponds to interactions of shorter range than $1/r^2$ if there is to be a thermodynamic limit. Such systems should be physically realizable. These properties include a set of invariant subspaces, and the center-of-mass operator R that creates exact eigenstates when it operates on eigenstates. The energy eigenvalues are arranged in subsets, so that any eigenvalue for a particular filling factor is an eigenvalue for any lower filling factor.

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[14]S. T. Chui has recently proposed a variational state with CDW order that is estimated to have a lower energy than ψ_m even at $\nu=\frac{1}{3}$.

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