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Critical dynamics of the kinetic Ising model on fractal geometries

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The critical dynamics of the kinetic Glauber-Ising model on different fractal geometries is studied. The classes of fractals which are examined are the nonbranching Koch curves, the branching Koch curves, and the two-dimensional Sierpinski gasket. The critical dynamic exponent is calculated for these models using an exact renormalization-group transformation. The value $z=2.58$ for the two-dimensional Sierpinski gasket agrees with recent results from experiments performed in a percolating system.

In this paper we present a study of the critical relaxation in a dissipative spin system which has a fractal lattice structure.[1] The fractal lattices (fractals) have a nonintegral dimensionality and a self-similar geometry. The idea of critical behavior in a nonintegral dimensionality is not a new one. The dimensionality of a system which is close to the critical point plays a crucial role in determining its behavior.[2] The appearance of upper and lower critical dimensionalities for systems which have similar Hamiltonians has attracted people to perform formal expansions around these dimensions.[3]

The fractal lattices, being well determined geometrical objects, introduce a new concept into the study of critical phenomena. Unlike usual lattices, fractals do not have translational invariance. Thus, it is difficult to have a straightforward equivalency between the critical behavior on fractals and on lattices having dimensionality which was obtained by standard analytic continuation. The existence of scale invariance without translational invariance in fractals results in an enriched critical behavior. It was found by Gefen *et al.*[4-6] that for a given symmetry of the order parameter and a range of the interactions, the dimensionality of the fractal is not the only property of the fractal determining the universality class of the critical behavior of the system. They found that in addition to the fractal dimensionality, D, the other topological characteristics of the lattice,[1] such as the topological dimensionality D_T, the order of ramification R, the connectivity, and the lacunarity,[1,4] are important in the determination of the critical behavior.

The self-similarity of these fractals can be used in order to analyze their critical behavior using renormalization-group (RG) methods.[7] Gefen *et al.*[4-6] show that the Ising spin system located on a fractal with a finite minimum order of ramification $R_{\min}$ has a $T_c=0$. However, the same T_c, and even the same dimensionality for some of the fractals, does not imply that the Ising models on these fractals would belong to the same universality class. The study of the static magnetic behavior in these fractals[4,5] reveals different values of ν (the exponent which describes the correlation length). For example, ν has the values $1/D$, $\ln 3/\ln 2$, and ∞ in the Ising systems on nonbranching Koch curves, branching Koch curves, and Sierpinski gasket, respectively.

Fractals are not merely an abstract geometrical construction. A wide range of physical systems have a self-similar structure and can be described using an effective fractal dimensionality.[1] In particular we would like to mention magnetic systems on percolating clusters,[6,8] to which we shall refer in the following. Fractals can be used as a reasonable model for real systems, and conclusions from the study of fractals can be applied to real systems.

Several months ago, Aeppli *et al.*[9] reported on a study of relaxation in a two-dimensional (2D) diluted antiferromagnet near the percolation threshold. They found that the dynamic exponent, z, that measures the dependence of the relaxation time scale, τ, on the correlation length $\xi, \tau=\xi^z$, has a value larger than the usual one for nonconserving relaxation. The critical dynamics of a dissipative system is assumed to be described using the Glauber model.[10] This model is exactly solvable in 1D, and gives for the dynamic exponent the value $z=2$. In 2D, a few approximations based on high-temperature expansion,[11] Monte Carlo renormalization-group methods,[12] and real-space time-dependent renormalization group[13,14] (TDRG) indicate that z in 2D has the value $z\approx 2.18$. On the other hand the conventional theory suggests for 2D the value $z=2-\eta=1.75$.

None of these values can serve as a starting value for extrapolation to the value $z=2.4^{+0.2}_{-0.1}$ which was found[9] to describe the time scale of the 2D Ising model on percolating lattices near the percolation threshold. (In 2D the fractal dimensionality of the backbone of the incipient infinite cluster is [8,6] $D\sim 1.5-1.6$.) Another naive attempt can be made by assuming that the system is basically a 1D one, and is thus described by $z=2$. However, this value describes the dependence on the correlation length measured along the system bonds. The measuring in the geometrical distance introduces a factor of $1/D$, which changes the value to $z=2$D. Substitution of the value $D\sim 1.55$ in the above relation gives an overestimate of z.

The above value $z=2D$ describes the critical dynamics of the kinetic Ising model on the nonbranching quasilinear Koch curves, but it does not fit other, more complex fractals. Simple arguments, as were used for the nonbranching Koch curves, cannot be applied trivially to other fractals. Thus, a more careful choice of the fractal geometry should be made. Here we use the Sierpinski gasket to model a system near the percolating critical concentration. This fractal and the backbone of the percolating system both have the same fractal dimensionality and have a finite maximum ramification and the same ordinary dimensionality.[8,6] The degree of similarity in the critical behavior of these two systems is an open question. Hence any physical quantity that can be calculated theoretically and can be measured experimentally has a large impact on this comparison.

We used the self-similarity of the fractals to perform a real-space TDRG (Refs. 13 and 14) study of some of them. Our study reveals different universality classes for the critical slowing down of the kinetic Ising model on these fractals. We found by an exact RG transformation that z has the following values: $z=2D$ in the nonbranching Koch curves, $z=D+1/\nu$ in the branching Koch curves (ν is the critical exponent of the correlation length), and $z=1+D$ in the 2D Sierpinski gaskets.

In the Sierpinski gasket, which is the relevant one for the experiment,[9] $z=2.58$. This value falls within the error margins of the experimental value: $z=2.4^{+0.2}_{-0.1}$. In spite of the good agreement between these z values, both of which are greater than for the regular lattice, caution must be exercised in concluding that Ising dynamics on a gasket is a good model for the experiment in view of the poor description of Ising statics on a 2D percolating lattice furnished by the gasket. The examination of our calculations shows that this high value is due to the $T_c=0$ quasi-first-order transition that the Sierpinski gasket exhibits. At such a transition the magnetization M is scaled as $M'=b^D \cdot M$ under a scaling of length by a factor of b. The scale factor b^D is larger than the usual one for a second-order phase transition, and it contributes the D in the relation, $z=1+D$.

In the rest of the paper we shall briefly describe the kinetic Ising model on the Sierpinski gasket, and the real-space TDRG transformation which was applied to it. The study of the other fractals which has been mentioned above is performed in a similar way and will not be discussed here. The reader is referred elsewhere[14] for a detailed description of the TDRG.

The kinetic Ising model is a generalization of the Glauber model to $D\neq 1$. It describes the time-dependent behavior of a large interacting spin system whose equilibrium is determined by the Hamiltonian

$$\overline{H}=K\sum_{(i,j)}\sigma_i\sigma_j \ . \qquad (1)$$

The spins $\{\sigma_i=\pm 1\}$ are located on the junctions of the Sierpinski gasket (see Fig. 1), and the sum is taken over (i,j) nearest neighbors. The system is brought into a state of constrained equilibrium. Then at time $t=0$ the constraint is removed, and the system relaxes towards the final equilibrium via an interaction with a heat bath. The heat bath is characterized by a bare time scale, τ_0. Only one spin is allowed to flip each time, with a transition probability rate, $W_i(\{\sigma\})$. This procedure can be described by an empirical master equation,

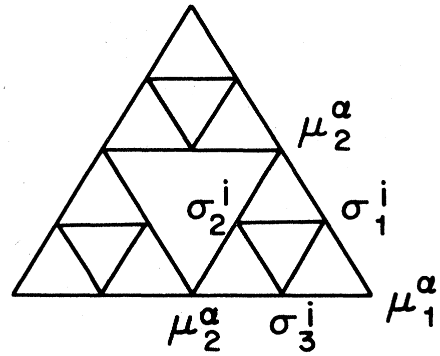

FIG. 1. Three stages of iterations in the Sierpinski gasket. In the RG transformation one traces over the spins which were created in the last stage in the midpoints of the edges of the previous smallest scale triangles (i.e., $\sigma_1^i,\sigma_2^i,\sigma_3^i$), leaving a new elementary cell (i.e., with the spins $\mu_1^\alpha,\mu_2^\alpha,\mu_3^\alpha$).

$$\tau_0\left[\frac{d}{dt}\right]P(\{\sigma\};t)=-\sum_i^N[(1+p_i)W_i(\{\sigma\}P(\{\sigma\};t))]$$
$$\equiv-\sum_i^N\tilde{L}_iP(\{\sigma\};t)$$
$$\equiv-\sum_i^N L_i\Phi(\mathbf{h},\{\sigma\};t)\ , \qquad (2)$$

where p_i is a spin flip operator, $p_if(\{\sigma_{j\neq i}\},\sigma_i)=f(\{\sigma_{j\neq i}\},-\sigma_i)$, and Φ is the perturbation from equilibrium, $\Phi\equiv P(\{\sigma\};t)/P_e$. P_e is the equilibrium probability distribution which is determined by the Hamiltonian $\overline{H}$, $P_e=(1/Z)\exp(\overline{H})$, where Z is a normalization factor. The transition probability satisfies the detailed balance which ensures the ergodicity of the system: $\tilde{L}_iP_e(\{\sigma\})=0$. This relation does not determine W_i uniquely. We use

$$W_i=[P_e(\{\sigma_{j\neq i}\},-\sigma_i)/P_e(\{\sigma_{j\neq i}\},\sigma_i)]^{1/2}\ . \qquad (3)$$

To study the critical slowing down, we can limit ourselves to the relaxation of an infinitely small perturbation from equilibrium. We study a "magneticlike" perturbation. Unlike the simpler case of the Koch curve, it is necessary to include other symmetry perturbations in addition to the magnetization, in order to have an equation of motion which is invariant under the RG transformation. The interactions between the three spins in the internal triangle which are summed out in the RG transformation create a triplet-spin interaction. This interaction has to be included in the perturbation from the equilibrium, Φ,

$$\Phi(\mathbf{h},\{\sigma\})=1+h_1\sum_i\sigma_i+h_3\sum_i\sigma_1^i\sigma_2^i\sigma_3^i\ , \qquad (4)$$

where $\sigma_1^i,\sigma_2^i,\sigma_3^i$ belongs to the ith triangle of the smallest scale. The calculations reveal that this perturbation spans an invariant subspace of the parameter space under the TDRG. Thus, we can substitute for $P(\{\sigma\};t)=P_e\Phi$ in the equation of motion (2), and then study its transformation under the TDRG.

The RG transformation which is used is the decimation transformation.[15,14,16] Each RG renormalized α upright triangle was composed, before the transformation, of three σ_i^α at its corners, and three spins in the middle of its edges. The equation of motion is multiplied by $T(\mu;\sigma)$,

$$T(\mu;\sigma) \equiv \prod_{\alpha} \prod_{i=1}^{3} \delta(\mu_i^\alpha - \sigma_i^\alpha) , \tag{5}$$

and a trace over the $\{\sigma\}$ is performed. Equation (2) becomes

$$\tau_0 \left[\frac{d}{dt} \right] \mathrm{Tr}_{\{\sigma\}} [T(\mu;\sigma) P(\{\sigma\};t)] = - \mathrm{Tr}_{\{\sigma\}} \left[T(\mu;\sigma) \sum_i^N L_i \Phi(\{\sigma\};t) \right] . \tag{6}$$

The left-hand side is nothing other than the standard static RG transformation,[7,15] which transforms P into P'. In the parameter space, (K, h_1, h_3), the RG transformation is described as

$$y' = y(1+4y) + O(y^3), \quad \mathbf{h}' = \mathbf{\Lambda h} , \tag{7}$$

where $y = \exp(-4K)$, and $\mathbf{\Lambda}$ has the largest eigenvalue b^D, corresponding to the eigenvector $h^1 = h_1 + h_3$. This eigenvalue has been already obtained by general arguments in Ref. 5, although the exact form of the eigenvector was not calculated there.

By performing the trace, the right-hand side of (6) becomes

$$- \sum_{\alpha}^{N/3} L_i \Phi(\mathbf{\Omega h}, \{\mu\}) , \tag{8}$$

where $\mathbf{\Omega} = \{\omega_{i,j}\}$, $i,j = 1 \rightarrow 2$, $\omega_{i,j} = \frac{1}{4}$. $\mathbf{\Omega}$ has the largest eigenvalue, $\omega^1 = \frac{1}{2}$, which corresponds to the same h^1 of $\mathbf{\Lambda}$.

For the slowest mode, h^1, Eq. (6) becomes

$$\tau_0 \left[\frac{d}{dt} \right] P'(\{\mu\};t) = -b^{-z} \sum_{\alpha}^{N'} L'_\alpha \Phi(h'^1\{\mu\};t) , \tag{9}$$

where

$$b^{-z} = \omega^1 / \lambda^1 . \tag{10}$$

By scaling the bare time scale, τ_0, by the time scaling factor, $\tau_0' = b^z \tau_0$, the renormalized equation of motion, (9), becomes of the same form as Eq. (6). By the standard RG arguments,[7,14] z is identified as the dynamic exponent. The substitution of the numerical values into (10) gives for the dynamic exponent $z = 1 + D$.

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