# Transverse Ising spin-glass model

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We discuss the zero-temperature behavior of the transverse Ising spin-glass ($\pm J_0$) model. The $d$-dimensional quantum model is shown to be equivalent to a classical $(d+1)$-dimensional Ising spin glass with correlated disorder. A renormalization-group treatment of the one-dimensional quantum model indicates the existence of a spin-glass phase. The Migdal-Kadanoff approximation is used to obtain the phase diagram of the quantum spin glass in two dimensions.

In a recent report, Chakrabarti[1] studied transverse Ising spin-glass models to assess the effects of quantum fluctuations[2,3] on the usual spin-glass properties.[4] In particular, the Edwards-Anderson[5] version of the transverse Ising model[6] (TIM) was considered, in which the exchange coupling between any pair of spins is a random variable symmetrically distributed about zero. Using Gaussian functional averages, Chakrabarti[1] showed that (a) for finite temperatures, the critical behavior of the $d$-dimensional transverse Ising spin-glass model is the same as that of the $d$-dimensional Ising spin glass, and (b) at zero temperature, however, quantum fluctuations drive the system to $(d+1)$-dimensional Ising spin-glass behavior. It is worth stressing that these results are the same as those for the nonrandom (pure) case.[2,7]

The purpose of this work is to point out that although Chakrabarti's analysis is essentially correct, it overlooks some important features of the corresponding classical model. To see this, we first briefly discuss how Suzuki's proof of equivalence between the TIM at zero temperature and the Ising model in one more spatial dimension is affected by bond randomness. Secondly, a renormalization-group (RG) analysis of the one-dimensional quantum spin-glass model is obtained. And, finally, a crude Migdal-Kadanoff approximation[8,9] is used to obtain qualitative information about the critical behavior in two dimensions.

The transverse Ising Hamiltonian is

$$H=-\Gamma\sum_i \sigma_i^x-\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z \ , \tag{1}$$

where the sums run over sites on a $d$-dimensional lattice, $\langle i,j\rangle$ stands for nearest-neighbor pairs only, and the $\sigma$'s are Pauli matrices; $\Gamma$ and $J_{ij}$ are the transverse field and exchange couplings, respectively.

At zero temperature, the critical properties of (1) follow from the ground-state energy

$$E_0(\{J_{ij}\})=\lim_{T\to 0}(-k_BT\ln \mathrm{Tr}\,e^{-H/k_BT}) \tag{2}$$

for a given bond configuration $\{J_{ij}\}$. Following along the same lines as in Ref. 7 we get

$$E_0(\{J_{ij}\})=-\Gamma \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{n}\ln\left[\left(\frac{1}{2}\sinh\frac{2}{m}\right)^{Nmn/2}\mathrm{Tr}\,e^{-H_{\text{eff}}}\right] \ , \tag{3}$$

where the $n$ and $m$ limits come from $T\to 0$ and from the noncommutation aspects of $H$, respectively; $N$ is the total number of sites in $d$ dimensions and an effective classical Ising Hamiltonian was introduced as

$$H_{\text{eff}}=-\frac{1}{m\Gamma}\sum_{\langle i,j\rangle}\sum_{k=1}^{mn} J_{ij}\sigma_{i,k}^z\sigma_{j,k}^z -\frac{1}{2}\left(\ln\coth\frac{1}{m}\right)\sum_i\sum_{k=1}^{mn}\sigma_{i,k}^z\sigma_{i,k+1} \ . \tag{4}$$

Note that the sums in $k$ lead to an effective extra spatial dimension; that is, the corresponding classical system consists of a stack of $d$-dimensional hyperplanes. The coupling between spins at positions $i$ and $j$ within a particular hyperplane is the same, irrespective to which hyperplane (i.e., which $k$) they belong. There is also a coupling between hyperplanes which is independent of $J_{ij}$. As argued in Ref. 7, the anisotropy along the extra dimension in (4) does not play any crucial role in the critical properties of the pure system.

The critical properties of the random system can, in principle, be obtained by performing a configurational average over $E_0(\{J_{ij}\})$. What is important to stress here is that, unlike the usual Ising spin glass, randomness is highly correlated along the extra dimension of the equivalent classical system. Also, there is no randomness whatsoever in the couplings between hyperplanes. For this reason one should not expect the critical behavior of a $d$-dimensional transverse Ising spin glass at zero temperature to be the same as that of a (usual) $(d+1)$-dimensional Ising spin glass. The fact that correlated randomness crucially alters the critical behavior of an otherwise uncorrelated system was also verified in the transverse Ising model in a random longitudinal field.[10]

At finite temperatures, however, the equivalent classical system is still finite in the $(d+1)$th dimension,[7] so that its critical behavior is characteristic of $d$ dimensions. The effect of correlated randomness is simply to shift the critical temperature.

Let us now specialize to one dimension and recall an exact scaling for the dilute transverse Ising chain, as derived by Stinchcombe:[9] upon rescaling, a "series" combination of $b$ bonds $(j_1,j_2,j_3,\ldots,j_b;j_i\equiv J_i/\Gamma$, with $J_i=0$ or $J)$ transforms as

$$j'=R_b(\{j_i\})=j_1j_2j_3\ldots j_b \tag{5}$$

at zero temperature. Note that if all $j$'s are equal, one has, in the usual way,[11] $j_c=\nu=1$, which are the exact results.[12]

To see that (5) is actually more general than originally proposed,[9] we perform a finite-size rescaling transformation[13] analysis in which a renormalization-group recursion relation is generated phenomenologically through the rescaling of the energy gap ($\Delta$) between the two lowest states.[13(b)] A system of size $n$ and coupling constant $j$ is related to a smaller system of size $n/b$ and coupling constant $j'$ through

$$\Delta_{n/b}(j') = b\Delta_n(j) \quad . \tag{6}$$

If we start off with a system of size $n=2b$, and allow for all possible configurations ($\{j\}$) in which each bond is $\pm j$, we can solve Eq. (6) numerically for each bond configuration to get $j'(\{j\})$. In Fig. 1 we show $g' \equiv 1/j'$ as a function of $g \equiv 1/j$ for several scaling factors, and for all possible bond configurations. The essential feature displayed in Fig. 1 is that, for a given scaling factor, all configurations with an even (including zero) number of negative bonds yield the same function $g'_+(g)$, whereas for an odd number of negative bonds we get $g'_-(g) = -g'_+(g)$. Although the recursion relations $g'_\pm(g)$ we obtain are only approximate, the fact that the presence of an odd or even number of negative bonds only affects the sign of the renormalized coupling (but not its magnitude) can hardly be regarded as fortuitous. We interpret this as signaling that Eq. (5) still holds [at least asymptotically, since the curves in Fig. 1 do not exactly fit $g'_\pm(g) = \pm g^b$] when $J_i = +J$, although we cannot at present prove it formally (as done by Stinchcombe[9] for the dilute TIM).

This result is consistent with an exact zero-temperature critical condition for a transverse Ising chain with (large) $N$ sites and arbitrary $J_i$'s[14]

$$\prod_{i=1}^{N} j_i = 1 = \prod_{k=1}^{N/b} j'_k \quad , \tag{7}$$

where the second equality follows from considering a chain with $N/b$ sites and couplings $j'_k \equiv (J'_k/\Gamma')$. Although Eq. (7) is only valid at criticality, it can be regarded as a "phenomenological" condition from which a generalized version of Eq. (5) follows (at least, at criticality).

Returning to the spin-glass case, consider a model in which each bond is distributed according to a binary distribution

$$p(j_i) = p\delta(j_i - j_0) + (1-p)\delta(j_i + j_0) \quad , \tag{8}$$

where $p$ and $(1-p)$ are the concentration of positive and negative couplings $\pm j_0$, respectively. Then, the renormalization-group (RG) transformations are defined through[15,16]

$$P'(j') = \int \prod_i dj_i P(j_i) \delta[j' - R_b(\{j_i\})] \quad . \tag{9}$$

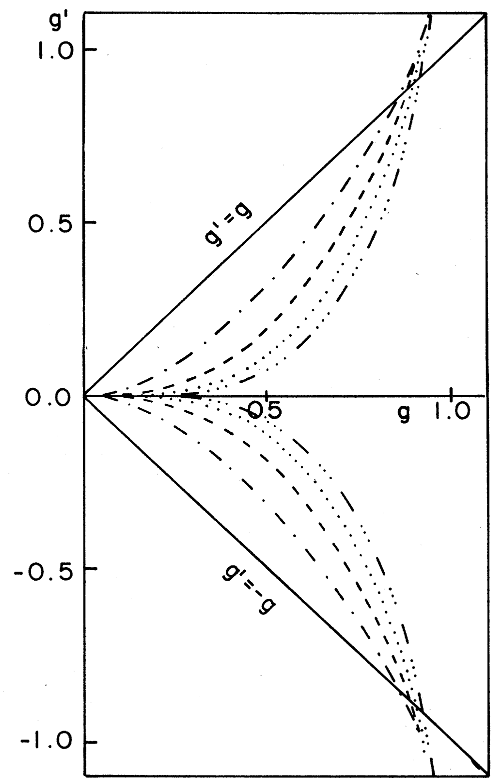


FIG. 1. Recursion relations obtained from Eq. (6) for $b=1.5(-\cdot-)$, $2(--)$, $2.5(\cdot\cdot\cdot)$, $3(-\cdot\cdot\cdot\cdot-)$. For a given $b$, all configurations with an even (odd) number of negative bonds lie on the same branch. Upper branches: $g'_+(g)$; lower branches: $g'_-(g)$.

Taking (5) and (8) into (9) yields a transformed distribution which is exactly binary (we take $b=3$, in order to preserve the symmetry between ferromagnetic and antiferromagnetic phases):

$$P'(j') = p'\delta(j' - j_0^3) + (1-p')\delta(j' + j_0^3) \quad , \tag{10}$$

with $p' = p^3 + 3p(1-p)^2$. The fixed distribution $P^*$ yields the following nontrivial fixed points ($g \equiv 1/|j|$ plays the role of temperature) and exponents:

$$p^*=0, \quad g^*=1 \text{ (antiferromagnetic critical point)}, \quad \nu_g = \nu_p = 1 \quad ,$$

$$p^*=1, \quad g^*=1 \text{ (ferromagnetic critical point)}, \quad \nu_g = 1; \quad \nu_p = 0 \ (\lambda_p = 0) \quad ,$$

$$p^*=\tfrac{1}{2}, \quad g^*=1 \text{ (spin-glass fixed point)}, \quad \nu_g = \nu_p = 1 \quad .$$

In Fig. 2 these fixed points are displayed, together with the trivial ones. Note that, while the ferromagnetic and antiferromagnetic phases are reduced to lines in the phase diagram (at $p=1$ and 0, respectively, for $g \leq 1$) a spin-glass phase is present for $g \leq 1$ and $0 < p < 1$, since all these points are driven towards the $p^*=\frac{1}{2}$, $g^*=0$ fixed point. From our previous discussion, the critical behavior at $g=1$ is the same as that of a two-dimensional correlated Ising spin glass at finite temperatures. Since it is now believed[4] that there is no finite temperature (true) spin-glass phase in two dimensions, this critical behavior is attributed to the correlations in the classical formulation, which do not generate frustration effects. If, on the other hand, we accepted Chakrabarti's results without restrictions we would be led to

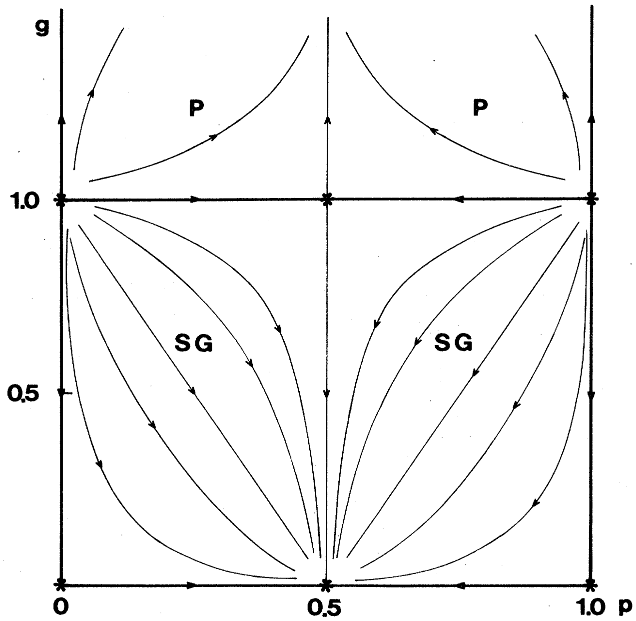


FIG. 2. Flow diagram for the one-dimensional transverse Ising spin glass at zero temperature in the field ($g \equiv \Gamma/J_0$)—concentration of positive bonds ($p$) space. The critical line $g \equiv 1$ separates the paramagnetic (P) and spin-glass (SG) phases.

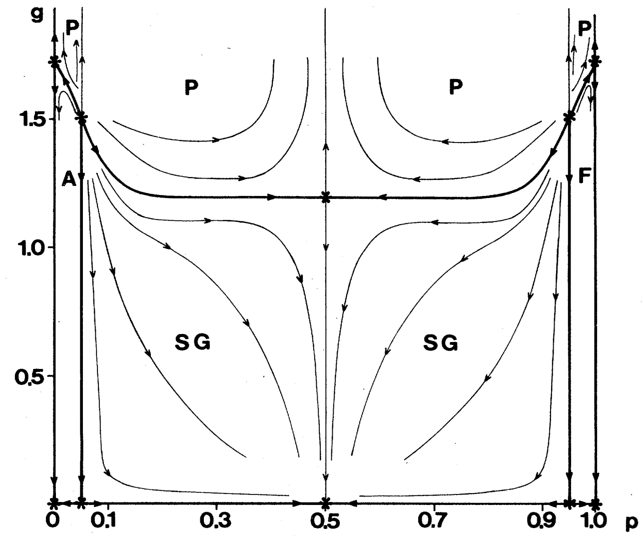


FIG. 3. Flow lines, fixed points, and critical curves for the two-dimensional transverse Ising spin glass at zero temperature (same parameters as in Fig. 2). Bold lines are critical frontiers between antiferromagnetic (A), spin-glass (SG), ferromagnetic (F), and paramagnetic (P) phases.

conclude that a spin-glass transition exists in two dimensions.

A qualitative description of what happens in a square lattice for the quantum model may be obtained by means of a Migdal-Kadanoff approximation.[8,9(b)] If we neglect commutation aspects in the bond-moving step of the approximation, we get for a "parallel" combination of bonds $\tilde{j}_i$, $\tilde{j}_2$, and $\tilde{j}_3$

$$j_p = \tilde{j}_1 + \tilde{j}_2 + \tilde{j}_3 \ , \tag{11}$$

since the exchange couplings just add amongst themselves in the Hamiltonian. With these, and the "series" combination given by (5) we get for a $b=3$ scaling[16]

$$j' \equiv R_3(\{j\}) = j_1 j_2 j_3 + j_4 j_5 j_6 + j_7 j_8 j_9 \ . \tag{12}$$

If one starts from a binary distribution (for each of these bonds) in two dimensions, the transformed distribution is no longer binary, but has four δ's. Since, under iteration these distributions will evolve to more complicated forms, one has to resort to further approximations. A particularly simple approximation,[16,17] which retains the essential physical features of the problem, consists in forcing the transformed distribution back into binary form by (1) grouping all positive (or negative) outcomes of $j'$ together, and (2) matching the average of $|j'|$ obtained from the (forced) binary and from the (actual) transformed distributions.[16] It is then straightforward to show that the (approximate) renormalization-group equations are

$$p' = f_1(p) + f_2(p) \tag{13}$$

where

$$f_1(p) = p^9 + 27p^5(1-p)^4 + 27p^3(1-p)^6 \ , \tag{14}$$

$$f_2(p) = 9p^8(1-p) + 9p^7(1-p)^2 + 57p^6(1-p)^3 + 99p^4(1-p)^5 + 27p^2(1-p)^7 \ , \tag{15}$$

and (again $g \equiv 1/j$)

$$g_0' = \frac{1}{1 + 2f_1(p) + 2f_1(1-p)} g_0^3 \ . \tag{16}$$

Solving (13) and (16) for fixed points and exponents,[11] we get the results displayed in Table I. In Fig. 3 we show the critical lines, i.e., those RG trajectories linking two nontrivial fixed points. Although these results should be re-

TABLE I. Results for the two-dimensional transverse Ising spin glass, obtained within a Migdal-Kadanoff approximation, for a scaling factor $b=3$. The eigenvalues are given by $\lambda_g = (\partial g'/\partial g)|_{g^*,p^*}$ and $\lambda_p = (dp'/dp)|_{p^*}$.

| Fixed points | | Eigenvalues | | Remarks |
|---|---|---|---|---|
| $p^*=1$ | $g^*=1.73$ | $\lambda_g=3$ | $\lambda_p=0$ | Ferromagnetic critical point |
| $p^*=0.95$ | $g^*=1.51$ | $\lambda_g=3$ | $\lambda_p=1.71$ | Tricritical point (Ref. 16) |
| $p^*=0.5$ | $g^*=1.20$ | $\lambda_g=3$ | $\lambda_p=0$ | Spin-glass fixed point |
| $p^*=0.05$ | $g^*=1.51$ | $\lambda_g=3$ | $\lambda_p=1.71$ | Tricritical point (Ref. 16) |
| $p^*=0$ | $g^*=1.73$ | $\lambda_g=3$ | $\lambda_p=0$ | Antiferromagnetic critical point |

garded as merely qualitative, Fig. 3 shows marked differences from the one-dimensional case. In particular, the FM and AFM phases are now present in a small region of the phase diagram, unlike the one-dimensional case. This feature was already observed in the Ising spin glass.[16] Again, a spin-glass region is present. It is worth mentioning that the fact that $\nu_g$ is the same for all nontrivial fixed points is an artifact of the Migdal-Kadanoff approximation for quantum systems. We are currently investigating this problem with a cluster technique[18] that will probably give the correct trend of $\nu_g$, as it was the case of dilution.[19] Also, other ways of determining an approximate binary distribution will be tested.

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