Solitons and electroacoustic interactions in ferroelectric crystals. II. Interactions of solitons and radiations

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Multiple-soliton solutions representative of the motion and interaction of walls in ferroelectric crystals of the same type as $NaNO_2$ are studied, both analytically and numerically, on the basis of a set of coupled nonlinear electroacoustic equations deduced in paper I [Phys. Rev. B **30**, 5036 (1984)] of the present series. Electromechanical couplings are duly taken into account and, in fact, are responsible for the nonlinear coupling between a d'Alembert wave equation for the transverse elastic displacement and a sine-Gordon equation that governs the orientation of the dipole-carrying molecular group in each lattice cell. To solve this rather complex problem, a singular perturbation technique associated with the methods used for solving nonlinear wave equations (Bäcklund transformations, etc.) is developed at the first order in the coupling parameter. This allows one to exhibit the first-order corrective radiative terms which superimpose on the nonlinear zeroth-order solution (multiple soliton), the free parameters of which are modulated in order to satisfy the secularity condition at the first order. Simultaneously, a numerical solution using a Lax-Wendroff finite-difference scheme is obtained which illustrates the analytical considerations and models the soliton-antisoliton collision, the soliton-soliton collision, the "oscillatory soliton," and the one-soliton solution (already exhibited in paper I on the basis of a double sine-Gordon equation) along with the accompanying elastic displacement field and the radiative contributions.

I. INTRODUCTION

In a previous paper,[1] referred to as paper I, having constructed a simple model of elastic ferroelectric bodies in which an electric polarization is associated with the orientable molecular group typical of a class of ferroelectric crystals for which $NaNO_2$ provides a prototype, we have described such ferroelectrics when large variations in the orientation of the polarization can occur, that is, essentially in the neighborhood of the ferroelectric phase transition and for strong spatial disuniformities observed in multidomain structures. Accounting, thus, for the necessary nonlinearities, it was shown that in the presence of electromechanical couplings the dynamical problem of the motion of a *single* ferroelectric wall could be represented, after elimination of the elastic displacement between the two equations at hand, by a *double sine-Gordon equation.* The corresponding physical interpretation found is that the stable solution of this equation represents the motion of a so-called 180° wall. Both the energy and the thickness of the wall are affected by electromechanical couplings. To conclude the previous paper, an attempt to explain, in the static case, the formation of domains in terms of an incommensurate-commensurate phase, was proposed on the basis of an approach of the Landau-Ginzburg type, including a Lifshitz invariant for the polarization, elastic terms, and electromechanical couplings. The writing of the fundamental dynamic equation as a double sine-Gordon equation could be achieved only for single-soliton solutions for which dynamical progressive solutions depend on *one* phase variable *only.* Obviously, however, ferroelectric structures most often contain more than a single domain wall, and more mathematically ambitious solutions that fit physical reality much better require considering *multiple-soliton solutions* on the basis of the nonlinear system of I if one wants to comprehend the dynamics of the creation and annihilation of ferroelectric domains in ferroelectric elastic crystals. It is the purpose of the present paper to develop the mathematical aspects of these multiple-soliton solutions both analytically (insofar as possible) and numerically.

In order to fulfill the program briefly sketched out above, note that the basic nonlinear system deduced in I and recalled in Sec. II below consists of the *nonlinear* coupling between a d'Alembert wave equation for the transverse elastic displacement and a (simple) sine-Gordon equation for the orientation of the molecular group of the ferroelectric crystalline cell, the coupling occurring through a small parameter representative of electromechanical couplings. That is, the zeroth-order solution in this small parameter is *already nonlinear* since it contains multiple (purely ferroelectric) -soliton solutions of the homogeneous sine-Gordon equation. To exploit the coupled case, in Sec. III we develop a perturbation scheme which amounts to determining corrective terms of the first order in the aforementioned small parameter by constructing a Green function associated with the zeroth-order nonlinear problem. This is a generalization to the case of a *system* of coupled nonlinear equations of a method already used by Hirota. In the process the nonlinear coupled solution is sought in the form of an *asymptotic expansion,* and it is the secularity condition for the first-order term in this expansion that imposes a modulation of the free parameters (wave number and phase velocity) of the zeroth-order solution, which would be left free otherwise. The fact that the zeroth-order solution is already nonlinear complicates the general solution, but it happens that the required Green function can be built with the help of *Bäcklund transformations*. The formal structure of these mathematical developments is given in Sec. III, while the ferroelectric problem in presence of electromechanical couplings is treated analytically in Sec. IV. In the latter, first the zeroth-order multiple-soliton solution is built by applying the Bäcklund transformation; next, the modulation of the free parameters of the zeroth-order solution is established and, finally, the *radiations*, or corrective terms of the first order, in elastic displacement and orientation angle of electric dipoles, are obtained.

The asymptotic behavior of the solution at the first order is established and proves to be useful, as border conditions, in Sec. V, where, upon using the Hamiltonian formulation of the coupled equations given in I and a "leap-frog" Lax-Wendroff finite-difference scheme—which is very efficient for such nonlinear hyperbolic systems—numerical solutions are obtained graphically for multiple-soliton solutions which represent the soliton-antisoliton collision, the soliton-soliton collision, and the breather or oscillatory soliton, all of these along with the accompanying elastic displacement field and the corresponding first-order radiations in displacement and orientation of electric dipoles. With the exception of the last one, these solutions can be interpreted in terms of the motion of *two* walls of various types in a ferroelectric sample. Finally, in Sec. VI we comment generally on the problem examined in this two-part work and illuminate other problems of nonlinear electroacoustics which could be treated on the basis of the available set of coupled nonlinear equations. Of particular interest would be the case accounting for the influence of external stimuli such as a stress or an applied electric field. This is also the case of the problem of reflection and diffraction of acoustic waves by a wall (or walls) in a ferroelectric crystal.

In all, it is thought that the present paper, along with the previous one, (i) introduces a relatively new point of view concerning the formation of domains, and the motion and interaction of ferroelectric walls in an elastic ferroelectric, (ii) clearly illustrates this dynamical behavior through the numerical solution of nonlinear hyperbolic equations, and (iii) offers an exemplary field of simultaneous applications of all methods recently developed for solving nonlinear wave equations.

II. GOVERNING EQUATIONS

In a previous paper (Ref. 1) the coupled equations which govern (in nondimensional form) the transverse elastic displacement v and the rotational motion of electric dipoles rigidly attached to the central molecular groups in ferroelectrics of the same type as $NaNO_2$ were deduced from a simple lattice model and then transformed to the long-wavelength limit. These equations read

$$\frac{\partial^2 v}{\partial \tau^2}-\hat{V}_{\rm T}^2\frac{\partial^2 v}{\partial X^2}+\eta\frac{\partial}{\partial X}\sin\phi=0\ , \tag{2.1}$$

$$\frac{\partial^2 \phi}{\partial \tau^2}-\frac{\partial^2 \phi}{\partial X^2}-\sin\phi-\eta\frac{\partial v}{\partial X}\cos\phi=0\ , \tag{2.2}$$

where X and τ are space and time coordinates, ϕ equals twice the variation in the orientation of electric dipoles, $\hat{V}_{\rm T}$ is a transverse-acoustic—wave speed that includes a *stiffening* of the elastic constant by the electrostatic dipole interactions, and η is an electromechanical coupling parameter. For functions v and ϕ, which depend on X and τ only through the phase combination $\xi=QX-\Omega\tau$, where Q and Ω are pseudo wave number and frequency, respectively, Eqs. (2.1) and (2.2) are equivalent to the equations

$$\frac{\partial^2 \phi}{\partial \tau^2}-\frac{\partial^2 \phi}{\partial X^2}-\sin\phi+\gamma\sin(2\phi)=0 \tag{2.3}$$

and

$$(\Omega^2-\hat{\Omega}_{\rm T}^2)\frac{dv}{d\xi}=-\eta Q\sin\phi\ , \tag{2.4}$$

where

$$\gamma(\Omega,Q)=\frac{\eta^2}{2}\frac{Q^2}{\Omega^2-\hat{\Omega}_{\rm T}^2}\ ,\quad \hat{\Omega}_{\rm T}^2=\hat{V}_{\rm T}^2Q^2\ . \tag{2.5}$$

Equation (2.3) is a sine-Gordon equation perturbed by the term with factor γ. Alternatively, it may be considered a double sine-Gordon equation. This equation is invariant under "Lorentz transformations" of the type

$$X\to X'=\frac{\xi}{(Q^2-\Omega^2)^{1/2}}\ ,\quad \tau\to\tau'=\frac{Q\tau-\Omega X}{(Q^2-\Omega^2)^{1/2}}\ . \tag{2.6}$$

A *stable* single-soliton of Eqs. (2.3) and (2.4) was found in the form

$$\phi=-2\tan^{-1}\left[\frac{\sinh\xi'}{\sqrt{1+2\gamma}}\right],\quad \xi'=\xi-\tfrac{1}{2}\ln\left[\frac{4(1+2\gamma)}{\delta}\right], \tag{2.7}$$

where δ is at our disposal, and Ω and Q are related by the "dispersion relation"

$$[\Omega^2-(\Omega_F^2-2)](\Omega^2-\hat{\Omega}_{\rm T}^2)+\eta^2Q^2=0\ , \tag{2.8}$$

where $\Omega_F^2=1+Q^2$ is the uncoupled ferroelectric mode in an harmonic analysis with parameters Ω and Q. Accompanying the "electric" soliton (2.7) are stresses and an elastic displacement field v given by

$$\sigma=\eta\frac{\Omega^2\sqrt{1+2\gamma}}{\Omega^2-\hat{\Omega}_{\rm T}^2}\frac{\sinh\xi'}{\sqrt{1+2\gamma}+\sinh^2\xi'} \tag{2.9}$$

and

$$v-v_0=\frac{2\eta Q}{\Omega^2-\hat{\Omega}_{\rm T}^2}\left[\frac{1+2\gamma}{-2\gamma}\right]^{1/3}\tanh^{-1}\left[\frac{\cosh\xi'}{\sqrt{-2\gamma}}\right], \tag{2.10}$$

Another solution can be found which is *not* stable and corresponds to

$$\phi=\pi-2\tan^{-1}\left[\frac{\sinh\xi'}{\sqrt{1-2\gamma}}\right],\quad \xi'=\xi-\tfrac{1}{2}\ln\left[\frac{4(1-2\gamma)}{\delta}\right] \tag{2.11}$$

with the "dispersion relation"

$$(\Omega^2-\Omega_F^2)(\Omega^2-\hat{\Omega}_{\rm T}^2)+\eta^2 Q^2=0 . \tag{2.12}$$

Note for numerical computational purposes that the system (2.1) and (2.2) can be given the following Hamiltonian form:

$$\begin{aligned}
&\frac{\partial v}{\partial t}=y ,\\
&\frac{\partial y}{\partial t}+\hat{V}_{\rm T}\frac{\partial}{\partial x}[\alpha+(\eta/\hat{V}_{\rm T})\sin\phi]=0 ,\\
&\frac{\partial \alpha}{\partial t}+\hat{V}_{\rm T}\frac{\partial y}{\partial x}=0 ,\\
&\frac{\partial \phi}{\partial t}=z ,\\
&\frac{\partial z}{\partial t}+\frac{\partial \beta}{\partial x}=-\sin\phi-(\eta/\hat{V}_{\rm T})\alpha\cos\phi ,\\
&\frac{\partial \beta}{\partial t}+\frac{\partial z}{\partial x}=0 ,
\end{aligned} \tag{2.13}$$

with obvious redefinitions.

III. INTERACTIONS OF SOLITONS IN THE PRESENCE OF PERTURBATIONS: GENERAL METHOD

A. Perturbation scheme

In a general manner the system (2.1) and (2.2) has no simple solution such as a single soliton for rather general initial conditions at $(X,0)$. However, if the electromechanical coupling is discarded ($\eta=0$), Eq. (2.1) will yield a propagative solution $v(X-\hat{V}_{\rm T}\tau)$, while Eq. (2.2) will reduce to an ordinary sine-Gordon equation which *does* admit soliton solutions and, in fact, multiple-soliton solutions.[2–7] In particular, if we consider the interaction of two such solutions, we may envisage "soliton-soliton" collisions, "soliton-antisoliton" collisions (or doublets), and "oscillatory solitons" (or "breathers"). This will physically correspond to the propagation of several domain walls or to stationary domain walls. If the electromechanical coupling is then reintroduced (small η), this may be considered as a perturbation which, in some manner, should alter the uncoupled solutions. As far as Eq. (2.1) is concerned, it remains linear for v even after coupling. Its solution is thus composed of a propagative solution and a particular solution induced by ϕ through the parameter η. That is, one may obtain v once ϕ is known. Regarding Eq. (2.2), it is nonlinear both before and after perturbation by the η term and, for $\eta\neq 0$, must account for the coupling with Eq. (2.1). A special perturbation scheme must be devised for treating the general solution (v,ϕ). In a first step, one must determine the "principal" solution for which the effect of perturbations is only to modulate the free parameters (phase and velocity) of the solution; this requires only general uncoupled solutions. In a second step, we shall determine corrective terms of the first order by constructing a *Green function* associated with the problem. To that purpose, the inverse-scattering method must be used. The first-order correction will represent the radiation of harmonic and/or soliton waves. The method employed has already been used with success in the solution of the Korteweg–de Vries and Schrödinger equations in the presence of perturbations,[8–11] and in treating the interactions of solitons for the double sine-Gordon equation.[12,13] Here the method is generalized to the case of two coupled nonlinear equations.

Consider the system (2.1) and (2.2) and, for questions of stability concerning the sine-Gordon equation, replace v by $-v$ and ϕ by $\phi+\pi$. The system obtained can be cast in the following operator form (T denotes transpose):

$$\frac{\partial}{\partial\tau}\mathbf{U}+\mathcal{N}(\mathbf{U})=\eta\mathbf{F}(\mathbf{U}) , \tag{3.1}$$

where we have set

$$\mathbf{U}=(v,v_\tau,\phi,\phi_\tau)^T ,\quad v_\tau=\frac{\partial v}{\partial\tau} ,\quad \phi_\tau=\frac{\partial\phi}{\partial\tau} , \tag{3.2a}$$

$$\mathcal{N}=\begin{bmatrix} 0 & -1 & 0 & 0\\ -\hat{V}_{\rm T}^2\dfrac{\partial^2}{\partial X^2} & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -\dfrac{\partial^2}{\partial X^2}+\sin(\cdots) & 0\end{bmatrix} , \tag{3.2b}$$

$$\mathbf{F}\equiv(0,f_1(v,\phi),0,f_2(v,\phi))^T=(0,-\partial\sin\phi/\partial X ,0,\cos\phi\,\partial v/\partial X)^T . \tag{3.2c}$$

For $\eta=0$ the system (3.1) admits solutions v_0 of the harmonic-wave type for v and ϕ_0 of the multiple-soliton type for ϕ. For small $\eta\neq 0$ we look for an *asymptotic* expansion

$$\mathbf{U}(X,\tau)=\mathbf{U}_0(X,\tau,\bar{\tau})+\eta\mathbf{U}_1(X,\tau,\bar{\tau})+\cdots , \tag{3.3}$$

where $\mathbf{U}_0$ is the solution of

$$\frac{\partial}{\partial\tau}\mathbf{U}_0+\mathcal{N}(\mathbf{U}_0)=\mathbf{0} \tag{3.4}$$

and

$$\lim_{\eta\to 0}\eta\mathbf{U}_1(X,\tau/\eta)=\mathbf{0} \tag{3.5}$$

for fixed X and τ, with $\bar{\tau}=\eta\tau$ a slow time scale. Equation

(3.4) is a *secularity condition* which guarantees that the expansion (3.3) is valid for time intervals of the order of η^{-1} (equivalently, the term $\eta \mathbf{U}_1$ remains bounded for large time intervals).[14] The first-order solution $\mathbf{U}_1$ satisfies the problem

$$[\hat{\mathcal{N}}(\mathbf{U}_0)]\mathbf{U}_1 \equiv \frac{\partial}{\partial \tau}\mathbf{U}_1 + [\mathcal{N}^{(p)}(\mathbf{U}_0)]\mathbf{U}_1 = \mathcal{F}(\mathbf{U}_0) , \quad \mathbf{U}_1(X,0) = \mathbf{0} \tag{3.6}$$

where $\hat{\mathcal{N}}(\mathbf{U}_0)$ is the linearization of Eq. (3.1) about $\mathbf{U}_0$, $\mathcal{N}^{(p)}$ is the perturbed operator $\mathcal{N}$, v_0, ϕ_0 are supposed to hold good initially, and

$$\mathcal{F}(\mathbf{U}_0) = \mathbf{F}(\mathbf{U}_0) - \frac{1}{\eta}\left[\frac{\partial}{\partial \tau}\mathbf{U}_0 + \mathcal{N}(\mathbf{U}_0)\right] . \tag{3.7}$$

This *effective source* term accounts for the fact that the condition (3.5) cannot be checked for an $\mathbf{U}_0$ which satisfies Eq. (3.4) exactly. Accordingly, a certain freedom must be granted to $\mathbf{U}_0$ which may be modulated on the time scale $\bar{\tau}$, so that this modulated $\mathbf{U}_0$ solution will satisfy Eq. (3.4) at the order η (and *not* zero.) Equation (3.6) can be solved by representing the inverse operator $[\hat{\mathcal{N}}(\mathbf{U}_0)]^{-1}$ by means of a Green function,[15,16] so that, formally,

$$\mathbf{U}_1(X,\tau) = \int_0^{\tau} (\mathcal{G}(X,\tau \mid \ldots, \tau'), \ \mathcal{F}(\mathbf{U}_0(\ldots,\tau')))d\tau', \tag{3.8}$$

where $\mathcal{G}$ is a linear operator on a certain Hilbert space $\mathcal{H}$ equipped with an inner product $(\ldots, \ldots)$ involving a spatial integration; $\mathcal{G}$ is such that

$$[\hat{\mathcal{N}}(\mathbf{U}_0)]\mathcal{G} = 0 , \quad \tau > \tau' \geq 0 , \quad \lim_{\tau \to \tau'} \mathcal{G} = 1 \tag{3.9a}$$

or

$$[\hat{\mathcal{N}}(\mathbf{U})]^T \mathcal{G}^T = 0 , \quad \tau' > \tau \geq 0 , \quad \lim_{\tau' \to \tau} \mathcal{G}^T = 1 \tag{3.9b}$$

where $[\hat{\mathcal{N}}(\mathbf{U}_0)]^T$ is the adjoint of $\hat{\mathcal{N}}(\mathbf{U}_0)$ and 1 is the identity in $\mathcal{H}$.

The efficiency of the above-sketched method relies heavily on the ease with which the inverse operator can be found for a given $\mathbf{U}_0$. In most cases this is a difficult task, but the *inverse-scattering method* allows one to find the appropriate Green function in the case of integrable equations of evolution (cf. Refs. 10 and 16).[17] Fortunately, here we do not need the technical details for the construction of $\mathcal{G}$. All we need is to know the structure of the kernel $\mathcal{K}(\hat{\mathcal{N}})$ of $\hat{\mathcal{N}}(\mathbf{U}_0)$ or the kernel of $\hat{\mathcal{N}}^T$. To briefly sketch out the required development, let $\{p_j\}$ denote collectively the free parameters that we ascribe to the $\mathbf{U}_0$ solution. For instance, these are the speeds and phases of the solitons in a multiple-soliton solution U_0 [e.g., Q and ξ_0 in the solution (2.7)—(2.10) above]. $\mathcal{K}$ consists of two parts: a *discrete* subspace $\mathcal{K}_d$ associated with dispersive waves and a *continuum subspace* $\mathcal{K}_c$ which, physically, corresponds to soliton solutions of $\mathbf{U}_0$. Because $\hat{\mathcal{N}}(\mathbf{U})$ results from the linearization of the operator $[1(\partial/\tau)+\mathcal{N}]\mathbf{U}_0$ about $\mathbf{U}_0$, elements of $\mathcal{K}(\hat{\mathcal{N}})$ are simply found by differentiating $\mathbf{U}_0$ with respect to its free parameters. Thus $\mathcal{K}_d(\hat{\mathcal{N}})$ is generated by the *finite* family of functions $\{\partial \mathbf{U}_0/\partial p_j\}$ with $j = 1,2,\ldots,2N$ if $\mathbf{U}_0$ contains N soliton solutions. Accordingly, one can decompose $\mathcal{G}$ as $\mathcal{G} = \mathcal{G}_d + \mathcal{G}_c$, and $\mathcal{G}_d$ admits a representation on the basis $\{\partial \mathbf{U}_0/\partial p_j\}$, while the "continuous" component $\mathcal{G}_c$ is composed of continuous wave trains (see below).

At this stage the problem consists in determining the modulation of the free parameters $\{p_j\}$. We know that, for large τ's, a multiple-soliton solution is built up of a sum of N single soliton (Refs. 2—7), each depending on X and τ through a phase $Q_j X - \Omega_j \tau$, $j = 1,2,\ldots,N$, the pseudo wave numbers Q_j and frequencies Ω_j satisfying the "dispersion relations" $\Omega_j = \tilde{\Omega}_j(Q_j)$, j fixed. The same space-time dependence will show up in the discrete component of $\mathcal{G}$, but secular terms may appear in $\mathbf{U}_1$ since both $\mathbf{U}_0$ and $\mathcal{G}$ are functions of $Q_j X - \Omega_j \tau$. In particular, with the above-recalled definitions of the inner product, the first-order corrective term (3.8) will, after dummy integration over the time variable, yield a term linear in τ in $\mathbf{U}_1$ which violates the secularity condition (3.5). In other words, we may also say that any part of the effective source (3.7) that is parallel to one of the discrete components resonates with the Green function and produces secular terms. In order to eliminate such disturbing perturbations, the selection of the modulation of the p_j's can be made such that the effective source (3.7) is *orthogonal* to $\mathcal{K}_d(\hat{\mathcal{N}}^T)$. Let $\{\mathbf{b}_j(X,\tau);\ j = 1,2,\ldots,2N\}$ be a basis which spans $\mathcal{K}_d(\hat{\mathcal{N}}^T)$. The orthogonality condition on $\mathcal{F}(\mathbf{U}_0)$ yields a system of ordinary differential equations

$$\sum_{j=1}^{2N} \left[\mathbf{b}_k, \frac{\partial}{\partial p_j}\mathbf{U}_0\right]\frac{dp_j}{d\bar{\tau}} = (\mathbf{b}_k, \mathbf{F}(\mathbf{U}_0)) , \tag{3.10}$$

where $\bar{\tau} = \eta\tau$ is the slow time scale and $(\ldots, \ldots)$ denotes the inner product in the space of square-summable functions. Equation (3.10) provides the looked-for modulation. In addition, it can be noted that the elements of $\mathcal{K}_d(\hat{\mathcal{N}}^T)$ may be generated by differentiating $\mathbf{U}_0$ with respect to the p_j's. In particular, if J is defined by

$$J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \tag{3.11}$$

we can generate the elements of $\mathcal{K}_d(\hat{\mathcal{N}}^T)$ with the $2N$ elements $\{J\,\partial \mathbf{U}_0/\partial p_j\}$, so that Eq. (3.10) can be replaced by

$$\sum_{j=1}^{2N} \left[J\frac{\partial}{\partial p_k}\mathbf{U}_0 , \frac{\partial}{\partial p_j}\mathbf{U}_0\right]\frac{dp_j}{d\bar{\tau}} = \left[J\frac{\partial}{\partial p_k}\mathbf{U}_0 , \mathbf{F}(\mathbf{U}_0)\right] , \tag{3.12}$$

where $k = 1,2,\ldots,2N$. According to Eq. (3.12), we only need the zeroth-order solution to proceed to the $p_j(\bar{\tau})$'s.

B. Green functions

We now must face the problem of finding a basis in the continuous space $\mathcal{K}_c(\hat{\mathcal{N}})$. Following previous works by other authors (Refs. 16 and 17), the inverse-scattering method provides a systematic way for constructing $\mathscr{K}_c$ for the subsequent construction of $\mathscr{G}$. Since our zeroth-order solution consists of a multiple-soliton solution, $\mathscr{G}$ here may be built with the help of Bäcklund transformations. In our case we need to extend the procedure for building the Green function of the coupled system (3.1), which includes a sine-Gordon equation.

The solution of the linear nonhomogeneous system (3.6) for $\mathbf{U}_1$, on account of the inner product and Eq. (3.8), can be written as

$$\mathbf{U}_1(X,\tau)=\int_0^{\tau}\int_{-\infty}^{+\infty}\mathscr{G}_c(X,\tau\,|\,X',\tau')\mathscr{F}(X',\tau')dX'\,d\tau' , \tag{3.13}$$

where only $\mathscr{G}_c$ intervenes by virtue of the orthogonality condition on $\mathscr{G}_d$. In general, the complete solution $\mathbf{U}$ will contain multiple solitons as well as a continuum of radiation at all wavelengths (continuous frequency spectrum). We should differentiate $\mathbf{U}$ with respect to the amplitudes of the radiation components to obtain the missing additional basis elements for the whole of $\mathscr{K}(\mathscr{N})$, that is, the infinite, but complete, set

$$\mathscr{S}=\left\{\frac{\partial}{\partial p_j}\mathbf{U}\ ,\ j=1,2,\ldots,2N\ ;\frac{\delta}{\delta\rho_+(\lambda,0)}\mathbf{U}\ ,\ \lambda\in R\right\} , \tag{3.14}$$

where $\rho_+(\lambda,0)$ represents the initial value (at $t=0$) of a "positive" or "direct" reflection coefficient that characterizes the continuous density of radiation at the wave number (Refs. 3, 4, and 10)

$$k(\lambda)=2\lambda-1/8\lambda . \tag{3.15}$$

We shall not give the detail of the construction of the continuous part of $\mathscr{S}$ here. With $\rho_-(\lambda,0)$ a "retrograde" reflection coefficient and $a(\lambda)$ the maximum transmission coefficient, noting (since we ultimately need $\delta\mathbf{U}/\delta\rho_\pm$ at the zeroth order for $\rho_\pm=0$)

$$\dot{\mathbf{U}}(X,\tau;\lambda)=\left.\frac{\delta\mathbf{U}(X,\tau)}{\delta\rho_\pm(\lambda,0)}\right|_{\rho_\pm=0} , \tag{3.16}$$

one finally arrives at the representation[18]

$$\mathscr{G}_c(X,\tau\,|\,X',\tau')=-\frac{\pi i}{4}\int_{-\infty}^{+\infty}\frac{\lambda}{[a(\lambda)]^2}\dot{\mathbf{U}}_+(X,\tau;\lambda)\times\dot{\mathbf{U}}_-^T(X',\tau';\lambda)J\,d\lambda . \tag{3.17}$$

We have yet to evaluate the derivatives (3.16). Two methods can be used for that purpose, one using the squared eigenfunctions of the inverse-scattering method (Refs. 10 and 16) and the other using Bäcklund transformations (Ref. 17). The second method requires less secondary data than the first, and the idea that $\mathscr{K}(\hat{\mathscr{N}})$ or $\mathscr{K}(\hat{\mathscr{N}}^T)$ can be spanned by differentiation of $\mathbf{U}_0$ with respect to the free parameters underlies it. We finally note that $\dot{\mathbf{U}}_\pm$ are such that

$$[\hat{\mathscr{N}}(\mathbf{U}_0)]\dot{\mathbf{U}}_\pm=\mathbf{0}\ ,\ \ X\in(-\infty,+\infty) \tag{3.18a}$$

$$\dot{\mathbf{U}}_\pm(X,\tau;\lambda)\simeq\frac{1}{\pi\lambda}\exp\{\mp[k(\lambda)X+\omega(\lambda)\tau]\}\ \text{ for } X\to\pm\infty \tag{3.18b}$$

with $k(\lambda)$ given by Eq. (3.15) and $\omega(\lambda)=2\lambda+(1/8\lambda)$. Once $\dot{\mathbf{U}}_\pm$ are determined for the problem (3.18), we have at hand all the ingredients needed to construct $\mathscr{G}_c$. In the next section, omitting details, we exploit the method exposed in the present section with particular attention to the soliton-antisoliton collision.

IV. INTERACTIONS OF SOLITONS IN THE PRESENCE OF PERTURBATIONS: SOLUTION

A. Zeroth-order solution

For $\eta=0$, Eq. (2.1) admits a solution of the plane-harmonic-wave type

$$v_0=\exp[i(KX-\bar{\omega}\tau+\xi_0)] , \tag{4.1}$$

where K and $\bar{\omega}$ satisfy the usual dispersion law for transverse elastic waves, $\bar{\omega}=\hat{V}_{\rm T}K$. Under the same condition, Eq. (2.1) reduces to a sine-Gordon equation, a conservative nonlinear, dispersive wave equation which admits multiple-soliton solutions with $2N$ free parameters (Refs. 2, 4, and 7). In particular, using Bäcklund transformations[19] or Hirota's method,[20,21] one may generate a general two-soliton solution as

$$\phi_0=4\tan^{-1}\varphi\ ,\ \ \varphi=\left|\frac{Q_1-Q_2}{\Omega_1+\Omega_2}\right|\frac{\cosh[(\xi_1'-\xi_2')/2]}{\sinh[(\xi_1'+\xi_2')/2]} \tag{4.2}$$

where

$$\xi_i'=\xi_i+\delta_i'\ ,\ \ \xi_i=Q_iX-\Omega_i\tau$$
$$\delta_i'=\delta_i-\tfrac{1}{2}\ln\left|\frac{Q_1-Q_2}{\Omega_1+\Omega_2}\right|\ ,\ \ Q_i^2-\Omega_i^2=1\ ,\ \ i=1,2 \tag{4.3}$$

in which the free parameters are the Q_i's and δ_i's (or δ_1' and δ_2'). Depending on the type or relation connecting these parameters, we may have three categories of solutions: the "soliton-soliton" collision, the "soliton-antisoliton" collision, and the "oscillatory soliton" or "breather." For the sake of illustration, in this analytical part of the paper we consider the second category for which $Q_1=-Q_2=Q$ and $\Omega_1=\Omega_2=\Omega$. Then the solution (4.2) assumes the form

$$\phi_0=-4\tan^{-1}[\gamma\beta(\tau)/\alpha(X)] , \tag{4.4}$$

with

$$\gamma=Q/\Omega\ ,\ \ \beta(\tau)=\sinh(\Omega\tau+\delta)\ ,\ \ \alpha(X)=\cosh(QX) . \tag{4.5}$$

One is left with *three* parameters, one of which is taken zero, so that there remains in the end the free parameters Q and δ, which will be modulated by the perturbation. We stress at this point that Eq. (2.1) for v is linear and its perturbation will contain only the solution in ϕ. Then the complete solution v is composed of a soliton in the absence of source terms and a particular solution depending on ϕ_0; it is useless to introduce a modulation of the free parameters K and ξ_0. We also note the following asymptotic behavior (Ref. 7):

$$\phi_0 \sim -4\tan^{-1}[\exp(QX-\Omega\tau-\delta-\tilde{\delta})] -4\tan^{-1}\{\exp[-(QX-\Omega\tau+\delta-\tilde{\delta})]\} \quad (4.6)$$

for $\tau \to \infty$, with $\tilde{\delta}=\ln\gamma$.

B. Modulation of the free parameters

One must deduce the modulation $Q(\bar{\tau})$ and $\delta(\bar{\tau})$, where $\bar{\tau}=\eta\tau$, from the orthogonality condition (3.12). To that purpose we first build a basis of $\mathscr{K}_d(\hat{\mathscr{N}})$, which, here, has dimension two. This space is spanned by $\partial\mathbf{V}/\partial Q$ and $\partial\mathbf{V}/\partial\delta$, where $\mathbf{V}=(\phi_0, \partial\phi_0/\partial\tau)^T$. The basis should be determined at the zeroth order. If we set

$$\Delta=\Omega\tau+\delta , \quad (4.7)$$

at zeroth order we have

$$\frac{\partial\Delta}{\partial\tau}\simeq\Omega , \quad \frac{\partial\mathbf{V}}{\partial\delta}\simeq\frac{1}{\Omega}\frac{\partial\mathbf{V}}{\partial\tau} . \quad (4.8)$$

At the same order, $J\,\partial\mathbf{V}/\partial Q$ and $\Omega^{-1}J\,\partial\mathbf{V}/\partial\tau$ thus form a basis for $\mathscr{K}_d(\dot{\mathscr{N}}^T)$. More precisely, we set

$$\mathbf{b}_1\simeq\begin{bmatrix}\phi_{0,\tau\tau}\\ -\phi_{0,\tau}\end{bmatrix} , \quad \mathbf{b}_2\simeq\begin{bmatrix}\phi_{0,Q\tau}\\ -\phi_{0,Q}\end{bmatrix} , \quad (4.9)$$

where the symbol $\simeq$ recalls the approximation at zeroth order. The Hamiltonian associated with the sine-Gordon equation governing ϕ_0 reads

$$H(\phi_0)=\int_{-\infty}^{+\infty}\left[\frac{1}{2}\left[\frac{\partial\phi_0}{\partial\tau}\right]^2+\frac{1}{2}\left[\frac{\partial\phi_0}{\partial X}\right]^2+1-\cos\phi_0\right]dX . \quad (4.10)$$

Then, using the exact form of the perturbation given in Eq. (3.2c), we can show that Eq. (3.12) for $k=1,2$ and $\{p_j\}=(Q,\delta)$ reads

$$\frac{dH(\phi_0)}{dQ}\frac{dQ}{d\bar{\tau}}=iK\exp[-i(\bar{\omega}\tau-\xi_0)]\frac{\partial}{\partial\tau}\times\int_{-\infty}^{+\infty}(\sin\phi_0)e^{iKX}dX ,$$

$$\frac{dH(\phi_0)}{dQ}\frac{d\delta}{d\bar{\tau}}=iK\exp[-i(\bar{\omega}\tau-\xi_0)]\frac{\partial}{\partial Q}\times\int_{-\infty}^{+\infty}(\sin\phi_0)e^{iKX}dX , \quad (4.11)$$

where the integrals are Fourier transforms of $\sin\phi_0$. We note that, for the solution (4.4),

$$H(\phi_0)=16Q . \quad (4.12)$$

A rather long calculation involving residues leads, from Eq. (4.11), to the following evolution system for Q and Δ:

$$\frac{dQ}{d\bar{\tau}}=-\frac{\pi\lambda}{2}\exp\left[-\frac{\pi\lambda}{2}-i(\bar{\omega}\tau-\xi_0)\right]\times\frac{\partial}{\partial\tau}\left[\frac{\cos(\lambda B+A)}{\cos A}\right] ,$$

$$\frac{d\Delta}{d\bar{\tau}}=\frac{\Omega}{\eta}+\frac{\pi\lambda\Omega}{2}\exp\left[-\frac{\pi\lambda}{2}-i(\bar{\omega}\tau-\xi_0)\right]\times\left[\frac{\pi\lambda}{2}+Q\frac{\partial}{\partial Q}\left[\frac{\cos\lambda B+A)}{\cos A}\right]\right] , \quad (4.13)$$

where we have set

$$\lambda=K/Q , \quad \Omega^2=Q^2-1 , \quad \bar{\beta}=\gamma\beta(\Delta)=\gamma\sinh\Delta ,$$
$$B=\ln[\bar{\beta}+(\bar{\beta}^2+1)^{1/2}]=\sinh^{-1}\bar{\beta} , \quad (4.14)$$
$$\tan A=\lambda\bar{\beta}/(\bar{\beta}^2+1)^{1/2} .$$

The modulations $Q(\bar{\tau})$ and $\Delta(\bar{\tau})$ may be real, imaginary, or complex depending on the expression of $\mathbf{U}_0(X,\tau)$. Here the latter is taken to be complex. Note that ϕ_0 is even in X. From the system (4.11) it follows that the right-hand sides will vanish if v_0 is odd in X. In this case the perturbation would not have any effect on Q and δ. Therefore, only the component of v_0 even in X will contribute to the modulation of Q and δ. In a general manner, the system (4.13) shows that Q and Δ are complex-valued functions of τ. The imaginary parts of Q and Δ are of the order of η; these parts therefore induce oscillating modulations in X with frequency $\eta\,\mathrm{Im}[Q(\tau)]$ and in τ with frequency $\eta\,\mathrm{Im}[\Delta(\tau)]$, which are both small. As to the real parts, they contain the principal parts (zeroth order) of Q and Δ, $Q=Q_0$ and $\Delta=\Omega_0\tau+\delta_0$, which must be such that the nature of the zeroth-order solution is left unchanged. In effect, Ω must not become imaginary for the solution ϕ_0 would then become oscillatory while we are considering the soliton-antisoliton—collision problem. This constraint forces us to select the initial value Q_0 such that $|Q(\tau)|\geq 1$ be satisfied for all times.

It is customary to study systems such as (4.13) in the (Q,Δ) plane (Ref. 17),[22] for various initial conditions Q_0,Δ_0. In all cases depicted (Fig. 1) we have oscillations along Q while Δ increases. For values of Q_0 close to *one*, we obtain curves (a) and (b), where the phase Δ is strongly oscillatory while increasing steadily in amplitude simultaneously. For Q_0 far from the value *one*, the phase Δ behaves like $\Omega\tau$, on which oscillations of the order of η are superimposed. If Δ is much larger than zero—curve (c)—then the oscillatory regime is well imposed. Other parameters such as K, $\hat{V}_T$, and ξ_0 do not influence the behavior of $Q(\tau)$ and $\Delta(\tau)$ much. The asymptotic behavior of Eqs. (4.11) is given in Appendix A.

C. Study of radiations (or corrective terms of the first order)

Here the ideas and methods recalled in Sec. III are applied to our specific problem. First, we note that the effective source $[\mathscr{F}=(\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3,\mathscr{F}_4)^T$, Eq. (3.7)] has components given by

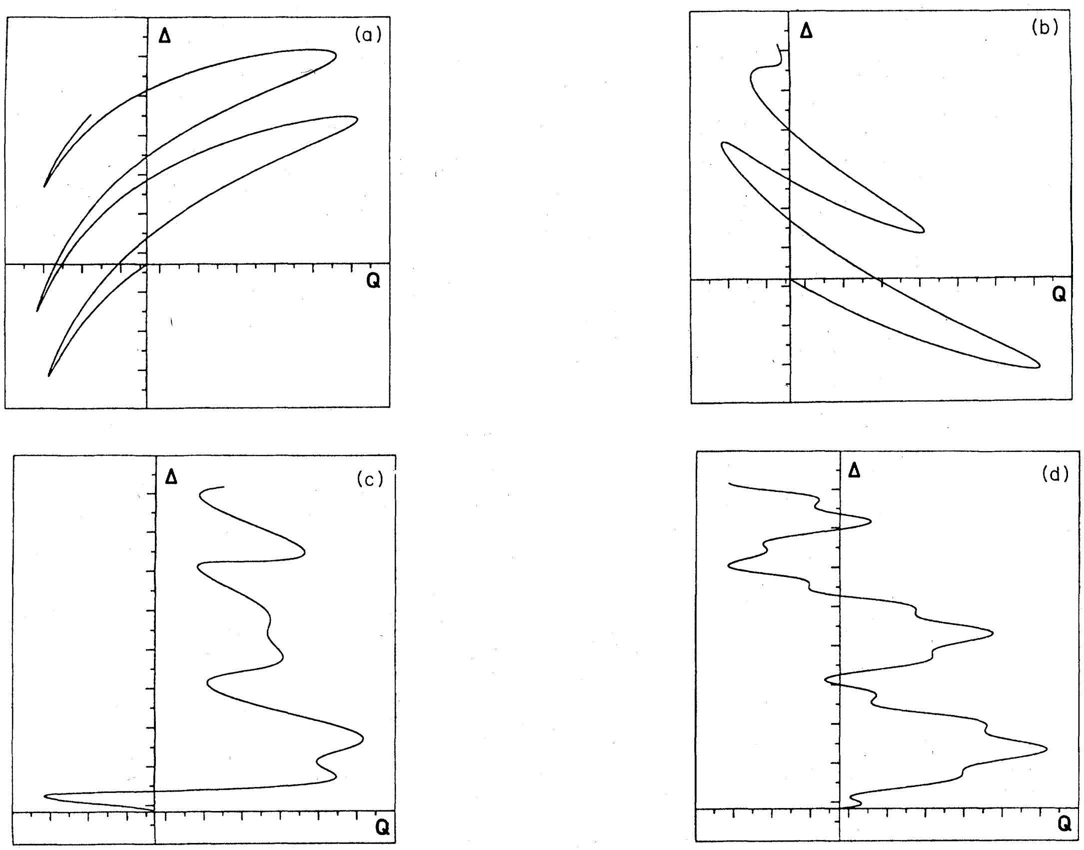

FIG. 1. (a)—(d) Modulation of parameters Q and Δ represented in the (Q,Δ) plane for various initial conditions (Q_0,Δ_0).

$$\mathcal{F}_1=0 , \tag{4.15a}$$

$$\mathcal{F}_2=f_1(\phi_0) , \tag{4.15b}$$

$$\mathcal{F}_3=-\left[\frac{\partial\phi_0}{\partial Q}\frac{dQ}{d\bar{\tau}}+\frac{\partial\phi_0}{\partial\Delta}\frac{d\Delta}{d\bar{\tau}}\right] , \tag{4.15c}$$

$$\mathcal{F}_4=f_2(\phi_0,v_0)-\left[\frac{\partial^2\phi_0}{\partial Q\,\partial\tau}\frac{dQ}{d\bar{\tau}}+\frac{\partial^2\phi_0}{\partial\Delta\,\partial\tau}\frac{d\Delta}{d\bar{\tau}}\right] , \tag{4.15d}$$

where f_1 and f_2 are defined by Eq. (3.2c). The derivatives $dQ/d\bar{\tau}$ and $d\Delta/d\bar{\tau}$ are provided by the modulation problem (4.13), while the partial derivatives involved in Eqs. (4.15) are readily computed from the solution (4.4). Of particular interest is the asymptotic behavior of the components of $\mathcal{F}$ for both X and τ going to infinity in a ratio $\xi=X/\tau>1$ (the X behavior is dominant). In these limit conditions, we obtain

$$\mathcal{F}_1=0 , \tag{4.16a}$$

$$\mathcal{F}_2\simeq-4Q\gamma\exp(\Omega\tau-QX+\delta) , \tag{4.16b}$$

$$\mathcal{F}_3\simeq-2(X/\Omega)(dQ^2/d\bar{\tau})\exp(\Omega\tau-QX+\delta) , \tag{4.16c}$$

$$\mathcal{F}_4\simeq iKv_0\{1-4\gamma^2\exp[2(\Omega\tau-QX+\delta)]\} -2X(dQ^2/d\bar{\tau})\exp(\Omega\tau-QX+\delta) , \tag{4.16d}$$

where $dQ/d\bar{\tau}$ is the asymptotic expression given in Appendix A. Expressions (4.16) are valid for $\xi>\gamma^{-1}=\Omega/Q$, so that both X and τ must go simultaneously to infinity along a characteristic line ξ where the wave has a speed greater than that of the soliton. The $\mathcal{F}_2$ component decreases for large X values, and thus behaves like the ϕ_0 solution at infinity. The $\mathcal{F}_2$ component decreases for large X values and thus behaves like the ϕ_0 solution at infinity. The component $\mathcal{F}_3$ decreases, but this decrease is accompanied by an oscillation (due to the presence of $dQ/d\bar{\tau}$). However, the $\mathcal{F}_4$ component has an oscillatory contribution v_0—an elastic wave—which dominates the other terms which are fastly decreasing.

1. Radiation in $v(X,\tau)$

The problem for v_1 can be stated thus:

$$\left[\frac{\partial^2}{\partial\tau^2}-\hat{V}_T^2\frac{\partial^2}{\partial X^2}\right]v_1=f_1(\phi_0) , \tag{4.17a}$$

$$v_1(X,0)=\frac{\partial v_1}{\partial\tau}(X,0)=0 . \tag{4.17b}$$

On account of the linearity of the problem, we should determine the derivatives $\dot{v}_\pm$ satisfying the problem (cf. Eq. (3.28)],

$$\left[\frac{\partial^2}{\partial\tau^2}-\hat{V}_T^2\frac{\partial^2}{\partial X^2}\right]\dot{v}_\pm=0\ ,\quad X\in(-\infty,+\infty) \tag{4.18a}$$

$$\dot{v}_\pm(X,\tau;\lambda)\simeq\frac{1}{\pi\lambda}\exp\{\mp i[k(\lambda)X+\omega(\lambda)\tau]\}\ ,\quad X\to\pm\infty \tag{4.18b}$$

where $k(\lambda)=\lambda$ and $\omega(\lambda)=\hat{V}_T\lambda$. The problem (4.18) here is simple because

$$\dot{v}_\pm(X,\tau;\lambda)=\frac{1}{\pi\lambda}\exp\{\mp i[k(\lambda)X+\omega(\lambda)\tau]\} \tag{4.19}$$

satisfies the problem and $O(\lambda)=1$. After noting that, here, $(d\lambda/\lambda)=\hat{V}_T(dk/\omega)$, and that both k and ω vary in the same manner as λ, and after a somewhat lengthy calculation, the Green function (3.17) is obtained in the form of a Fourier representation,

$$\mathcal{G}_c(X,\tau\,|\,X',\tau')=\frac{\hat{V}_T}{4\pi}\int_{-\infty}^{+\infty}\begin{bmatrix}\cos[\omega(\tau-\tau')] & (1/\omega)\sin[\omega(\tau-\tau')]\\ -\omega\sin[\omega(\tau-\tau')] & \cos[\omega(\tau-\tau')]\end{bmatrix}e^{-ik(X-X')}dk\ . \tag{4.20}$$

In order to obtain v_1, we especially need the component $G_{12}(X,\tau\,|\,X',\tau')$ of this Green function (cf. Appendix B). It is finally found that

$$v_1(X,\tau)=\tfrac{1}{4}\int_0^\tau\sin\phi_+(X,\tau;\tau')d\tau'+\tfrac{1}{4}\int_0^\tau\sin\phi_-(X,\tau;\tau')d\tau'\ , \tag{4.21}$$

where ϕ_+ and ϕ_- are zeroth-order solutions of ϕ_0 given by

$$\phi_\pm=\phi_0(Y_\pm,\tau')=-4\tan^{-1}\varphi_\pm\ , \tag{4.22}$$

with

$$\varphi_\pm=\gamma\frac{\sinh[\Delta(\tau')]}{\cosh[Q(\tau')Y_\pm]}\ ,\quad Y_\pm=X\pm\hat{V}_T(\tau-\tau')\ . \tag{4.23}$$

Clearly, v_1 consists of two radiations, v_1^+ and v_1^-, which correspond to the first and second integrals, respectively, in Eq. (4.21). From Eq. (4.22) we see that ϕ_+ is a soliton which propagates with speed $+\hat{V}_T$ (of transverse elastic waves), while ϕ_- is a soliton which propagates with speed $-\hat{V}_T$. Consequently, both $\sin\phi_+$ and $\sin\phi_-$ have the same features as ϕ_+ and ϕ_-, respectively. This means that, through electromechanical couplings, a "soliton-antisoliton" collision generates radiations of solitons v_1^+ and v_1^- of speeds $\hat{V}_T$ and $-\hat{V}_T$. Note that the speed of these radiated elastic solitons is larger than the one of the soliton in ϕ. These results are later illustrated in the numerical study. Finally, we note that the asymptotic behavior of v_1 for large X values can be examined either by looking for the asymptotic expression of $\sin\phi_\pm$ from the integral representations given in Appendix B or by considering the asymptotic form of Eqs. (4.17). Using the latter method, for $X\to\infty$ one thus obtains

$$v_1\simeq-\frac{8\gamma Q}{\Omega^2-\bar{\omega}^2}\{\sinh(\Omega\tau+\delta)-(\Omega/\bar{\omega})(\cosh\delta)[\sinh(\bar{\omega}\tau)]-(\sinh\delta)[\cosh(\bar{\omega}\tau)]\}e^{-QX}\ , \tag{4.24}$$

where $\bar{\omega}=\hat{V}_TQ$. If $\tau\to\infty$, then X and τ go simultaneously to infinity in a finite ratio $\xi=X/\tau>\hat{V}_T$. Therefore, the radiation v_1 behaves asymptotically like a soliton having two components, one propagating at a speed $\pm\gamma^{-1}=+\Omega/Q$—hence at the speed of the soliton waves in ϕ—and the other at a speed $\pm\hat{V}_T$ of elastic waves. By replacing Q by $-Q$ in Eq. (4.24), one obtains the behavior for $X\to-\infty$.

2. *Radiation in* $\phi(X,\tau)$

Here the problem is markedly more difficult than for v since both the operator $\mathcal{N}$ and the perturbing term are nonlinear, and the latter contains *both* v_0 and ϕ_0. First, one must search for the derivative $\dot{\boldsymbol{\phi}}_\pm(\dot{\phi}_\pm,\partial\dot{\phi}_\pm/\partial\tau)$, which must satisfy a problem of the type of (3.18). In particular, since $\dot{\boldsymbol{\phi}}_\pm$ is an element of $\mathcal{K}(\mathcal{N})$, the component $\dot{\phi}_\pm$ satisfies the linear problem

$$\left[\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial X^2}+\cos\phi_0\right]\dot{\phi}_\pm=0\ ,\quad X\in(-\infty,+\infty) \tag{4.25a}$$

$$\dot{\phi}_\pm(X,\tau;\lambda)\simeq\frac{1}{\pi\lambda}\exp\{\mp i[k(\lambda)X+\omega(\lambda)\tau]\}\ ,\quad X\to\pm\infty \tag{4.25b}$$

where the previously defined $k(\lambda)$ and $\omega(\lambda)$ check the dispersion relation $\omega^2(\lambda)=k^2(\lambda)+1$. The detail of the obtainment of the Green function $\mathcal{G}_c$ in this case is reported in Appendix C. The following remarks can be noted. The zeroth-order solution ϕ_0, a two-soliton solution, can be generated by means of a Bäcklund transformation. We also use such a transformation to obtain $\dot{\phi}_\pm$. However, while the solution ϕ_0 is defined in the absence of radiation, one must differentiate it with respect to the amplitude of the radiation of wave number k so as to generate solutions of Eqs. (4.25).

One therefore needs a generalization of solitons in which radiations are superimposed on solitons. Then, in applying the Bäcklund transformations, the idea is to start, not from a vanishing fundamental state, but rather from a zero state that is one of pure radiation at wave number k. Then the resulting wave solution can be differentiated with respect to the amplitude of this radiation and, only thereafter, is this amplitude set equal to zero.

The general Green function deduced by applying Eq. (3.17) is obtained in Appendix C. The radiation ϕ_1 can be written as

$$\phi_1=\int_0^{\tau}\int_{-\infty}^{+\infty} G_{11}(X,\tau\,|\,X',\tau')\mathscr{F}_3(X',\tau')dX'\,d\tau'+\int_0^{\tau}\int_{-\infty}^{+\infty} G_{12}(X,\tau\,|\,X',\tau')\mathscr{F}_4(X',\tau')dX'\,d\tau' , \tag{4.26}$$

where the components G_{11} and G_{12} are defined in Appendix C and $\mathscr{F}_3$ and $\mathscr{F}_4$ are defined in Eq. (4.15). A general study of the behavior of ϕ_1 here is out of the question on account of the complexity of the components of the Green function. However, an asymptotic estimate of ϕ_1 as $|X|$ goes to infinity is reasonable. Referring to the asymptotic behavior of the source terms—Eqs. (4.16)—we see that $\mathscr{F}_3$ goes to zero while $\mathscr{F}_4 \simeq iKv_0(X,\tau)$ for $|X|\to\infty$ and $\tau\to\infty$, which means that the contribution of $\mathscr{F}_3$ can be neglected compared to that of $\mathscr{F}_4$, so that only G_{12} will intervene in the asymptotic behavior of expression (4.26). However, as for v_1, we prefer to argue directly from the differential equation obtained by letting $|X|\to\infty$. Indeed, Eq. (2.2) becomes ($\phi\simeq\pi-\phi_1$)

$$\frac{\partial^2\phi_1}{\partial\tau^2}-\frac{\partial^2\phi_1}{\partial X^2}+\phi_1=iKv_0(X,\tau) , \tag{4.27a}$$

$$\frac{\partial\phi_1}{\partial\tau}(X,0)=\phi_1(X,0)=0 . \tag{4.27b}$$

Direct computations yield the following result ($|X|\to\infty$ with fixed τ):

$$\phi_1\simeq\frac{iK}{\Omega^2-\varpi^2}[(\omega/\Omega)i\sin(\Omega\tau)-\cos(\Omega\tau)-e^{+i\varpi\tau}]e^{iKX} , \tag{4.28}$$

where we have set $\varpi=\hat{V}_{\rm T}K$ and $\Omega^2=1+K^2$. In the case where there exists K such that $\varpi=\Omega$, we have resonance, and we shall have secondary secular terms. In reality, in order to smooth out this effect one should introduce relaxation terms for ϕ_1 (physically, an orientational relaxation of molecular groups). We note that when $|X|$ goes to infinity, the radiation ϕ_1 consists of a component which propagates at the speed $\hat{V}_{\rm T}$ of transverse elastic waves and another component which propagates at the speed Ω/K of the ferroelectric mode (or, in our model, the libration mode of the molecular groups).[23,24] This radiation superimposes itself on the solitons by means of electromechanical couplings.

To conclude this section it is of interest to gather, in one place, the common remarks regarding the radiations v_1 and ϕ_1. The essential point is that the perturbing term (small parameter η) plays the role of an electromechanical *resonance* coupling, since we have exchange of the nature of waves at the level of radiations. Concerning the elastic behavior, one can notice that $v_1(X,\tau)$ is a radiated soliton generated by the "soliton-antisoliton" collision through the coupling. This radiation can be schematized by the collision of two solitons traveling with speeds of $\pm\hat{V}_{\rm T}$. This is sketched out in the foregoing numerical study. Regarding the radiation ϕ_1, it behaves essentially like an harmonic wave traveling at the elastic speed $\hat{V}_{\rm T}$, but also possesses components with a speed $\Omega/K<\hat{V}_{\rm T}$. This radiation is generated by the electromechanical coupling and superimposes itself on solitons as will be shown in the upcoming numerical study.

V. NUMERICAL STUDY

A. Numerical scheme

In order to illustrate the interaction problem between solitons and harmonic waves, we shall treat the system (2.1) and (2.2) numerically. Intending to compare the present results with the analytical ones of Sec. IV, we shall first consider the "soliton-antisoliton" collision with coupling to an elastic wave. In this problem it is assumed that, initially, $v(X,0)$ and $\phi(X,0)$ are given by Eqs. (4.1) and (4.4). The numerical scheme must be selected in order to treat the system of nonlinear hyperbolic equations at hand in the easiest manner. All reasoning will be done on the Hamiltonian form (2.13), for which a "leap-frog" Lax-Wendroff scheme[25,26] is well adapted. If m and n denote vertical and horizontal nodes, then for a system such as (2.13), which reads.

$$\frac{\partial}{\partial t}\mathbf{U}+\frac{\partial}{\partial x}\mathbf{F}(\mathbf{U})=\mathbf{G}(\mathbf{U}) , \tag{5.1}$$

with

$$\mathbf{U}=(v,y,\alpha,\phi,z,\beta)^T ,$$

$$\mathbf{F}(\mathbf{U})=\left[0,\,\hat{V}_{\rm T}\left[\alpha+\frac{\eta}{\hat{V}_{\rm T}}\sin\phi\right],\,\hat{V}_{\rm T}y,0,\beta,z\right]^T , \tag{5.2}$$

$$\mathbf{G}(\mathbf{U})=(y,0,0,z,\,-\sin\phi-(\eta/\hat{V}_{\rm T})\cos\phi,\,0)^T ,$$

the discretization scheme is given by

$$\hat{U}^{\,n}_{m}=\hat{U}((m+\tfrac{1}{2})h_x,(n+\tfrac{1}{2})h_t)=\tfrac{1}{2}(U^n_m+U^n_{n+1})-\frac{\sigma}{2}(F^n_{m+1}-F^n_m)+\frac{h_t}{2}G^n_m ,$$
$$U^{n+1}_{m+1}=U((m+1)h_x,(n+1)h_t)=\tfrac{1}{2}(\hat{U}^{\,n}_{m}+\hat{U}^{\,m}_{m+1})-\frac{\sigma}{2}(\hat{F}^{\,n}_{m+1}-\hat{F}^{\,n}_{m})+\frac{h_t}{2}\hat{G}^{\,n}_{m} , \tag{5.3}$$

where $\sigma = h_x / h_t$, and h_x and h_t are steps in space and time, respectively. The caret over U, F, and G indicates that the function is evaluated at the intermediate point $((m+\frac{1}{2})h_x, (n+\frac{1}{2})h_t)$. The numerical scheme is stable for $\sigma \hat{V}_T < 1$ (Ref. 25), which yields the stability for Eqs. (2.13), since, in a general manner, $\hat{V}_T > 1$. The scheme is accurate to second order in h_x and h_t, and is started with the knowledge of $\mathbf{U}(x,0)$, the initial value that is assumed to equal the uncoupled solution. Theoretically, the problem takes place in the entire (x,t) plane. Obviously, conditions at x_R and x_L (the borders at the right and left, respectively), must be specified. It is naturally assumed that for sufficiently large $|X|$ values the solutions v and ϕ are very close to the uncoupled solutions. On the other hand, on account of the very expression of Eq. (2.1), we can write $v = v_0 + v_p$, where v_0 is the uncoupled solution in v and v_p is a particular solution such that $v_p(X,0) = [\partial v_p(X,0)/\partial t] = 0$. Similarly we have $\alpha = \alpha_0 + \alpha_p$, so that we will solve the problem for $v_p = v - v_0$, where v_p is of the order of η (the border condition can be of the same order). For $|X|$ sufficiently large, we consider the asymptotic expression obtained for v_1 in Sec. IV.

B. Results and comments

The above-described scheme allows one to compute and draw the curves $\phi(X,\tau)$ and $v_p(X,\tau) = v(X,\tau) - v_0(X,\tau)$ in perspective. Figure 2 gives the soliton-antisoliton—collision case that was studied analytically in Sec. IV. The first panel, Fig. 2(a), gives the evolution of the orientation ϕ (or $\theta = \phi/2$) of dipoles. In this case two solitons of equal but opposite speed meet, and, after the collision, each goes on, but with a reversed amplitude. In the same figure oscillations are superimposed and we recover the harmonic-wave radiation due to electromechanical couplings and the prediction in Sec. IV. The corresponding radiation of elastic waves is given in Fig. (9b), where curves are odd with respect to X. The principal peaks which constitute the graph follow the evolution of the solitons of Fig. 2(a). We also have a "collision" of these humps which behave like solitons, but these are "elastic displacement" solitons because, after collision, they pursue their way unaltered. In addition, we note two small humps at the left and right of the graph. These are the radiations of solitons at speeds $-\hat{V}_T$ and $+\hat{V}_T$, respectively. This numerical study illustrates efficiently the analytical results of Sec. IV, particularly insofar as the radiations ϕ_1 and v_1 are concerned. However, it is clearly difficult to comment on the modulation in speed and phase on these figures.

Cases which were not studied analytically in Sec. IV can be treated numerically. This is the case of the soliton-soliton collision in Fig. 3, where, in (a), we give the solution ϕ after filtering of the harmonic-wave radiation. The v_1 radiation is given in Fig. 3(b). Here the radiation is even with respect to X, and we also note the collision phenomenon between peaks as well as the radiation of solitons with speeds $-\hat{V}_T$ and $+\hat{V}_T$ (left- and right-hand bumps, respectively). If we consider the solution $\pi + \phi$ and then the rotation angle of dipoles $\theta = (\pi + \phi)/2$, then the soliton-soliton collision corresponds to a domain

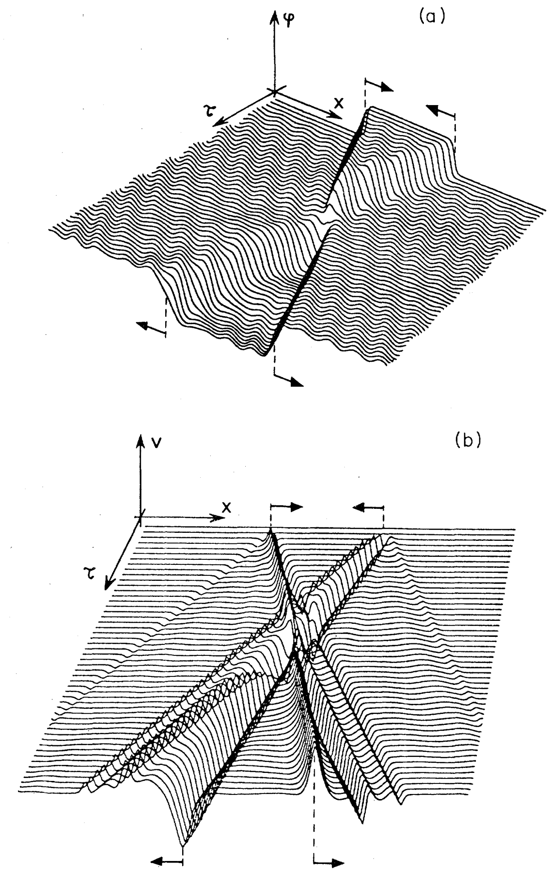

FIG. 2. Soliton-antisoliton collision. (a) Rotation $\phi(X,\tau)$ and radiation of harmonic waves; (b) associated radiation of elastic soliton waves.

oriented at $3\pi/2$, then at $\pi/2$, and finally at $-\pi/2$, while the electric dipoles rotate in the indirect sense on the passing of the wall. In the soliton-antisoliton collision we had a sequence of a domain oriented at $\pi/2$, then $3\pi/2$, and finally at $\pi/2$, but with the dipoles rotated in the direct sense on the passing of the first wall and in the indirect sense for the second wall. The first series of panels of Fig. 4 illustrates the case of an oscillating soliton ("breather"), i.e., waves localized in X which evolve in time, but with a wave number smaller than 1. It seems that no interpretation of this class of solitons in terms of domains and walls exists. Figure 4(b) gives the radiation in v, while the wave evolves along X, which is not the case in Fig. 4(a), which illustrates the ϕ solution. This is explained by the fact that $\phi(X,\tau)$ is localized by its very nature, whereas the v radiation is obtained after application of a wave operator which creates components which travel at the speed of elastic waves. Finally, Figs. 5(a) and 5(b) illustrate the case of a single soliton already examined in a previous paper (paper I) on the basis of the double sine-Gordon equation (2.3). However, we note a different behavior from this previous solution, because, in Ref. 1, we studied the motion of a single soliton in steady regime by looking for propagative-solution functions of

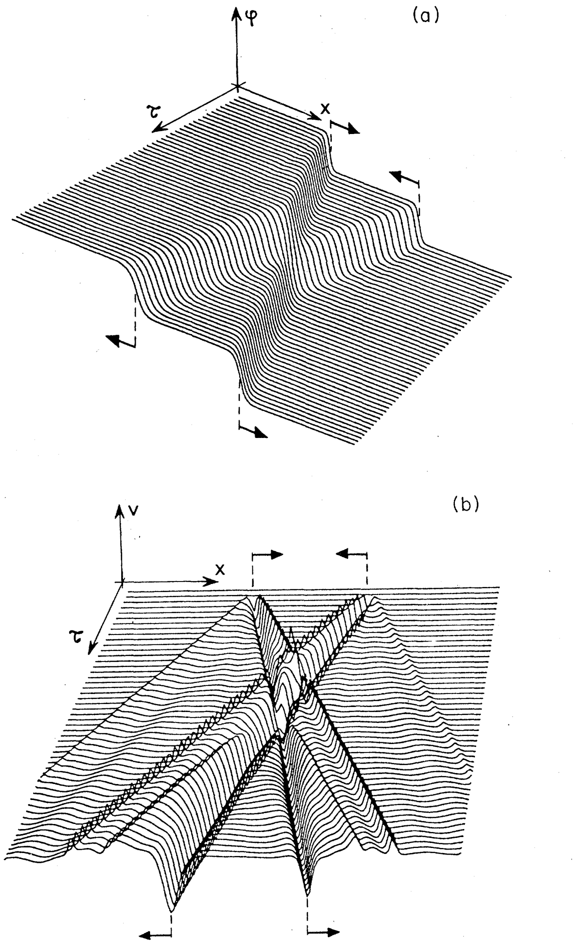

FIG. 3. Soliton-soliton collision. (a) Rotation $\phi(X,\tau)$; (b) associated radiation of elastic solitons.

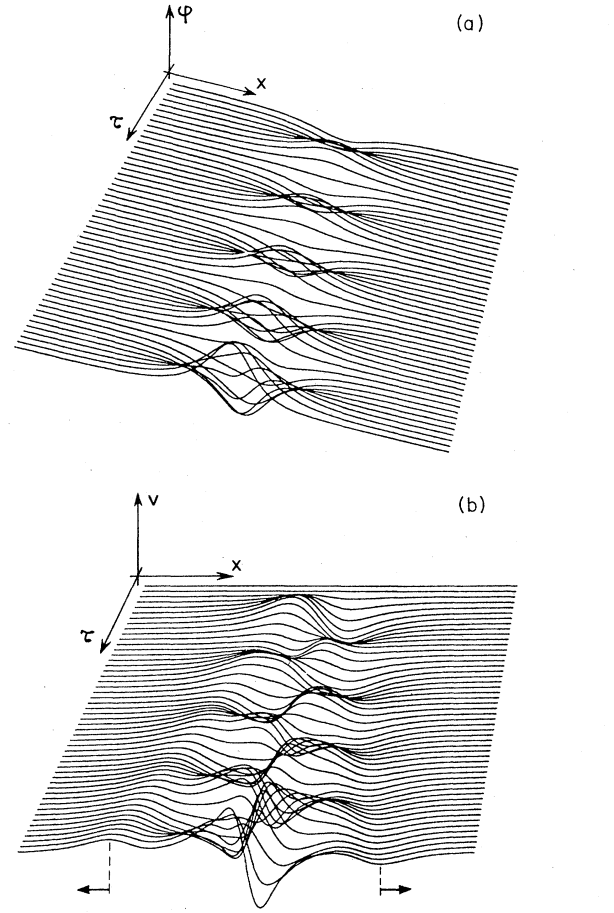

FIG. 4. Oscillatory soliton or breather. (a) Rotation $\phi(X,\tau)$; (b) associated radiation of elastic solitons.

$\xi = QX - \Omega\tau$ only. Here, the medium is originally at rest, containing an initial wall (at $t=0$, $v(X,0)=0$, $v_{,\tau}(X,0)=0$, and $\phi(X,0)=4\tan^{-1}\{\exp[-Q(X-X_0)]\}$), and then the wall is suddenly put in motion traveling from left to right, as shown in Fig. 5(a). Then the electromechanical coupling induces a nonlinear elastic wave, as represented in Fig. 5(b). In the latter panel we notice two types of waves, one made of a soliton (principal part of the curves) moving from left to right with the speed of a wall, and the second, a kind of soliton radiation, moving from right to left with the speed $\hat{V}_T$. Furthermore, in Fig. 5(a) the stationary perturbation increases, to some ex-

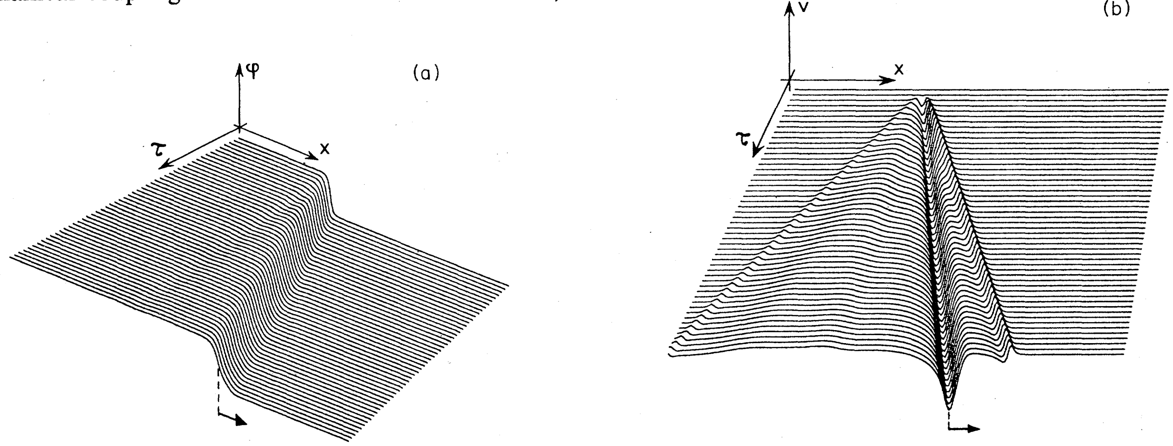

FIG. 5. Single-soliton case. (a) Rotation $\phi(X,\tau)$, motion of a 180° wall; (b) associated radiation of elastic solitons.

tent, the depth of the single soliton (not very marked). To explain the presence of this deepening effect, one should examine the correction of second-order ϕ_2. Tv sub {, tau} he present numerical study takes its place in a series of works[27,28] which use other numerical schemes to study a variety of equations (sine-Gordon equation, equation in ϕ^4, etc.). To end this point we note that the results illustrated in Fig. 5 may be related to the phenomenon of *acoustic emission* generated by the motion of a wall in a ferroelectric (cf. Ref. 29), but a theoretical support is missing from this conjecture.

VI. CONCLUDING SUMMARY AND PROSPECTS

In this two-part work we pursued several purposes within the framework of mechanics and physics. In the first part (Ref. 1) we first constructed a model of ferroelectric bodies in which a polarization is associated with the orientable molecular group (a prototype of which is provided by $NaNO_2$). This microscopic model was then used to elaborate on a more sophisticated model including nonlinearities related to the rotation of electric dipoles. Thereby, we described ferroelectrics where large variations in the electric polarization could be accounted for—hence a description well adapted to media with strong spatial disuniformities—which is the case of ferroelectrics with a *multidomain structure*. The domains are separated by walls which can be set in motion under the action of external stimuli (e.g., an electric field). A crucial point in the model obtained is the reciprocal influence of electromechanical couplings on the motion of walls and the mechanical behavior of the solid. This model necessarily implies couplings between perturbations in polarization (via the orientation of dipoles) and the elastic perturbations. Paper I concluded with the study of the relatively simple case provided by the motion of *one* ferroelectric wall. In the present part, of an obvious mathematical orientation, we have considered the problem of the interactions between a harmonic elastic wave and solitons in a broad sense (i.e., multiple-soliton solutions) by using a theory of perturbations. A zeroth-order solution was thus constructed with a temporal modulation of its speed and phase at the origin. Furthermore, first-order terms have allowed us to exhibit radiation phenomena, and a concluding numerical analysis has proved the correctness of the analytical considerations.

The *single-soliton* solution of paper I, insofar as the stable solution is concerned, can be interpreted as the motion of a so-called 180° wall. That is, starting from a globally paraelectric (no global remanent polarization) configuration, by moving inside the specimen, the wall motion transforms the specimen to a ferroelectric configuration. In other words, we can also say that one ferroelectric domain was created where there were initially two. This phenomenon is obviously closely related to the phase transition of such crystals. Indeed, from a macroscopic point of view, the paraelectric phase is characterized by a vanishing global polarization for $T > T_C$ (T_C denotes transition temperature). However, for $T < T_C$, a "remanent" or spontaneous polarization is present. The precise relationship between wall motion and phase transition is not simple. Nonetheless, if there is a sufficient input of energy (including caloric energy), we should have formation of domains and motion of walls so as to reach a ferroelectric phase. More precisely, if the energy input (electrostatic energy where the electric field is involved) is larger than the depolarization energy,[30,31] then domains form. Beyond these energy considerations, there is also an important "dynamic" point of view, namely the explanation of the anomaly in the speed of elastic waves in the neighborhood of the phase transition[32] (see also Refs. 30 and 31). This anomaly is carried to the level of the frequency spectrum in spectroscopic experiments. This is the "central-peak" phenomenon for which the wall motion is essential.[33] Experimentalists are interested in measuring a response function which is expressed by the correlation between elementary oscillations, and hence, that between quantities involving the solutions ϕ and v of the problem. Accordingly, it is in the neighborhood of the phase transition that the effects are of interest. However, it is also in this neighborhood that nonlinearities play a dominant role. In addition to the physical problem, we also think that we gave a very interesting example of the use of the double sine-Gordon equation. Finally, the model presented can be adapted to ferroelectrics with molecular groups [such as $NaNO_2$ or $SC(NH_2)_2$] in which an electromechanical coupling exists between modes of rotation of dipoles and the acoustic modes.[34,37] However, ferroelectric crystals such as $NaNO_2$ present more complex phase transitions for which a so-called *incommensurate,* intermediate phase fits in a narrow temperature interval between the disordered and ordered phases.[38,39] An attempt at the explanation, in the static case, of the formation of domains in terms of the incommensurate-commensurate phase transition, was given in the last section of paper I. In order to do this we introduced a continuous model in the manner of Landau, with the polarization as the order parameter, the model being completed by the elastic potential, electromechanical couplings in the form of electrostriction, and piezoelectricity when necessary, and a term of interaction between the polarization and its spatial gradient[40,41] that is necessary to account for the modulated incommensurate phase and which may be considered the analog of the Lifshitz invariant.[42] In this description we have a *vectorial* order parameter, the polarization with two components which allows one to model the evolution of the polarization within a wall. The model thus obtained provides an explanation of the formation of domains (in which the polarization is well defined) as the emergence of the ferroelectric phase within the incommensurate phase.

The present work has been devoted to a full exploitation of the coupled equations (2.1) and (2.2) by considering the coupling to yield a perturbation. A perturbation scheme has been devised which follows from the theory of singular perturbations. The idea consists of the elimination of secularly terms which result from resonances between the source (coupling term) and the zeroth-order solution. This is achieved by adjusting the free parameters of the zeroth-order solution. The use of the inverse-scattering method is avoided at this level, but its use becomes necessary in order to obtain the corrective terms at the first order. It also allows one to build an adequate basis on which the necessary Green function can be represented. The nature of the corrective term, for the elastic displacement, can be interpreted as the radiation of elastic waves in the form of solitons traveling at speeds of $\pm\hat{V}_T$, while, for the libration of electric dipoles, it can be interpreted as a radiation of harmonic waves traveling with the speed of solitons and elastic waves. As a matter of fact, only the asymptotic behavior of these radiations can be reasonably examined. The analysis is exemplified by the soliton-antisoliton collision, where the interaction between the harmonic elastic wave and the solitons is characterized, at the first order in the corrective term by an *exchange* of nature between the radiated waves. The soliton-antisoliton collision can be interpreted as the motion of two walls meeting one another in a three-domain specimen (two domains at 90° and one at −90°). The simple numerical scheme of Sec. V yields excellent results which have been easily generated to cases which were not studied analytically in the present work (e.g., soliton-soliton collision, breather soliton, and single soliton with radiation). One can envisage the adaptation of this numerical scheme to other cases of perturbation: forced oscillations, and relaxation of elastic waves (viscoelasticity), orientational relaxation of electric dipoles, and also perturbation of the motion of one or several walls by a defect (dislocation, point lattice defect, defect in polarization, etc.). This last point is essential in solid-state physics.

Obviously, the model of ferroelectrics with domain structure we have considered remains "ideal" since we have only two or three domains, but a real ferroelectric crystal presents many domains of various sizes and orientations (but always along crystallographic axes;[43–45] also see Ref. 30). In solid-state physics this real multidomain structure is taken into account by the introduction of a statistical theory using partition functions[46,47] (Ref. 33). In such an approach the case of a crystal with two domains is extended to a multidomain structure by computing the correlation function corresponding to the response of a specimen in a neutron-diffraction experiment. An altogether different view of the matter accounts for the periodic distribution of domains in many ferroelectric crystals. In this view it is possible to define a cell comprised of two 180° domains (Ref. 30) and then envisage a method of "homogenization" (i.e., obtaining a "macroscopic" behavior) by a method akin to the one now used in continuum mechanics. Other generalizations of the present model are, first, a *three*-dimensional lattice theory for molecular crystals of which the molecular group may be subjected to rotations of large amplitude (Refs. 35—37). Staying in the domain of the continuum, one can also envisage a *nonlinear "micropolar"* description of crystals with molecular groups.[48,49] In the latter case nonlinearities in the micropolar variables and electroacoustic interactions must be taken into account (ponderomotive force and couple).

In conclusion, let us emphasize several problems of practical and theoretical interest, the solution of which can be based on the equations provided in the present work. One is the problem of the reflection and diffraction of an acoustic wave by a wall (or walls) in a ferroelectric crystal. This is certainly a complex problem since one must study the superimposition of small signals (acoustic waves) on a nonhomogeneous state, taking nonlinear effects into account. For this one needs Eqs. (2.1) and (2.2), for which it was shown in paper I that a static solution $[v(X),\ \phi(X)]$ could be found which represents a wall separating two 180° domains; here, $v(X)$ is the elastic displacement induced by the inhomogeneous domain structure. Perturbations are considered about such a state by

$$v(X,\tau)=v_0(X)+\hat{v}(X,\tau),\quad \phi(X,\tau)=\phi_0(X)+\hat{\phi}(X,\tau)\ . \tag{6.1}$$

Then, Eqs. (2.1) and (2.2), linearized in the perturbations, read

$$\frac{\partial^2\hat{v}}{\partial\tau^2}-\hat{V}_T^2\frac{\partial^2\hat{v}}{\partial X^2}=-\eta\frac{\partial}{\partial X}(\hat{\phi}\cos\phi_0) \tag{6.2}$$

and

$$\frac{\partial^2\hat{\phi}}{\partial\tau^2}-\frac{\partial^2\hat{\phi}}{\partial X^2}=\left[\cos\phi_0+\eta\sin\phi_0\frac{\partial v_0}{\partial X}\right]\hat{\phi}+\eta(\cos\phi_0)\frac{\partial\hat{v}}{\partial X}\ . \tag{6.3}$$

Solutions are sought in the form

$$(\hat{v},\hat{\phi})=(V(X),\Phi(X))e^{-i\omega\tau}\ , \tag{6.4}$$

yielding

$$\hat{V}_T^2\frac{d^2V}{dX^2}=-\omega^2V+\eta\frac{d}{dX}(\Phi\cos\phi_0)\ , \tag{6.5a}$$

$$\frac{d^2\Phi}{dX^2}=-\omega^2\Phi-\left[\cos\phi_0+\eta\sin\phi_0\frac{dv_0}{dX}\right]\Phi-\eta\cos\phi_0\frac{dV}{dX}\ , \tag{6.5b}$$

which is an eigenvalue problem for which one must look for a complete basis of orthogonal eigenfunctions, with the help of which V and Φ can be represented. Then several particular problems can be envisaged[50–54]

(i) In the expansion of Eqs. (2.1) and (2.2) it is possible to introduce terms nonlinear in the perturbations $\hat{v}$ and $\hat{\phi}$, up to the third order, for instance, so that this anharmonic model allows one to study the interaction of phonons with ferroelectric domain walls. This leads to a problem of reflection and transmission of phonons by walls. The triggering of domain-wall motion by phonons resorts to the same approach. An example of this type of problem has already been given for a double-well potential.[55]

(ii) Another problem is the influence of external perturbations on the solutions v and ϕ for a stationary or moving perturbation [the transformations (2.6) can be used to return to the first case]. This leads to the study of the wall resonance[56,57] in the presence, or the absence, of dissipative processes. This problem can also be tackled with the aid of the perturbation scheme developed in Sec. III.

(iii) If the electromechanical coupling is envisioned as a perturbing effect, we can also view the interaction of phonons with a stationary or moving wall as acoustic perturbations $v(X,\tau)$ induced by the external stimulus. In this case the problem can be expanded in terms of η and the small signals $\hat{v}$ and $\hat{\phi}$. To this aim one considers an uncoupled solution (v_0,ϕ_0). Then an external perturbation is introduced (applied field, defect, etc.) which modifies the zeroth-order solution and generates the fields v_1 and ϕ_1, the v_1-v_0 and $\phi_1-\phi_0$ thus being the generated alterations. Then we account for the electromechanical couplings (terms in η) and look for perturbations induced by η, i.e., $\hat{v}$ and $\hat{\phi}$. The latter two fields are influenced by v_1 and ϕ_1, which are themselves consequences of the external stimulus, so that $\hat{v}$ and $\hat{\phi}$ finally correspond to the solution affected by both the external stimulus and the electromechanical couplings. Nonlinear terms can be included in such calculations if needed. This procedure, at least in theory, allows one to account for the mutual influence of the effects of external sources and couplings, and, thus, an action on the field of mechanical stresses (for instance, defects) will have an effect, through the electromechanical coupling, on the motion of a wall (alteration in the speed, for instance). The study of the interaction between a wall and a defect is of the utmost importance for a good knowledge of domain structure and crystallography.[58] Another problem involves the ultrasonic investigation of domains in ferroelectric crystals: if such a crystal is excited by an ultrasonic source, we can detect signals transmitted and reflected by moving or stationary walls. The detected mode is altered by the wall features (thickness, speed) and this, in theory, provides a nondestructive means of measurement.

APPENDIX A: ASYMPTOTIC BEHAVIOR OF THE SYSTEM (4.13)

For $\tau\to\infty$, $\beta\simeq\frac{1}{2}e^{\Delta}$ since $\Delta\to\infty$, and the system (4.13) yields

$$\frac{dQ}{d\bar{\tau}}\simeq-\frac{\pi\lambda}{2}\exp\left[-\frac{\pi\lambda}{2}-i(\bar{\omega}\tau-\xi_0)\right]\frac{\partial}{\partial\tau}[(1+\lambda^2)^{1/2}\cos(\lambda\Delta+\lambda\ln\gamma+A_l)]\,, \tag{A1}$$

$$\frac{d\Delta}{d\bar{\tau}}\simeq\frac{\Omega}{\eta}+\frac{\pi\lambda\Omega}{2}\exp\left[-\frac{\pi\lambda}{2}-i(\bar{\omega}\tau-\xi_0)\right]\left[\frac{\pi\lambda}{2}+Q\frac{\partial}{\partial Q}\left[\frac{(1+\lambda^2)^{1/2}}{Q}\cos(\lambda\Delta+\lambda\ln\gamma+A_l)\right]\right]\,, \tag{A2}$$

where A_l is such that $\tan A_l=\lambda$.

APPENDIX B: GREEN FUNCTION FOR RADIATIONS IN v

On account of f_1 given by Eq. (3.26), the G_{12} component of Eq. (4.20) is given by

$$G_{12}(X,\tau\,|\,X',\tau')=\frac{1}{4\pi}\int_0^\infty\sin(a_+k')\frac{dk}{k}-\frac{1}{4\pi}\int_0^\infty\sin(a_-k)\frac{dk}{k}\,, \tag{B1}$$

where

$$a_\pm=(X-X')\pm\hat{V}_T(\tau-\tau')\,. \tag{B2}$$

A tedious calculation yields

$$G_{12}=\tfrac{1}{4}\quad\text{for }(\tau',X')\in[0,X/\hat{V}_T]\times[\hat{V}_T(\tau'-\tau)+X\,,\;-\hat{V}_T(\tau'-\tau)+X]\,, \tag{B3a}$$

$$G_{12}=-\tfrac{1}{4}\quad\text{for }(\tau',X')\in[X/\hat{V}_T,\tau]\times[\hat{V}_T(\tau'-\tau)+X,\;-\hat{V}_T(\tau'-\tau)+X]\,, \tag{B3b}$$

$$G_{12}=0\quad\text{otherwise}\,.$$

APPENDIX C: GREEN FUNCTION FOR RADIATIONS IN ϕ (CF. Ref. 17)

Consider the coordinate change

$$X\to Z=\tfrac{1}{2}(X-\tau)\,,\quad \tau\to T=\tfrac{1}{2}(X+\tau) \tag{C1}$$

so that the sine-Gordon equation reads

$$\frac{\partial^2\phi}{\partial Z\,\partial T}=\sin\phi\,. \tag{C2}$$

Let ϕ_r be one solution of this equation. One can generate another solution ϕ through the Bäcklund transformation (Ref. 2),

$$\frac{1}{2}\frac{\partial}{\partial Z}(\phi-\phi_r)=4i\zeta\sin[\tfrac{1}{2}(\phi+\phi_r)]\,,\quad \frac{1}{2}\frac{\partial}{\partial T}(\phi+\phi_r)=\frac{1}{4i\zeta}\sin[\tfrac{1}{2}(\phi-\phi_r)]\,, \tag{C3}$$

where ζ is a parameter related to k and ω by

$$k=2\zeta-1/8\zeta\,,\quad \omega=2\zeta+1/8\zeta\,. \tag{C4}$$

In the case of a *two-soliton solution,* the integrals of Eqs. (C3) can be algebraically obtained from the commutativity of the Bäcklund transformation. Thus,

$$\tan[\tfrac{1}{4}(\phi_2-\phi_r)]=\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\tan[\tfrac{1}{4}(\phi_A-\phi_B)]\ , \tag{C5}$$

where $\phi_A(\phi_B)$ represents a single-soliton solution with parameter $\zeta_A(\zeta_B)$ and wave number $k_A=2\zeta_A-(8\zeta_A)^{-1}$ (k_B), and ϕ_2 is a Bäcklund transformation of ϕ_A (ϕ_B) through the parameter ζ_B (ζ_A). For the present problem, we assume that $\phi_r(X,\tau;\lambda,\rho)$ is a state of pure radiation with wave number $k(\lambda)$ and fixed amplitude ρ. Since ϕ_r depends on ρ, so does its Bäcklund transformation ϕ_2, which satisfies Eq. (C5). Therefore, we note

$$\dot\phi_r(X,\tau;\lambda)=\frac{\partial}{\partial\rho}\phi_r(X,\tau;\lambda,\rho)\bigg|_{\rho=0}\ ,\quad \dot\phi(X,\tau;\lambda)=\frac{\partial}{\partial\rho}\phi(X,\tau;\lambda,\rho)\bigg|_{\rho=0}\ , \tag{C6}$$

where $\dot\phi_r$ and $\dot\phi$ satisfy the equations

$$\frac{\partial^2}{\partial Z\,\partial T}\dot\phi_r=\dot\phi_r \tag{C7}$$

and

$$\frac{\partial^2\dot\phi}{\partial Z\,\partial T}+(\cos\phi_0)\dot\phi=0\ , \tag{C8}$$

respectively, where ϕ_0 is the two-soliton solution (4.4). Upon differentiating Eq. (C5) with respect to ρ and setting $\rho=0$, we obtain

$$\dot\phi_2=\dot\phi_r+\left[\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\frac{\cos^2(\phi_0/4)}{\cos^2[(\phi_{0A}-\phi_{0B})/4]}\right](\dot\phi_A-\dot\phi_B)\ , \tag{C9}$$

where ϕ_{0A} (ϕ_{0B}) represents a pure soliton with parameter ζ_A (ζ_B) and $\dot\phi_A$ $(\dot\phi_B)$ is the derivative of ϕ_A (ϕ_B) with respect to ρ. In fact, by differentiation of Eq. (C3) we obtain a linear first-order system in $\dot\phi_A$ and $\dot\phi_B$ which is easier to solve than Eq. (C8). Furthermore, since $\dot\phi_r$ satisfies Eq. (C7), any solution of this linear system will also satisfy Eq. (C8). In order to build the Green function, we must determine $\dot\phi_2^{\pm}$ from Eq. (C9). We can select a radiation in the form

$$\dot\phi_r^{\pm}=\frac{1}{\pi\lambda}f_{AB}(\lambda)\exp\{\mp i[k(\lambda)X-\omega(\lambda)\tau]\}\ , \tag{C10}$$

with

$$f_{AB}(\lambda)=\left[1+\left[\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\right]\left[\frac{\lambda+\zeta_A}{\lambda-\zeta_B}-\frac{\lambda+\zeta_B}{\lambda-\zeta_A}\right]\right]^{-1}. \tag{C11}$$

Then the linear system in $\dot\phi_A$ and $\dot\phi_B$ provides $\dot\phi_A^{\pm}$ and $\dot\phi_B^{\pm}$, and Eq. (C9) finally produces $\dot\phi_2^{\pm}$ as

$$\dot\phi_2^{\pm}=\left[1+\left[\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\right]\left[\frac{\cos(\phi_0/4)}{\cos[(\phi_{0A}-\phi_{0B})/4]}\right]^2\left[\frac{\zeta_A^2+\lambda^2\pm2\lambda\zeta_A\tanh\widetilde X_A}{\lambda^2-\zeta_A^2}-\frac{\zeta_B^2+\lambda^2\pm2\lambda\zeta_B\tanh\widetilde X_B}{\lambda^2-\zeta_B^2}\right]\right]\dot\phi_r^{\pm}\ , \tag{C12}$$

where $\widetilde X_A$ and $\widetilde X_B$ are such that

$$\cos(\phi_{0A(B)}/2)=\tanh\widetilde X_{A(B)}\ . \tag{C13}$$

Now applying the general representation (3.17), one obtains the Green function

$$\mathscr{G}_c(X,\tau\,|\,X',\tau')=\frac{1}{4i\pi\gamma^2}\int_{-\infty}^{+\infty}\begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\exp\{-i[k(\lambda)(X-X')-\omega(\lambda)(\tau-\tau')]\}f_{AB}^2(\lambda)\frac{d\lambda}{\lambda}\ . \tag{C14}$$

Being essentially interested in ϕ_1, we need the components G_{11} and G_{12}. On account of the expression of the components $\mathscr{F}_3$ and $\mathscr{F}_4$ of $\mathscr{F}$, we obtain

$$G_{11}=\dot\phi_2^{+}\frac{\partial\dot\phi_2^{-}}{\partial\tau}=-i\{1+F_{AB}(X,\tau)[G_A^{+}(\lambda)-G_B^{+}(\lambda)]\}\bigg[\omega(\lambda)-F_{AB}(X',\tau')[\widetilde G_A^{-\prime}(\lambda)-\widetilde G_B^{-\prime}(\lambda)] + \frac{\partial F_{AB}}{\partial\tau'}(X',\tau')[G_A^{-\prime}(\lambda)-G_B^{-\prime}(\lambda)]\bigg] \tag{C15}$$

and

$$G_{12}=-\dot{\phi}_2^+\dot{\phi}_2^-=-\{1+F_{AB}(X,\tau)[G_A^+(\lambda)-G_B^+(\lambda)]\}\{1+F_{AB}(X',\tau')[G_A^{-\prime}(\lambda)-G_B^{-\prime}(\lambda)]\}\ , \tag{C16}$$

where we have set $[\zeta=\zeta_A=-(16\zeta_B)^{-1}]$

$$G_{A(B)}^{\pm}(\lambda)=\frac{\zeta_{A(B)}^2+\lambda^2\pm 2\lambda\zeta_{A(B)}\tanh\widetilde{X}_{A(B)}}{\lambda^2-\zeta_{A(B)}^2}\ , \tag{C17}$$

$$\widetilde{G}_A^{\pm}=\pm\omega(\lambda)G_A^{\pm}\mp\frac{\partial}{\partial\tau}G_A^{\pm}=\pm\frac{1}{\lambda^2-\zeta^2}[\omega(\lambda)(\zeta^2+\lambda^2\pm 2\lambda\zeta\tanh\widetilde{X}_A)-2\lambda\zeta\Omega\,\mathrm{sech}^2\widetilde{X}_A]\ , \tag{C18}$$

$$\widetilde{G}_B^{\pm}=\pm\omega(\lambda)G_B^{\pm}\mp\frac{\partial}{\partial\tau}G_B^{\pm}=\pm\frac{1}{(16\zeta\lambda)^2-1}\{\omega(\lambda)[1+(16\zeta\lambda)^2\pm 32\lambda\zeta\tanh\widetilde{X}_B]+32\lambda\zeta\Omega\,\mathrm{sech}^2\widetilde{X}_B\}\ , \tag{C19}$$

$$F_{AB}(X,\tau)=\gamma\frac{1+\varphi^2\gamma^2}{1+\varphi^2}\ , \tag{C20}$$

and

$$\frac{\partial F_{AB}}{\partial\tau}(X,\tau)=-\frac{\gamma^2}{Q^3}\frac{\varphi^2}{(1+\varphi^2)^2}\coth\Delta\ , \tag{C21}$$

and we recall that sech $=1/\cosh$, $\varphi=\gamma\beta(\tau)/\alpha(X)$, and $G_A^{-\prime}$ ($G_B^{-\prime}$), are obtained by replacing $\widetilde{X}_A$ ($\widetilde{X}_B$) by $\widetilde{X}'_A$ ($\widetilde{X}'_B$), and, hence, (X,τ) by (X',τ').

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