PHYSICAL REVIEW B VOLUME 31, NUMBER 7 1 APRIL 1985

Crossover functions by renormalization-group matching: Three-loop results

J. F. Nicoll
Pangaro Incorporated, 800 Third Street, Washington, D.C. 20002

P. C. Albright*
Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742
(Received 7 June 1984; revised manuscript received 9 November 1984)

Crossover functions are obtained to three-loop order in an expansion around dimension $d=4$ for the ϕ^4 theory. Expressions for the internal energy, susceptibility, and correlation length are given for the general n-component system in zero magnetic field and above the critical temperature. The full Helmholtz free energy, magnetization equation of state, and susceptibility are given for the Ising-like $n=1$ case. Corrected values for the critical amplitude ratios as well as for the correction-to-scaling amplitude ratios are given to three-loop order.

I. INTRODUCTION

In this work, we extend earlier work[1–3] on crossover functions for thermodynamic functions in the critical region. These are obtained by a match point applied to the results of a conventional perturbation-theory expansion of the functions in question. Match-point methods were introduced by Nelson[4] as a simplification of the differential generator approach to nonlinear renormalization-group equations.[5] As such, they appeared to be limited to first-order, one-loop calculations. However, Bruce and Wallace[6] showed that the same match-point idea could be used with field-theoretic perturbation methods and could thereby be carried out to any desired order. Applications of the approach also have been made by Theumann.[7]

The method introduced in Ref. 6 utilizes a perturbation expansion calculated with a smooth cutoff, discarding effects which are only important when the correlation length approaches the effective cutoff distance (which models physical quantities such as the mean intermolecular spacing of a fluid or lattice spacing of a magnet). In that limit, relatively simple renormalization-group equations apply and can be solved to give a renormalized theory with a different "mass" or correlation length. By reducing the renormalized correlation length to a size comparable to the cutoff distance, a manageable perturbation theory is obtained, with the singularities resummed into crossover functions. In Ref. 1 it was pointed out that the choice of match point needs to be carefully considered to resum the series in the most effective way. For example, for n-component systems, the spherical model gives an exact result in the large-n limit which can be used as a guide for selecting a match point. The smooth cutoff approach was used by Refs. 1–3 (utilizing, as does Ref. 6, the diagrams calculated by Nickel[8]), but varied the match-point values for each function of interest in an attempt to capture the best result to two-loop order. At three loops, this procedure could again have been followed, but it was found to be convenient to use dimensional regularization[9] of the theory and choose the precise values of the renormalization factors to simplify the match-point values for the functions of interest. Technical details are given in Appendix A.

The crossover functions calculated by any of the match-point variants described above incorporate the asymptotic critical behavior and provide a resummation of the Wegner expansion[10] (of a limited number of Wegner correction-to-scaling operators, here one). Thus, in applications where the convergence of the Wegner expansion is in doubt (that is, where the critical region is apparently small), crossover functions are necessary. Far out from the critical region they must fail, due to the approximations made in the evaluation of the perturbation theory. These may be phenomenologically repaired to some degree; cf. Ref. 11.

Beyond giving a crossover form to the quantities of interest, the result of the present work also extends to higher order in the ϵ expansion (at $n=1$) values for the asymptotic critical ratios Γ_0^-/Γ_0^+ and $(H_0M_0^{\delta-1}/\Gamma_0^+)^{-1/\delta}$.[12] We provide corrected values for previously published ratios E_0^+/E_0^-, R_ξ^+,[13,14] a_m^-/a_χ^+, and a_c^+/a_ξ^+.[15] We confirm the results for a_χ^+/a_ξ^+ (Ref. 15) and the asymptotic equation of state given by Wallace and Zia.[16]

In Sec. II the results for the general n-component model are given for the internal energy, susceptibility, and correlation length in zero magnetic field, above the critical temperature. This particular choice is governed by the relations between their diagrammatic expansions, which allow a single match point to be useful for all three. Section III gives the complete Helmholtz free energy for the Ising-type $n=1$ system. Crossover functions as well as a parametric representation of the free energy including the first Wegner correction. The related single-component models of the Sak compressible-ferromagnet model[17] (the Fisher-renormalization[2,18] counterpart of the Ising model) and the asymmetric fluid model[3] will be discussed elsewhere. Appendix A describes the renormalization method employed in more detail, contrasting the smooth cutoff and dimensionally regularized approaches, and provides the formalism for the solution of the renormalization-group equations. Appendix B derives the transformation equations for parametric models under change of the parameters governing the temperature variable t.

II. RESULTS FOR n-COMPONENT SYSTEMS

For all the calculations of this work, we use the Wilson Hamiltonian in d dimensions for an n-component spin field:

$$H=\int d^dx\left[\tfrac{1}{2}t\boldsymbol{\phi}^2(x)+\tfrac{1}{2}\,|\nabla\boldsymbol{\phi}(x)|^2+\frac{u_0}{4!}\,|\boldsymbol{\phi}^2(x)|^2-\mathbf{h}\cdot\boldsymbol{\phi}(x)\right]. \quad (2.1)$$

The parameter t is proportional to the reduced temperature difference $(T-T_c)/T_c$, and h is the magnetic field. All initial evaluation of diagrams is performed without a cutoff, using dimensional regularization. The theory is made finite at $4-d=0$ by making scale changes in t, M, and u, and by adding a constant (proportional to t^2) to the Helmholtz free energy $A(M,t)$. (Note that we are always treating the reduced Helmholtz free energy A/k_BT.) The rescaled or (initially) renormalized quantities are given by

$$t\to Z_t(u)t\ , \quad (2.2a)$$

$$M^2\to Z_m(u)m^2\ , \quad (2.2b)$$

$$u_0=\Lambda^{\epsilon}uZ_u(u)\ , \quad (2.2c)$$

$$A\to A+\tfrac{1}{2}C(u)t^2\Lambda^{-\epsilon}\ . \quad (2.2d)$$

The parameter Λ is introduced to render u dimensionless. There is a relatively large amount of freedom in the choice of the Z's and $C(u)$. Each acceptable choice must make each perturbation diagram finite for fixed nonzero mass and $d=4$. It is convenient to choose them so that the diagrams for the internal energy E, susceptibility [χ^{-1} is given by the value of the two-point vertex function $\Gamma_2(k)$ at zero wave vector $k=0$], and correlation length ξ [obtained by $\xi^2\chi^{-1}=\partial\Gamma_2/\partial(k^2)\,|_{k^2=0}$] are individually zero at unit mass for all d. Any remaining freedom is used to simplify the four-point vertex function Γ_4. The diagram expansions for the internal energy, the two-point function, and the four-point function are given in Fig. 1. For convenience, a factor of $(S/2)^L$, where L is the number of loops and S is defined by

$$S=\Gamma(1+\tfrac{1}{2}\epsilon)\Gamma(1-\tfrac{1}{2}\epsilon)2^{1-d}\pi^{-d/2}/(\Gamma\tfrac{1}{2}d)\ ,$$

is taken out of each diagram (the factor of 2^L is included in the combinatorial factor for the diagram in question). This corresponds to making the transformations $u\to 2u/S$ with uA and uM^2 fixed. Note that this differs

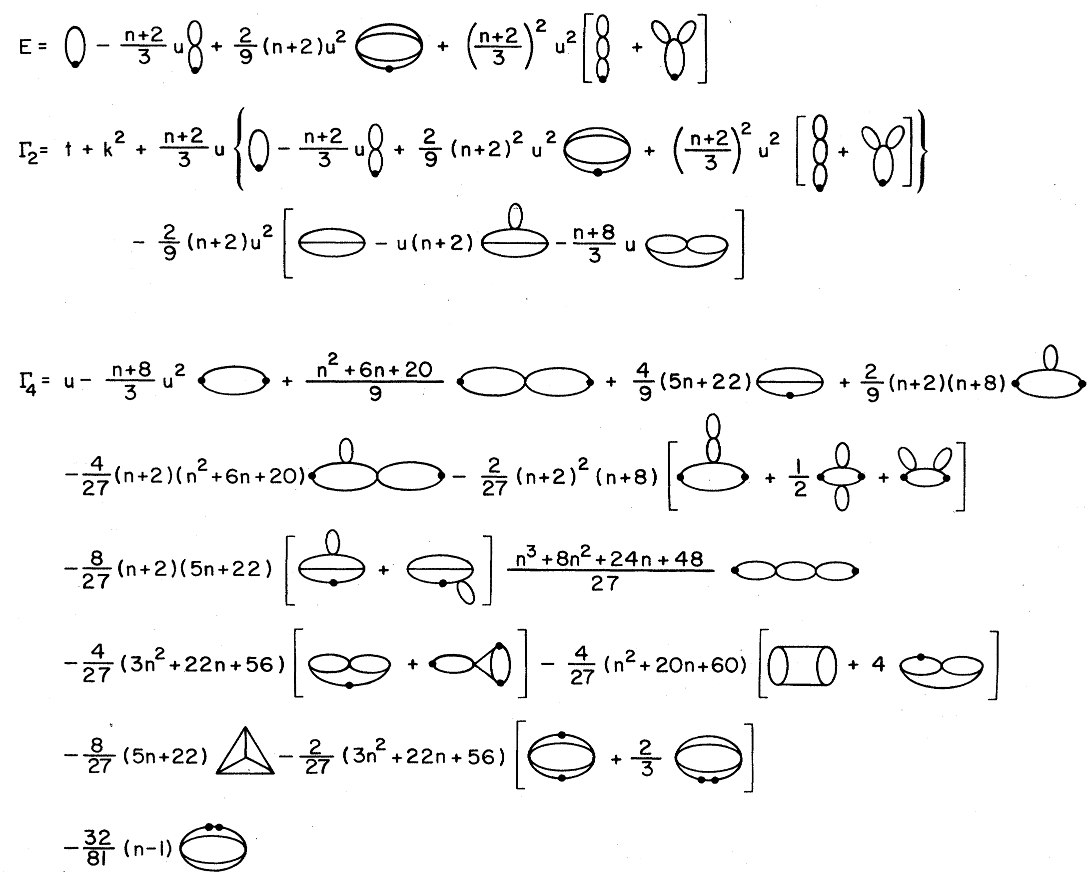


FIG. 1. Diagram expansions of the internal energy E, two-point function Γ_2, and four-point function Γ_4.

TABLE I. Renormalization constants.

$$Z_t = 1+\frac{n+2}{3}\frac{u}{\epsilon}+\left[\frac{n+2}{9}\right][(n+2)(1-2d_1)+2e_1\epsilon]\left[\frac{u}{\epsilon}\right]^2 + \left[\frac{n+2}{27}\right]\left[\frac{(n+2)^2}{2}(1+4d_6)+2(n+8)[d_3+\epsilon(e_2-2e_1)]\right]\left[\frac{u}{\epsilon}\right]^3$$

$$Z_u = 1+\frac{n+8}{3}\frac{u}{\epsilon}+\left[\frac{n^2+6n+20}{9}-\frac{4(5n+22)}{27}d_1-\frac{4(n+2)e_1\epsilon}{9}\right]\left[\frac{u}{\epsilon}\right]^2$$
$$+\left[\frac{n^3+8n^2+24n+48}{27}-\frac{4}{27}(3n^2+22n+56)\left[d_4+\frac{d_1}{3}-\frac{d_6}{6}\right]+\frac{4}{27}(5n+22)\zeta(3)\epsilon^2 - \frac{32}{81}(n-1)d_8\epsilon+\frac{4}{27}(n^2+20n+60)d_3-\frac{4(n+2)(n+8)}{27}(3e_1-e_2)\epsilon\right]\left[\frac{u}{\epsilon}\right]^3$$

$$Z_m = 1+\frac{2}{9}(n+2)e_1\frac{u^2}{\epsilon}+\frac{2(2e_1-e_2)}{27}(n+2)(n+8)\frac{u^3}{\epsilon^2}$$

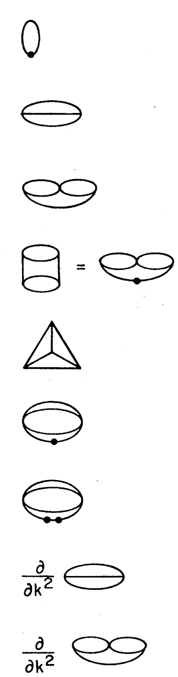
$$\epsilon C(u) = 1+\frac{n+2}{3}\frac{u}{\epsilon}+\left[\frac{n+2}{9}\right](2d_6+n+2)\left[\frac{u}{\epsilon}\right]^2$$

from the value of S given in Ref. 13 by the inclusion of the factor of $\Gamma(1+\frac{1}{2}\epsilon)\Gamma(1-\frac{1}{2}\epsilon)$. This change of scale in A needs to be taken into account in the two-scale universality factor. The renormalization constants obtained under this constraint are given in Table I and the values of the renormalized diagrams are given in Fig. 2; the coefficients c_i, d_i, and e_i are listed in Table II. The leading term of each (determined by c_i) is, of course, the value as given by dimensional regularization.

The redundancy of Λ leads to the renormalization-group equation:

$$\{\Lambda\partial_\Lambda+\beta(u)\partial_\mu+[2-1/\nu(u)]t\partial_t-\tfrac{1}{2}\eta(u)M\partial_M\}A = -\frac{nB(u)t^2\Lambda^{-\epsilon}}{2} . \quad (2.3)$$

Similar equations apply for the various correlation functions.[19] The functions appearing in Eq. (2.3) are determined by the renormalization factor, as described in Appendix A. Explicit expressions are given in Table III for the n-component system. Whenever they are written without an argument, the critical value is to be understood. Having rescaled the Hamiltonian in this discrete manner via the renormalization constants [Eqs. (2.2)] in order to make it finite at $d=4$, we may now solve Eq. (2.3) by the method of characteristics to study the behavior under more delicate scale changes. This is, in fact, formally equivalent to choosing a different set of Z's and a different value of C. However, we can allow the renormalization-group equation to choose these factors because of the existence of the fixed point $u=u^*$ [at which $\beta(u)=0$].[18] Using a reference value of unity for Λ,



$$-\frac{\kappa^2(\kappa^{-\epsilon}-1)}{\epsilon}$$

$$\kappa^2\frac{(c_1\kappa^{-2\epsilon}+3\kappa^{-\epsilon}+d_1)}{\epsilon^2}$$

$$\frac{\kappa^2(c_2\kappa^{-3\epsilon}-2c_1\kappa^{-2\epsilon}+d_2\kappa^{-\epsilon}+d_3)}{\epsilon^2}$$

$$\frac{c_4\kappa^{-3\epsilon}+\frac{2}{3}c_1(1-\epsilon)\kappa^{-2\epsilon}+(1-\frac{\epsilon}{2})\kappa^{-\epsilon}+d_4}{\epsilon^3}$$

$$\frac{\zeta(3)(\kappa^{-3\epsilon}-1)}{2\epsilon}$$

$$\frac{\kappa^2[c_3\kappa^{-3\epsilon}+3(1-\frac{\epsilon}{2})\kappa^{-2\epsilon}+d_5\kappa^{-\epsilon}+d_6]}{\epsilon^3}$$

$$\frac{-\frac{5}{12}\kappa^{-3\epsilon}+\frac{3}{4}(1-\frac{\epsilon}{2})\kappa^{-2\epsilon}+d_7\kappa^{-\epsilon}+d_8}{\epsilon^3}$$

$$\frac{e_1(\kappa^{-2\epsilon}-1)}{\epsilon}$$

$$\frac{e_2\kappa^{-3\epsilon}-2e_1\kappa^{-2\epsilon}+2e_1-e_2}{\epsilon^2}$$

FIG. 2. Values for the renormalized diagrams. c_i, d_i, e_i are given in Table II; $\zeta(3)=1.202\ldots$, ζ is the Riemann zeta function.

TABLE II. Coefficients for Fig. 2.

$$c_1=-\frac{3}{2}\left[1+\frac{\epsilon}{2}+\frac{\epsilon^2}{2}(1-\lambda)\right],\quad c_2=-\left[1+\frac{4\epsilon}{3}-\frac{3}{2}\lambda\epsilon^2+\frac{25}{12}\epsilon^2\right],$$

$$c_3=-\left[1-\frac{\epsilon}{3}-\frac{\epsilon^2}{4}\right],\quad c_4=\frac{1}{3}\left[1-\frac{\epsilon}{2}-\frac{\epsilon^2}{4}-\frac{3\lambda}{2}\epsilon^2\right],$$

$$d_1=-\frac{3}{2}\left[1-\frac{\epsilon}{2}-\frac{\epsilon^2}{2}(1-\lambda)\right],\quad d_2=-3+\epsilon+\epsilon^2(1-\lambda),$$

$$d_3=1-\tfrac{7}{6}\epsilon+(\lambda-\tfrac{5}{12})\epsilon^2,\quad d_4=-\frac{1}{3}\left[1-\frac{\epsilon}{2}-\frac{\epsilon^2}{4}\right],$$

$$d_5=-3+2\epsilon+\frac{5}{4}\epsilon^2-\frac{7\lambda}{6}\epsilon^2,\quad d_6=1-\frac{5}{6}\epsilon-\frac{3}{2}\epsilon^2+\frac{7\lambda}{6}\epsilon^2,$$

$$d_7=-\frac{1}{4}\left[1-\epsilon+\frac{2\lambda}{3}\epsilon\right],\quad d_8=-\frac{1}{2}+\frac{\epsilon}{8}+\frac{\lambda}{6}\epsilon,$$

$$e_1=-\frac{1}{8}\left[1-\frac{\epsilon}{4}-\frac{4\lambda}{3}\epsilon\right],\quad e_2=-\frac{1}{6}\left[1-\frac{\epsilon}{4}-2\lambda\epsilon\right]$$

the solution of Eq. (2.3) is of the form

$$A(t,u,M^2,\Lambda=1)=A(t\mathscr{T},u\mathscr{U}\exp(\epsilon l),\ \mathscr{D}M^2,\Lambda=\exp(-l))=-\tfrac{1}{2}nt^2\mathscr{K}\,.$$

The general expressions for the crossover functions $\mathscr{T}$, $\mathscr{D}$, $\mathscr{U}$, and $\mathscr{K}$ in Eq. (2.4) are given in Appendix A, Eq. (A15). To three-loop order they can be written as follows:

$$\mathscr{T}=Y^{(2-1/\nu)/\omega}\exp[\tau_1(p-\bar{u})+\tau_2(p^2-\bar{u}^2)]\,, \tag{2.5a}$$

$$\mathscr{D}=Y^{-\eta/\omega}\exp[d_1(p-\bar{u})+d_2(p^2-\bar{u}^2)]\,, \tag{2.5b}$$

$$\mathscr{U}=Y^{\epsilon/\omega}\exp[u_1(p-\bar{u})]\,, \tag{2.5c}$$

$$\mathscr{K}=\frac{k_0B(u^*)}{\bar{u}}\left[\frac{Y^{-\alpha/\omega\nu}-1}{\alpha/\epsilon\nu}+\frac{Y^{1-\alpha/\omega\nu}-1}{1-\alpha/\omega\nu}[\epsilon/\omega-1+k_1](1-\bar{u})+\frac{Y^{2-\alpha/\omega\nu}-1}{2-\alpha/\omega\nu}k_2(1-\bar{u})^2\right]. \tag{2.5d}$$

In Eqs. (2.5), $\bar{u}=u/u^*$, $p=u(l)/u^*$, $u(l)=\mathscr{U}\exp(\epsilon l)$, and $Y=(1-p)/(1-\bar{u})$. As $l\to\infty$, $p\to1$ and $u(l)$ approaches the fixed point.

Those constants not expressed as exponents are given in

TABLE III. Functions appearing in the renormalization-group equation.

$$B(u)=1+\frac{n+2}{18}b^*u^2$$

$$\beta(u)=-\epsilon u+\frac{n+8}{3}u^2+\beta_2u^3+\beta_3u^4$$

$$\text{where }\beta_2=\frac{2}{9}\left[\frac{3(3n+14)}{2}+\epsilon\left[\frac{41n+178}{8}-\frac{13n+62}{3}\lambda\right]\right]$$

$$\beta_3=\left[\frac{33n^2+922n+2960}{216}+\frac{n+8}{27}\left[\frac{41n+178}{8}-\frac{13n+62}{3}\lambda\right]+\frac{4}{9}(5n+22)\zeta(3)\right]$$

$$2-1/\nu(u)=\frac{n+2}{3}u+\frac{n+2}{3}\left[-\frac{5}{6}-\frac{25}{24}\epsilon+\frac{7\lambda}{9}\epsilon^2\right]^2+\frac{n+2}{3}\left[\frac{n+8}{9}\left[\frac{9}{2}-\frac{\lambda}{3}\right]+\frac{n+2}{18}b^*\right]u^3$$

$$\eta(u)=\frac{n+2}{18}\left[1-\frac{\epsilon}{4}-\frac{4\lambda}{3}\epsilon\right]u^2+\frac{2(n+2)(n+8))}{81}\lambda u^3$$

where $b^*=-\frac{11}{2}+\frac{14}{3}\lambda$, $\lambda\doteq1.171\,854\ldots$

TABLE IV. Constants appearing in the crossover functions.

$$\tau_1=\frac{n+2}{n+8}\epsilon\left\{\frac{13n+44}{2(n+8)^2}+\epsilon\left[\frac{-18n^2+691n+2072}{4(n+8)^3}+\frac{9(3n+14)(19n+72)}{2(n+8)^4}-\frac{24(5n+22)}{(n+8)^3}\zeta(3)-\frac{1}{(n+8)^2}\left[\frac{41n+178}{8}-\frac{13n+62}{3}\lambda\right]\right]\right\}$$

$$\tau_2=\epsilon^2\left[\frac{n+2}{n+8}\right]\left[-\frac{3n^2+460n+1184}{16(n+8)^3}+\frac{3(3n+14)(13n+44)}{4(n+8)^4}-\frac{6(5n+22)}{(n+8)^3}\zeta(3)+\frac{1}{(n+8)^2}\left[-\frac{57n+156}{16}+\frac{19n+68}{6}\lambda\right]\right]$$

$$d_1=-\frac{n+2}{2(n+8)^2}\epsilon\left[1+\frac{-n^2+92n+440}{4(n+8)^2}\epsilon\right],\quad d_1=-\frac{n+2}{4(n+8)^2}\epsilon^2\left[\frac{3(3n+14)}{(n+8)^2}+\frac{4\lambda}{3}\right]$$

$$u_1=\epsilon^2\left[\frac{9(3n+14)^2}{(n+8)^4}-\frac{33n^2+922n+2960}{8(n+8)^3}-\frac{12(5n+22)}{(n+8)^3}\zeta(3)-\frac{1}{(n+8)^2}\left[\frac{41n+178}{8}-\frac{13n+62}{3}\lambda\right]\right]$$

$$B(u^*)=1+\eta b^*,\quad k_0=\frac{1}{\epsilon}\exp[(2\tau_1-u_1)(1-\bar{u})+2\tau_2(1-\bar{u}^2)]$$

$$k_1=2\frac{\epsilon}{\omega}(\tau_1+2\tau_2+\eta b^*),\quad k_2=2\tau_1\left[1-\frac{\epsilon}{\omega}\right]-2\frac{\epsilon}{\omega}\left[\frac{\eta b^*}{2}+\tau_1^2+\tau_2\right]-u_1$$

Table IV. The exponent ω [$\equiv\beta'(u)$ at $u=u^*$] determines the value of the first Wegner correction-to-scaling exponent: $\Delta=\omega\nu$ ($=0.5$ at $d=3$ and $n=1$).[20] The variable p is a renormalization-group global invariant in the sense of Refs. 1–3. The use of the critical-point exponents in Eqs. (2.5) is partly arbitrary, since within this calculation, the exponents are only consistently known to third order. However, the promiscuous expansion of the exponents is dangerous as well, as will be illustrated in the discussion of the correction to scaling amplitude ratios below. Furthermore, if the crossover functions are to be applied in three dimensions, realistic values of the exponents can be used in place of the poorly convergent third-order expressions.[21] Note that Eqs. (2.5a)—(2.5c) are exact for $n=-2$ and $n=\infty$, as might be expected.

We now choose the match point $\exp(-l)=\kappa$, where κ^2 is the renormalized value of the mass parameter, $\kappa^2=\tau$. With this match point, the effective mass appearing in the diagrams is unity. Thus we have:

$$\chi^{-1}=t\mathscr{T}\mathscr{D}\ , \tag{2.6a}$$

$$\xi^{-2}\equiv\kappa^2=t\mathscr{T}\ , \tag{2.6b}$$

$$\frac{2E}{S}=-nt\mathscr{K}\ . \tag{2.6c}$$

Each of the functions of interest is given entirely in terms of the crossover functions, and the match-point mass κ is identical to the inverse correlation length. This is the advantage of the particular choice of the renormalization constants and match point. Not all functions can be simultaneously simplified; the value of the four-point function at the same match point is (with $\lambda\approx1.171\,954$)

$$\Gamma_4=\mathscr{D}^2\kappa^{-\epsilon}u^*p\left[1+\frac{u^*p}{6}(n+8)+\frac{(u^*p)^2}{9}\left[\frac{n^2+6n+20}{4}-(5n+22)[1+\epsilon(1-\lambda)]\right]+\frac{(u^*p)^3}{27}\left[\frac{n^3+8n^2+24n+48}{8}-\frac{3n^2+22n+56}{2}-(n^2+20n+6)(\tfrac{9}{2}\lambda)-\tfrac{2}{3}(3n^2+22n+56)(1-\tfrac{7}{12}\lambda)\right]\right]\ . \tag{2.6d}$$

Although the crossover functions are of interest in their complete form, it is also useful to expand them around the critical point. Giving only the first Wegner correction, they are given by

$$\mathscr{T}=T_0\kappa^{2-1/\nu}(1+T_1p_1\kappa^{\omega})\ , \tag{2.7a}$$

$$\mathscr{D}=D_0\kappa^{-\eta}(1+D_1p_1\kappa^{\omega})\ , \tag{2.7b}$$

$$\mathscr{U}=\frac{\kappa^{-\epsilon}}{\bar{u}}(1-p_1\kappa^{\omega})\ , \tag{2.7c}$$

$$\mathscr{K}=B(u^*)\frac{\alpha}{\nu}T_0^2\kappa^{-\alpha/\nu}(1+\bar{K}_1p_1\kappa^{\omega})\ . \tag{2.7d}$$

The value of the constant term in $\mathscr{K}$ has been dropped since it is only relevant in the full crossover function. The values of the coefficients are given in Table V. T_1, D_1, and K_1 may be considered universal quantities. The values of the universal amplitudes of both leading and nonleading singularities can now be easily determined. We define

$$\chi^{-1}=\Gamma_0^+ t^{\gamma}(1-a_\chi^+ t^{\Delta}) , \quad (2.8a)$$

$$E=\frac{E_0^+}{\alpha(1-\alpha)}t^{1-\alpha}\left[1+\frac{1-\alpha}{1-\alpha+\Delta}a_c^+ t^{\Delta}\right] , \quad (2.8b)$$

$$\xi^{-1}=(\xi_0^+)^{-1}t^{\nu}(1-a_\xi^+ t^{\Delta}) . \quad (2.8c)$$

The notation for the correction-to-scaling terms conforms with that of Ref. 13, giving precedence to the correction for the specific heat rather than the internal energy quoted. The Wegner expansion correction terms have amplitudes that have a simple form in terms of the expansions of the crossover functions and the critical-point exponents:

$$a_\chi^+ = -(\gamma T_1 + D_1) p_1 T_0^{\Delta} , \quad (2.9a)$$

$$a_c^+ = \frac{1-\alpha+\Delta}{1-\alpha}(\bar{K}_1-\alpha T_1)p_1 T_0^{\Delta}$$

$$= \frac{1-\alpha+\Delta}{1-\alpha}\left[\frac{2\alpha}{\alpha-\Delta}(T_1-\eta b^*)-\alpha T_1\right]p_1 T_0^{\Delta} , \quad (2.9b)$$

$$a_\xi^+ = -\nu T_1 p_1 T_0^{\Delta} . \quad (2.9c)$$

These forms are valid to all orders in terms of the expansion coefficients of the crossover functions (2.7), if the contribution ηb^* is recognized as $u(\partial/\partial u)B(u)$ at $u=u^*$. If the exponents and the crossover functions coefficients are expanded, a more opaque and partly misleading result is obtained, as illustrated in Table VI. Here we can see the value in retaining the exponents in these expressions. At $n=-2$, the susceptibility, correlation length, and internal energy are known exactly and have no Wegner corrections, as is shown in Eqs. (2.7). However, the ϵ expansion for the correction to the internal energy appears to be finite to lowest order and to have poles at $n=-2$ in the higher-order term. This arises from the expansion of $1-\alpha/\omega\nu=2(n+2)/(n+8)+O(\epsilon)$, which appears in the denominator of Eq. (2.9b) and Eq. (2.5d). We note that the expressions using the exponents are exact for both $n=-2$ an $n=\infty$, in sharp contrast to the corresponding ϵ expansions. This suggests that those expressions which must represent exponents should be kept in that form whenever possible. A somewhat similar attitude is taken in Ref. 21, which evaluates correction to scaling amplitudes directly at $d=3$.

The only universal ratio involving the leading-order behavior of these functions is the two-scale universality factor given by

$$(R_\xi^+)^d = E_0^+(\xi_0^+)^d = \frac{nS\nu(1-\alpha)}{2}B(u^*)$$

$$= \frac{nS\nu(1-\alpha)}{2}(1+\eta b^*) . \quad (2.10)$$

TABLE V. Coefficients of the expanded crossover functions for general n and for $n=1$.

$$T_0=\bar{u}^{(\nu^{-1}-2)/\epsilon}\exp(\{\tau_1-[(2-1/\nu)/\epsilon](1-u_1)\}(1-\bar{u})+\tau_2(1-\bar{u}^2))$$

$$T_1=-\{[(2-\nu^{-1})/\epsilon](1-u_1)+(\tau_1+2\tau_2)\}$$

$$=-\frac{n+2}{n+8}\left[1+\frac{13n+44}{(n+8)^2}\epsilon-\left[\frac{3(11n^3-48n^2-312n-922)}{4(n+8)^4}+\frac{12(5n+22)\zeta(3)}{(n+8)^3}-\frac{19n+68}{3(n+8)^2}\lambda\right]\right]\epsilon$$

$$=-\tfrac{1}{3}\{1+\tfrac{19}{27}\epsilon+[\tfrac{1271}{8748}+\tfrac{29}{81}\lambda-\tfrac{4}{9}\zeta(3)]\epsilon^2\} \quad (n=1)$$

$$D_0=\bar{u}^{\eta/\epsilon}\exp[(d_1+\eta/\epsilon)(1-\bar{u})+d_2(1-\bar{u}^2)]$$

$$D_1=\frac{\eta}{\epsilon}-(d_1+2d_2)=\frac{n+2}{(n+8)^2}\epsilon\left[1+\left[\frac{2\lambda}{3}+\frac{-n^2+92n+440}{4(n+8)^2}\right]\epsilon\right]=\frac{\epsilon}{27}\left[1+\left[\frac{2\lambda}{3}+\frac{59}{36}\right]\epsilon\right] \quad (n=1)$$

$$\bar{K}_1=2\left[\frac{\alpha}{\alpha-\Delta}\right](T_1-\eta b^*)=-\left[\frac{n-4}{n+8}\right]\left[1+\frac{24(n^3+6n^2+3n-28)}{(n+2)(n-4)(n+8)^3}\epsilon\right.$$

$$-\left[\frac{31n^6-102n^5-4788n^4-41\,615n^3-166\,560n^2-274\,860n-828}{2(n-4)(n+2)^2(n+8)^4}\right.$$

$$\left.\left.+\frac{12(5n+22)(5n-24)}{(n-4)(n+8)^3}\zeta(3)-\frac{2(13n+62)}{3(n+8)^2}\lambda\right]\epsilon^2\right]$$

$$=\tfrac{1}{3}\{1+\tfrac{16}{243}\epsilon-[\tfrac{3523}{2187}+\tfrac{76}{27}\zeta(3)-\tfrac{50}{81}\lambda]\epsilon^2\}$$

$$P_1=\frac{1-\bar{u}}{\bar{u}^{\omega/\epsilon}}e^{-(\omega/\epsilon)u_1(1-\bar{u})}$$

TABLE VI. Leading and correction-to-scaling amplitudes.

$$\xi_0^+ = T_0^\nu$$

$$a_\xi^+ = -\nu T_1 p_1 T_0^\Delta$$

$$= \frac{n+2}{2(n+8)}\left[1+\frac{n^2+36n+104}{2(n+8)^2}\epsilon+\left[\frac{n^4+26n^3+798n^2+3200n+5434}{4(n+8)^4}+\frac{19n+68}{3(n+8)^2}\lambda-\frac{12(5n+22)}{(n+8)^3}\zeta(3)\right]\epsilon^2\right]p_1 T_0^\Delta$$

$$= \tfrac{1}{6}\{1+\tfrac{47}{54}\epsilon+[\tfrac{1051}{2916}+\tfrac{29}{81}\lambda-\tfrac{4}{9}\zeta(3)]\epsilon^2\}p_1 T_0^\Delta \quad (n=1)$$

$$\Gamma_0^+ = D_0 T_0^\gamma$$

$$a_\chi^+ = -(\gamma T_1 + D_1)p_1 T_0^\Delta$$

$$= \frac{n+2}{n+8}\left[1+\frac{n^2+34n+88}{2(n+8)^2}\epsilon+\left[\frac{n^4+26n^3+696n^2+1928n+1696}{4(n+8)^4}+\frac{17n+52}{3(n+8)^2}\lambda-\frac{12(5n+22)}{(n+8)^3}\zeta(3)\right]\epsilon^2\right]p_1 T_0^\Delta$$

$$= \tfrac{1}{3}\{1+\tfrac{41}{54}\epsilon+[\tfrac{483}{2916}+\tfrac{23}{81}\lambda-\tfrac{4}{9}\zeta(3)]\epsilon^2\}p_1 T_0^\Delta \quad (n=1)$$

$$E_0^+ = T_0^{2-\alpha}\nu(1-\alpha)(1+\eta b^*)$$

$$a_c^+ = \frac{1-\alpha+\Delta}{1-\alpha}(\overline{K}_1-\alpha T_1)p_1 T_0^\Delta = \frac{1-\alpha+\Delta}{1-\alpha}\left[\frac{2\alpha}{\alpha-\Delta}(T_1-\eta b^*)-\alpha T_1\right]p_1 T_0^\Delta$$

$$= -\left[\frac{n-4}{n+8}\right]\left[1+\frac{n^4+35n^3+150n^2-112n-992}{(n+2)(n-4)(n+8)^2}\epsilon\right.$$

$$+\left[\frac{2(13n+62)}{3(n+8)^2}\lambda-\frac{12(5n+22)(5n-24)}{(n+2)(n-4)(n+8)^2}\zeta(3)\right.$$

$$+\frac{1}{4(n-4)(n+2)^2(n+8)^4}(2n^7+77n^6+2526n^5+25\,412n^4+127\,574n^3+339\,168n^2$$

$$\left.\left.+\;357\,464n-76\,000)\right]\epsilon^2\right]p_1 T_0^\Delta$$

$$= +\tfrac{1}{3}\{1+\tfrac{34}{27}\epsilon+[\tfrac{50}{81}\lambda-\tfrac{76}{9}\zeta(3)-\tfrac{9583}{8748}]\epsilon^2\}p_1 T_0^\Delta \quad (n=1)$$

$$a_c^- = \frac{Q_1^0(1-\frac{5}{12}Q_1^0)[1+(2-\alpha)T_1]-(u^*/\alpha-\Delta)\nu(1\ \eta b^*)(1-Q_1^0/2)^2[T_1(2-\alpha+\Delta)-\eta b^*]}{Q_1^0(1-\frac{5}{12}Q_1^0)-\dfrac{u^*}{\alpha}\nu(1+\eta b^*)(1-Q_1^0/2)^2}\left[\frac{1-\alpha+\Delta}{1-\alpha}\right]2^\Delta p_1 T_0^\Delta$$

$$= \tfrac{1}{9}\{1-\tfrac{13}{54}\epsilon-[\tfrac{10\,697}{17\,496}+\tfrac{31}{81}\lambda-\tfrac{5}{9}\zeta(3)]\epsilon^2\}2^\Delta p_1 T_0^\Delta$$

$$a_\chi^- = -\left[D_1+\gamma T_1-\frac{\Delta Q_1^0}{1+(Q_1^0/2)(2\beta-1)}\left[\beta T_1+\frac{1-D_1}{2}\right]\right]2^\Delta p_1 T_0^\Delta = \tfrac{1}{3}\{1+\tfrac{61}{27}\epsilon-[\tfrac{34}{243}-\tfrac{23}{81}\lambda+\tfrac{4}{9}\zeta(3)]\epsilon^2\}2^\Delta p_1 T_0^\Delta$$

$$a_m^- = \left[\beta T_1+\frac{1-D_1}{2}-\frac{\epsilon^2}{6}\right]2^\Delta p_1 T_0^\Delta = \tfrac{1}{3}\{1-\tfrac{13}{54}\epsilon-[\tfrac{16\,229}{26\,224}+\tfrac{35}{162}\lambda-\tfrac{2}{9}\zeta(3)]\epsilon^2\}2^\Delta p_1 T_0^\Delta$$

where $Q_1^0 = 3\left\{1-\frac{\epsilon^2}{12}\left[1+\frac{\epsilon}{3}\left[\frac{127}{36}-3\zeta(3)-\frac{\lambda}{2}\right]\right]\right\} = 2b^2$

Expressed in this form, it is exact for $n=-2$ and $n=\infty$. This differs from the value given in Ref. 13 (which is equivalent to a value for b^* of $2\lambda-\frac{2}{3}\pi^2-\frac{7}{2}=3.96$), but agrees with Ref. 22.

There are two *independent* universal ratios involving the first-order corrections to scaling:

$$\frac{a_\chi^+}{a_\xi^+}=2\left[1-\frac{\epsilon}{n+8}-\left[\frac{2\lambda}{3(n+8)}-\frac{n^2-15n-124}{2(n+8)^3}\right]\epsilon^2\right], \tag{2.11}$$

$$\frac{a_c^+}{a_\xi^+}=-2\left[\frac{n-4}{n+2}\right]\left[1+\frac{n^3+28n^2+52n-144}{2(n+2)(n-4)(n+8)}\epsilon + \left[\frac{7\lambda}{3(n+8)}-\frac{24(5n+22)(2n^2+9n-92)}{(n+2)(n-4)(n+8)^3}\zeta(3) - \frac{1}{4(n-4)(n+2)^2(n+8)^4}(23n^6+80n^5-5236n^4-66\,884n^3-328\,832n^2 - 620\,048n-250\,560)\right]\epsilon^2\right]. \tag{2.12}$$

The Riemann zeta function $\zeta(3)\approx 1.202\,05\ldots$ Equation (2.12) differs from the results of Ref. 15.

III. RESULTS FOR THE ISING-TYPE CASE

In this section we give complete crossover expressions and leading-order Wegner expansions for the Helmholtz free energy, magnetization equation of state, and susceptibility of the $n=1$, Ising-type case. A parametric representation of the Wegner expansion is given to facilitate comparison with earlier results and the parametric classification scheme of Ley-Koo.[23] We use the model Hamiltonian

$$H=\int d^dx\left[\frac{1}{2}t\phi^2+\frac{1}{2}|\nabla\phi|^2+\frac{1}{4!}u_0\phi^4-h\phi\right]. \tag{3.1}$$

The diagram expansion to third order is given in Fig. 3.

Using the crossover functions of Sec. II for $n=1$, one finds that at the match point, the Helmholtz free energy is given by

$$A=\frac{t\mathscr{T}\mathscr{D}M^2}{2}+\frac{u\mathscr{D}^2M^4}{4!}-\frac{t^2}{2}\mathscr{K}-\frac{\kappa^d}{2d}B(u^*p). \tag{3.2}$$

Note that the match-point variable κ is equal to the inverse correlation length only in the disordered phase (however, in that phase, it is the complete and not merely the asymptotic singular part of the correlation length). A complete crossover calculation of the inverse correlation length will be given elsewhere. Here, κ is simply the solution of the implicit equation:

$$\kappa^2=t\mathscr{T}+\frac{u\mathscr{U}\mathscr{D}M^2}{2}. \tag{3.3}$$

The magnetization equation and susceptibility at the same matching point are given by

$$\frac{h}{M}=t\mathscr{T}\mathscr{D}+\frac{u\mathscr{U}\mathscr{D}^2M^2}{6}+\frac{q\mathscr{D}\mathscr{U}^2(u^*p)}{4}\left[u^*p(4-\lambda)+qu^*p\left[\zeta(3)-\frac{2\lambda+1}{4}\right]-[1+\epsilon(1-\lambda)]\right], \tag{3.4a}$$

$$\chi^{-1}=\mathscr{D}\kappa^2\left[1+\frac{u}{4}\left\{\left[\frac{7}{2}u^2q^2\left[\zeta(3)-\frac{2\lambda+1}{2}\right]+\frac{u^2q}{2}(47-9\lambda)+\frac{u^2}{3}\left[-\frac{13}{2}+\frac{14}{3}\lambda\right] - \tfrac{5}{2}uq[1+\epsilon(1-\tfrac{7}{5}\lambda)]+u+2\right] +(2-q)\left[5u^2q\left[\zeta(3)-\frac{2\lambda+1}{4}\right]-3u^2(-4+\lambda)-3u[1+\epsilon(1-\lambda)]\right]\right\}\right]. \tag{3.4b}$$

In addition to the previously defined renormalization-group invariant p, which measures the approach to asymptotic behavior, there is a new invariant q, giving the relative size of the order parameter:

$$q=\frac{u\mathscr{U}\mathscr{D}M^2}{\kappa^2}. \tag{3.5}$$

The invariants p and q and the scaling field κ provide a parametric representation of the free energy and equation of state. However, the invariant q, although identically equal to 0 in the disordered phase and equal to 2 on the critical isotherm ($t=0$), is not a constant on the coexistence surface. This is a consequence of the match point used; if the h/M match point had been used, it would have had the value 3 on the entire coexistence surface, at the price of complicating the functional dependence of the crossover functions such as $\mathscr{D}$. On the coexistence surface, q has the crossover expression

$$q_{\rm cxs}=3[1-\tfrac{3}{4}u^2(l)([1+\epsilon(1-\lambda)] +u(l)\{[\tfrac{5}{2}\lambda-3\zeta(3)-\tfrac{13}{4}]\})]. \tag{3.6}$$

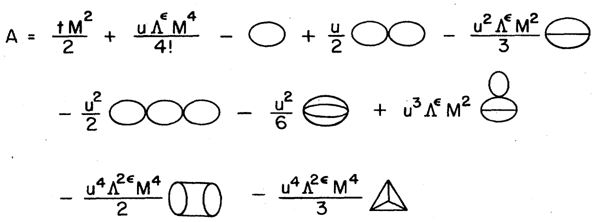


FIG. 3. Helmholtz free energy.

We define a new parametric variable $\theta^2=\pm 1$ on the coexistence surface by $\theta^2=q/q_{\rm cxs}$. Then, the h/M full crossover equation becomes

$$h/M=\mathscr{D}\kappa^2(1-\theta^2)(1+cp^3\theta^2)\ , \tag{3.7a}$$

where the constant c is given (in agreement with Wallace and Zia[16]) by

$$c=-\frac{\epsilon^3}{12}\left[\zeta(3)-\frac{2\lambda+1}{4}\right]\simeq -0.03\epsilon^3\ . \tag{3.7b}$$

The order parameter M is proportional to θ throughout the crossover region. Therefore, if c were equal to 0 the θ dependence would be that of the linear model. As pointed out by Ref. 16 for the asymptotic limit ($p=1$), a different choice of q can eliminate the correction term from the h/M equation and convert the model to the cubic model with an $O(\epsilon^2)$ cubic model parameter. The form used in Eqs. (3.7) is more convenient for our present purposes, but we will use this correspondence to refer to it as a "cubic" model.

Note that the deviations from the cubic model are small, even at $\epsilon=1$. Furthermore, preliminary analyses of fluid systems (cf. Ref. 11) indicate that the effective value of the coupling u is small compared to the fixed-point value, corresponding to a small asymptotic critical region. Thus, as the distance from the critical point increases, the invariant p decreases from unity, further diminishing the deviations from the linear model form. The remaining crossover effects in the h/M are also small, arising from the crossover function $\mathscr{D}$ and proportional to η/ω. Finally, note that the ϵ-dependent crossover effects in terms of the invariant p are determined to $O(\epsilon^N)$ for an N-loop expansion, but the crossover functions themselves are only determined to $O(\epsilon^{N-1})$.

These expressions are quite compact as written; some of their complexity is hidden in the crossover functions, which are themselves relatively complex. The full crossover expressions will be necessary in some circumstances (cf. Ref. 11); however, many applications require only an asymptotic expansion of the complete crossover functions. To simplify the exposition in the latter case, we may Wegner-expand the full crossover expressions to first order. The parametric expressions for t and M become

$$t=T_0^{-1}\kappa^{1/\nu}(1-b^2\theta^2)[1+T_1(\theta)\kappa^{\omega}]\ , \tag{3.8a}$$

$$M=M_0\kappa^{\beta/\nu}\theta[1+M_1\kappa^{\omega}]\ . \tag{3.8b}$$

This gives the magnetization equation of state

$$h/M=D_0\kappa^{2-\eta}(1-\theta^2)(1+c\theta^2)[1+H_1(\theta)\kappa^{\omega}]\ . \tag{3.9}$$

Although we have an explicit three-loop calculation for the thermodynamic function, and thus an $O(\epsilon^3)$ expression for the amplitude ratios, etc., this does not yield correspondingly explicit values for the parameters b^2 and c, which depend upon the details of the choice of parametric variables. Since we have chosen κ as both the distance parametric variable and the match-point value of $\exp(-l)$, rather than considering all possible distance variables compatible with the diagram expansion as in Ref. 16, we obtain a specific value for the parameter $b^2=q_{\rm cxs}(p=1)/2$:

$$b^2=\frac{3}{2}\left[1-\frac{\epsilon^2}{12}\right]\ . \tag{3.10}$$

Equation (3.10) is, of course, compatible with the conditions of Ref. 16. In fact, even if the defining equations (3.8) are retained, a freedom in the choice of the distance variable permits any value of b^2 as long as a corresponding freedom is permitted in Eq. (3.9). The parameters which define the correction to scaling portions of (3.9) are

$$T_1(\theta)=-\left[T_1+\frac{\theta^2\epsilon^2/4}{1-b^2\theta^2}\right]p_1\ , \tag{3.11a}$$

$$M_1=\left[\frac{1-D_1}{2}+\frac{\epsilon^2}{12}\right]p_1\ , \tag{3.11b}$$

$$H_1(\theta)=(D_1-3c\theta^2)p_1\ . \tag{3.11c}$$

There is a similar ambiguity in the values of these parameters. An exploration of the transformation of the equation of state under parameter change is given in Appendix B.

With these parametric variables, the Helmholtz free energy is given in parametric form by

$$A=\frac{\kappa^d}{u^*}\left\{\left[b^2\theta^2(1-\tfrac{5}{6}b^2\theta^2)-\frac{\alpha u^*}{2\nu}(1-b^2\theta^2)^2\right]\right.$$
$$\left.+p_1\kappa^{\omega}\left[b^2\theta^2(1-\tfrac{5}{6}b^2\theta^2)\left[1+\frac{\epsilon^2}{6}-\frac{(5\epsilon^2/24)\theta^2}{1-\frac{5}{6}b^2\theta^2}\right]+\frac{\alpha u^*}{2\nu}(1-b^2\theta^2)^2\left[\bar{K}_1-2T_1-\frac{\epsilon^2\theta^2/2}{1-b^2\theta^2}\right]\right]\right\}\ . \tag{3.12}$$

The universal critical amplitudes can be obtained from either the crossover or parametric expressions. We define

$$\chi_{\pm}^{-1}=\Gamma_0^{\pm}|t|^{\gamma}(1-a_{\chi}^{\pm}|t|^{\Delta})\ , \tag{3.13a}$$

$$E_{\pm}=\frac{E_0^{\pm}}{\alpha(1-\alpha)}|t|^{1-\alpha}\left[1+\frac{1-\alpha}{1-\alpha+\Delta}a_c^{\pm}|t|^{\Delta}\right]\ , \tag{3.13b}$$

$$M=M_0|t|^{\beta}(1+a_m|t|^{\Delta})\ . \tag{3.13c}$$

The results at $N=1$ are given in Table VI. We further define H_0 as the leading amplitude of the applied field along the critical isotherm:

$$H=H_0M^{\delta} . \quad (3.13d)$$

We thus have the leading universal amplitude ratios:[12]

$$\left[\frac{H_0M_0^{\delta-1}}{\Gamma_0^+}\right]^{-1/\delta} =3\left[\frac{2^{1-2\beta}}{27}\right]^{(\delta-1)/2\delta}\left[1+\frac{\epsilon^3}{54}\left[\zeta(3)-\frac{2\lambda+1}{4}\right]\right] , \quad (3.14a)$$

$$\frac{\Gamma_0^-}{\Gamma_0^+}=\frac{2(1+c)}{(b^2-1)^{\gamma-1}[1-b^2(1-2\beta)]} =2^{\gamma-1}\frac{\gamma}{\beta}\left[1+\frac{\epsilon^3}{36}\left[\zeta(3)+\frac{6\lambda+1}{4}\right]\right] , \quad (3.14b)$$

$$\frac{E_0^+}{E_0^-}=\frac{2^{\alpha}}{4}\left[1+\epsilon+\epsilon^2\left[\frac{43-9\lambda}{54}-\zeta(3)\right]\right] . \quad (3.14c)$$

Equation (3.14c) differs from that given in Ref. 13, which gives

$$\frac{E_0^+}{E_0^-}=\frac{2^{\alpha}}{4}\left[1+\epsilon+\epsilon^2\left[\frac{43-9\lambda}{54}-\zeta(3)+\frac{\zeta(2)}{6}\right]\right] ,$$

but agrees with Ref. 22.

We also have calculated the following correction-to-scaling amplitude ratio,

$$\frac{a_m^-}{a_\chi^+}=\left[1-\epsilon+\epsilon^2\left[\frac{15\,779}{26\,244}+\frac{\lambda}{2}-\frac{2\zeta(3)}{3}\right]\right]2^{\Delta} . \quad (3.15)$$

Reference 15 finds a_m^-/a_χ^+ to be $1-0.65\epsilon+2.23\epsilon^2$ at $n=1$, which we find to be

$$\frac{a_m^-}{a_\chi^+}=2^{\Delta}(1-\epsilon+0.39\epsilon^2)=1-0.65\epsilon-0.06\epsilon^2 .$$

There is too little information in these ϵ expansions to allow any sophisticated resummation procedure. It is possible, of course, to calculate elementary Padé approximants, and compare with the experimental and series-expansion results. Γ_0^-/Γ_0^+ we find to be 4.9; Pestak and Chen[24] find for N_2 and Ne the values 4.8 ± 0.6 and 4.8 ± 0.9, respectively. Sengers quotes a value of 4.89 for steam and 5.31 for ethylene,[25] Beysens *et al.*[26] quote 4.3 ± 0.3. The high-temperature series (HTS) expansion result is 5.07.[27] In experimental work, the ratio $H_0M_0^{\delta-1}/\Gamma_0^+$ is usually quoted rather than Eq. (3.14). We find $H_0M_0^{\delta-1}/\Gamma_0^+$ to be 1.67. Experimental results are 1.6 for steam and ethylene,[25] 1.75 ± 0.5 and 2.05 ± 0.8 for N_2 and Ne, respectively;[24] Ref. 26 gives 1.75 ± 0.3. The HTS result is 1.75.[27] E_0^+/E_0^- is less well behaved; the [1,1] Padé approximant gives a value of 0.44. Reference 25 quotes the experimental values to be 0.53 for steam and ethylene; the HTS result[27] is 0.51. The two-scale universality factor is given by $(R_\xi^+)^d=S\nu(1-\alpha)(1+\eta b^*)/2$ as before. The exceptionally small value of b^* is probably accidental. At higher order in ϵ the effective value could be larger but is expected to be of order unity. Thus the correction should always be $O(\eta)$. If we neglect the $O(\eta)$ piece, we find $R_\xi^+\approx0.2816$; Ref. 23 quotes results extending between 0.25 and 0.32. Finally, the correction-to-scaling amplitude ratio a_m^-/a_χ^+ is determined experimentally in Ref. 12 to be 0.95 ± 0.13 and 0.9 ± 0.2, but Pittman *et al.*[28] arrive at a value of 0.43 ± 0.08 for ^{3}He. We find (keeping the 2^{Δ} term explicit) Padé values ranging from 0.40 to 0.56.

ACKNOWLEDGMENTS

This work was supported by National Science Foundation Grant No. DMR-82-05356. The authors wish to express their gratitude to J. V. Sengers and M. Ley-Koo for many useful discussions. Computer time for this project was provided by the Computer Science Center of the University of Maryland.

APPENDIX A: SMOOTH CUTOFFS AND DIMENSIONAL REGULARIZATION

In this appendix, some remarks about the use of smooth cutoffs to regularize the field theory (as used in Refs. 1–3) as opposed to the dimensional regularization used in the present work are given. The latter method has allowed various Bruce-Wallace smooth cutoffs to be simulated and the nonuniversal aspects to be clarified.

A typical smooth cutoff Hamiltonian is given by

$$H=\int\left[\frac{1}{2}t\phi^2+\frac{1}{2}|\nabla\phi|_\Lambda^2+\frac{1}{4!}u\Lambda^{\epsilon}\phi^4\right]d^dx , \quad (A1)$$

with $|\nabla\phi|_\Lambda^2=|\nabla\phi|^2+|\nabla^2\phi|^2/\Lambda^2$. The existence of a renormalized theory in the infinite-Λ limit implies that the Helmholtz free energy A satisfies an equation of the form

$$\{\Lambda\partial_\Lambda+\beta\partial_u+[2-1/\nu(u)]t\partial_t-\tfrac{1}{2}\eta M\partial_M\}A=-nB(u) , \quad (A2)$$

when terms of order t/Λ^2 and $u\Lambda^{\epsilon}M^2/\Lambda^2$ are neglected. The functions $2-1/\nu$, η, β, and B are functions of u alone in that limit, thus facilitating the solution of Eq. (A2). This contrast with the differential generator approach (cf. Ref. 5) which, at least initially, includes such terms.

The renormalization-group equation for the free energy is solved by the method of characteristics. The renormalized parameters are defined by

$$\frac{\partial t}{\partial l}=-(2-1/\nu)t , \quad \frac{\partial u}{\partial l}=-\beta , \quad \frac{\partial M}{\partial l}=\frac{\eta}{2}M , \quad \Lambda=\exp(-l) , \quad (A3)$$

where, for convenience, the initial ($l=0$) value of Λ is chosen to be unity. Any other value can be introduced by conventional dimensional analysis. We define the formal solutions by

$$\mathscr{T}\equiv\exp-\int_0^l(2-1/\nu)dl,\quad u(l)=u\,\mathscr{U}\exp(\epsilon l)\ ,$$
$$\mathscr{D}=\exp\left[\int_0^l\eta\,dl\right],\quad \mathscr{K}=\int_0^l B[u(s)]\mathscr{T}^2(s)\exp(\epsilon s)ds\ . \tag{A4}$$

The free-energy relation Eq. (A2) becomes

$$A(t,u,M,\Lambda=1)=A(t\mathscr{T},u(l),\mathscr{D}^{1/2}M,\Lambda=\exp(-l))-\frac{t^2}{2}\mathscr{K}\ . \tag{A5}$$

The match-point method (not just for smooth cutoffs) consists in evaluating the right-hand side at a convenient value of $l=l^*$. This match point may be chosen in a variety of ways; in Refs. 1–3, the choice was dictated by the desire to have the perturbation series for thermodynamic functions of interest vanish at l^*.

It is well known that the critical part of the free energy is universal (within scale factors). Moreover, the entire crossover function for this Hamiltonian [neglecting the $O(t/\Lambda^2)$ terms] is universal within a single additional parameter measuring the distance of the actual value of u from its fixed point value u^*. However, it need not *appear* universal. To illuminate this issue, consider a theory without cutoff ($\Lambda\to\infty$ with $u\equiv u_0\Lambda^\epsilon$ fixed). If dimensional regularization is used then there will be no infinite Λ^2 mass counterterms. This theory could also be obtained by taking the infinite limit of any smooth-cutoff theory and is unique. However, it is divergent at $d=4$. These divergences can be removed by scale changes: the renormalization factors. For example, the bare free energy to two loops is given by

$$A_B=\frac{tM^2}{2!}+\frac{u_0M^4}{4!}-\frac{(\kappa^2)^2\kappa^{-\epsilon}}{2\epsilon(1-\epsilon/4)}+\frac{u_0(\kappa^2)^2\kappa^{-2\epsilon}}{2\epsilon^2}-\frac{\tilde{c}u_0^2M^2\kappa^2\kappa^{-2\epsilon}}{3\epsilon^2}\ , \tag{A6}$$

where $\kappa^2=t+(u_0/2)M^2$ and

$$\tilde{c}=-\frac{3}{2}\left[1+\frac{\epsilon}{2}+\frac{\epsilon}{2}(1-\lambda)\right],$$

with $\lambda\approx1.17$. The renormalized free energy A is given by

$$A(t,u,M,\Lambda)=A_B(z_tt,u_0=\mathscr{U}\Lambda^\epsilon Z_u,Z_m^{1/2}M)+\frac{C}{2}t^2\Lambda^{-\epsilon}\ . \tag{A7}$$

In Eq. (A7), a parameter Λ has been introduced simply to render u dimensionless. It is convenient to define $Z_1=Z_tZ_m$ and $Z_2=Z_uZ_m^2$. Then the choices,

$$Z_1=1+\frac{B_0u}{\epsilon}+\frac{B_0^2u^2}{\epsilon^2}\left[1-\frac{2\tilde{d}}{3}\right],$$
$$Z_2=1+3\frac{B_0u}{\epsilon}+\frac{B_0^2u^2}{\epsilon^2}(3-4\tilde{d})\ , \tag{A8}$$
$$C=\frac{B_0}{2\epsilon}\left[1+\frac{uB_0}{\epsilon}\right],$$

give a renormalized A given by

$$A=\frac{tM^2}{2}+\frac{u\Lambda^\epsilon M^2}{4!}-\frac{(\kappa^2)^2}{2\epsilon}\left[\frac{\kappa^{-\epsilon}}{1-(\epsilon/4)}-B_0\Lambda^{-\epsilon}\right]+\frac{u}{2}(\kappa^2)^2\Lambda^{-\epsilon}\frac{(\kappa/\Lambda)^{-\epsilon}-B_0}{\epsilon}-\frac{u^2}{3}M^2\kappa^2\frac{\tilde{c}(\kappa/\Lambda)^{-2\epsilon}+3B_0(\kappa/\Lambda)^{-\epsilon}+B_0^2\tilde{d}}{\epsilon^2}\ . \tag{A9}$$

For A to be finite at $d=4$, it suffices that $B=1+O(\epsilon)$, $3+\tilde{c}+\tilde{d}=0+O(\epsilon^2)$. The specific values encountered in the smooth cutoff of Refs. 1–3 are given by $B_0=1+\epsilon/2$ and $3+\tilde{c}+\tilde{d}=\frac{3}{8}(f-1)\epsilon^2$.

To determine Z_m, it is convenient to consider

$$R(k,\kappa^2)\equiv\partial\Gamma_2(k,\kappa^2)/\partial k^2\ .$$

For zero magnetization, one finds

$$R(k^2=0,\kappa)=1-\tfrac{2}{3}u_0^2\kappa^{-2\epsilon}\frac{e}{\epsilon}\ ,$$
$$R(k^2,\kappa=0)=1-\tfrac{2}{3}u_0^2k^{-2\epsilon}\frac{e'}{\epsilon}\ , \tag{A10}$$

with

$$e=-\frac{1}{8}\left[1-\frac{\epsilon}{4}-\frac{4\lambda\epsilon}{3}\right]$$

and

$$e'=-\tfrac{1}{8}(1+\tfrac{5}{4}\epsilon)\ .$$

Choosing

$$Z_m=1-\frac{2}{3}\frac{u^2}{\epsilon}\tilde{e}B_0^2\ , \tag{A11}$$

with $\tilde{e}=\frac{1}{8}+O(\epsilon)$ suffices to keep R finite at $d=4$. For the Bruce-Wallace smooth cutoff $\tilde{e}+e=\epsilon/4$.

Now the redundancy of the parameter Λ leads to the renormalization-group equation (A2) with

$$2-1/\nu-\eta\equiv-\beta\partial_u\ln Z_1\ ,$$
$$\epsilon+\beta/u-2\eta\equiv\beta\partial_u\ln Z_2\ ,$$
$$\eta\equiv\beta\partial_u\ln Z_m\ , \tag{A12}$$
$$B\equiv C[\epsilon-2(2-1/\nu)-\beta\partial_u\ln C]\ .$$

For the Z's given above, this leads to

$$2-1/\nu-\eta=B_0u+\frac{B_0^2u^2}{\epsilon}\left[-\frac{4\tilde{d}}{3}-2\right]\sim B_0u-u^2[1+O(\epsilon)]\ ,$$
$$\epsilon+\beta/u-2\eta=3B_0u+\frac{B_0^2u^2}{\epsilon}(-8\tilde{d}-12)\sim3B_0u-6u^2[1+O(\epsilon)]\ , \tag{A13}$$
$$\eta=\frac{4u^2B_0^2\tilde{e}}{3}\sim\frac{u^2}{6}[1+O(\epsilon)]\ .$$

The second line of each equation reflects the use of the ϵ expansion to control the perturbation theory. In this way the nonuniversal (and unspecified) portions of the constants $\tilde{d}$ and $\tilde{e}$ are lost in the flow equations and their solutions. The consequences may be seen by considering the formal solution to the flow equations. Writing

$$\ln Z_t = \sum_i Z_{ti} \left[\frac{u}{\epsilon}\right]^i ,$$

$$\ln Z_u = \sum_i Z_{ui} \left[\frac{u}{\epsilon}\right]^i , \qquad \text{(A14)}$$

$$\ln Z_m = \sum_i Z_{mi} \left[\frac{u}{\epsilon}\right]^i ,$$

then the solutions are given by

$$\mathcal{T} = \left[\frac{1-p}{1-\bar{u}}\right]^{(2-1/\nu)/\omega} \exp\left\{\sum_i \left[\frac{1}{i}(2-1/\nu)/\omega - Z_{ti}\left[\frac{u^*}{\epsilon}\right]^i\right](p^i-\bar{u}^i)\right\} ,$$

$$\mathcal{U} = \left[\frac{1-p}{1-\bar{u}}\right]^{\epsilon/\omega} \exp\left\{\sum_i \left[\frac{1}{i}\left[\frac{\epsilon}{\omega}\right] - Z_{ui}\left[\frac{u^*}{\epsilon}\right]^i (p^i-\bar{u}^i)\right]\right\} , \qquad \text{(A15)}$$

$$\mathcal{D} = \left[\frac{1-p}{1-\bar{u}}\right]^{-\eta/\omega} \exp\left\{\sum_i \left[-\frac{1}{i}\left[\frac{\eta}{\omega}\right] - Z_{mi}\left[\frac{u^*}{\epsilon}\right]^i\right](p^i-\bar{u}^i)\right\} ,$$

where $\bar{u}=u/u^*$ and $p=u(l)/u^*$. These equations simply state that, for example, $\mathcal{T}=Z_t(0)/Z_t(l)$, with the expected singularity explicitly displayed. The requirement of a finite $d=4$ limit shows that

$$\frac{(2-1/\nu)/\omega}{i} - Z_{ti}\left[\frac{u^*}{\epsilon}\right]^i \sim O(\epsilon^i) ,$$

$$-\frac{1}{i}\left[\frac{\eta}{\omega}\right] - Z_{mi}\left[\frac{u^*}{\epsilon}\right]^i \sim O(\epsilon^i) , \qquad \text{(A16)}$$

$$\frac{1}{i}\left[\frac{\epsilon}{\omega}\right] - Z_{ui}\left[\frac{u^*}{\epsilon}\right]^i \sim O(\epsilon^{i+1}) .$$

The evaluation of the differences in Eqs. (A15) in the ϵ expansion requires the $(i+1)$-loop terms for the first two and an $(i+2)$-loop calculation for the third. Thus, the difference

$$\frac{2-1/\nu-\eta}{2\omega} - \left[\frac{\hat{u}}{\epsilon}\right]^2 \left[\frac{1}{2}-\frac{\hat{d}}{3}\right], \quad \hat{u}=\frac{u^*}{B_0}$$

has no specific value at two-loop order, even if $\tilde{c}$ is known exactly, if the exponents are ϵ expanded. The nonuniversal features such as the number f do not appear. On the other hand, they do show up in the perturbation series. For example, in Ref. 1, the match point used for the equation of state was determined to be

$$\exp(-l^*) = \kappa^{\epsilon}(l)B_0\left[1-\epsilon\frac{\hat{u}}{4}p[(f-1)+\tfrac{1}{2}q(f+1)]\right]$$

where $q=u\mathcal{U}\mathcal{D}M^2/\kappa^2$. B_0 appears as a scale factor (corresponding to a cutoff parameter different from unity), while the actual value of f is irrelevant and will cancel in any universal ratio. Indeed, it could be adjusted to any value by means of a finite scale change. This freedom is exploited in the text to choose the Z's so that the match-point properties of the interval energy, susceptibility, and correlation length are simple. If such a standardization is made (and it can always be done), the crossover functions are standardized as well as the perturbation contributions, thus providing a completely universal crossover procedure.

This should not be taken as implying that these constants (such as f) have no physical meaning. Although they may be set to a prescribed value by scale changes, they are the remnant of the cutoff integrals of a physical cutoff in the large-Λ limit. They are the traces of genuine nonuniversal behavior when terms such as t/Λ^2 *cannot* be neglected.

Nonuniversal crossover functions may be simulated by choosing the Z's to be functions of κ^2/Λ^2 as well as u. This yields renormalization-group equations similar to those of differential generator approaches and can be used to consider systems with two different masses (cf. Ref. 29).

APPENDIX B: PARAMETER VARIATION IN THE PARAMETRIC EQUATION OF STATE

In this appendix the class of equivalent parametric representations for a given representations is studied. The existence of these classes shows that it is difficult to determine the values of the parameters precisely by perturbative means. A related discussion of these problems was given in Ref. 16. We begin with a one-term Wegner expansion in parametric form:

$$t=\kappa^{1/\nu}[(1-b^2\theta^2)+a\kappa^{\omega}(1-w\theta^2)] , \qquad \text{(B1a)}$$

$$M=M_0\kappa^{\beta/\nu}(1+aM_1\kappa^{\omega}) , \qquad \text{(B1b)}$$

$$\frac{h}{M}=\kappa^{2-\eta}(1-\theta^2)[C(\theta^2)+a\kappa^{\omega}D(\theta^2)] . \qquad \text{(B1c)}$$

For simplicity, we will consider variations of the parameters that leave the behavior at $\theta^2=0$ fixed. In the main text, this would correspond to maintaining the identification of κ with the inverse correlation length for zero-order parameter. Thus the parameter a in Eqs. (B1) is to be kept fixed. It is, of course, not universal. It is a measure of the relative size of the Wegner corrections and will vary from substance to substance. In the same way, the $C(0)$ and $D(0)$ are invariant. For simplicity the units have been chosen so that the susceptibility amplitude is unity at $\theta^2=0$. The parameters M_0 and M_1 determine the scales of the order parameter. The former is neither universal nor invariant, while the latter depends only on the values of the remaining parameters, b^2 and w. We now consider how the functions C and D vary when b^2 and w are varied. We must also allow the scale factors M_0 and M_1 to be adjusted in compensation. We take

$$b^2 \to b^2+db \ , \tag{B2a}$$

$$w \to w+dw \ , \tag{B2b}$$

$$M_0 \to M_0(1+dM_0) \ , \tag{B2c}$$

$$M_1 \to M_1+dM_1 \ . \tag{B2d}$$

These changes in the parameters require a corresponding redefinition of the parametric variables:

$$\kappa \to \kappa(1+\theta^2 d\kappa_0 + a\kappa^{\omega}\theta^2 d\kappa_1) \ , \tag{B2e}$$

$$\theta \to \theta[1+(1-\theta^2)d\theta_0 + a\kappa^{\omega}\theta^2 d\theta_1] \ . \tag{B2f}$$

Requiring that the form of Eq. (B1) remain invariant leads to

$$dM_0 = \frac{\beta}{b^2-1}db \ , \tag{B3a}$$

$$dM_1 = \frac{\beta}{b^2-1}dw + \left[\frac{\beta(1+\omega\nu)(1-w)}{(b^2-1)^2} + \frac{\omega\nu M_1}{b^2-1}\right]db \ . \tag{B3b}$$

These may be understood as simple rescaling to keep the Wegner expansion of the order parameter fixed on the coexistence surface. It proves convenient to define

$$R(\theta^2) = 1+(2\beta-1)b^2\theta^2 \ . \tag{B4}$$

Restricting ourselves to the calculation for C (the calculation for D is similar, albeit more tedious), we find, after some straightforward algebraic manipulations, that the change in C and D is given by

$$\frac{\partial C}{\partial b} = \frac{\theta^2 C(\theta^2)\alpha(2\beta-1)(b^2-\hat{b}^2)+2\beta\theta^2(1-\theta^2)C'(\theta^2)}{(1-b^2)R(\theta^2)} \ . \tag{B5}$$

The quantity $\hat{b}^2$ is the extremum value of b^2 for the linear model, at which, for example, the ratio of the susceptibility above and below the critical point is a maximum:

$$\hat{b}^2 = \frac{\delta-3}{(\delta-1)(1-2\beta)} \sim \tfrac{3}{2} + O(\epsilon^2) \ . \tag{B6}$$

These equations limit the validity of the values for the parametric variables found in perturbation theory, as may be seen by inserting those values into the variation equations. For example, the extremum values differ only at $O(\epsilon^2)$ from the value used in the text to fit the three-loop equation of state. In the first term of Eq. (B5), it is multiplied by $2\beta-1$, itself of $O(\epsilon)$. Moreover, the second term in Eq. (B5) is proportional to the derivative of C and is therefore of order ϵ^3. Thus, there is very little sensitivity to the value of b^2 to linear order. By taking the derivative of Eq. (B5), one sees that the second-order variation will have an important contribution. Thus an $O(\epsilon)$ variation in b^2 will have no linear effect, but the second-order variation will cause the parameter c in the function C to increase.[16] Thus, $O(\epsilon)$ contribution to b^2 may be fixed by minimizing the deviation from the linear model, but any $O(\epsilon^2)$ variation is acceptable, including the extremum value.

---

*Present address: National Bureau of Standards, Thermophysics Division 774, Gaithersburg, MD 20899.

[1]J. F. Nicoll and J. K. Bhattacharjee, Phys. Rev. B **23**, 389 (1981).
[2]J. F. Nicoll, Phys. Rev. B **24**, 388 (1981).
[3]J. F. Nicoll, Phys. Rev. A **24**, 2203 (1981).
[4]D. R. Nelson, Phys. Rev. B **11**, 3504 (1975); D. R. Nelson and E. Domany, *ibid.* **13**, 236 (1976).
[5]For reviews of the differential generator method, see J. F. Nicoll and T. S. Chang, Phys. Rev. A **17**, 2083 (1978); D. D. Vvedensky, T. S. Chang, and J. F. Nicoll, *ibid.* **27**, 3311 (1983).
[6]A. D. Bruce and D. J. Wallace, J. Phys. A **9**, 1117 (1976).
[7]W. K. Theumann, Phys. Rev. B **21**, 1930 (1980).
[8]B. G. Nickel (unpublished).
[9]For a treatment of dimensional regularization, see, for example, D. J. Amit, *Field Theory, the Renormalization-Group, and Critical Phenomena* (McGraw-Hill, New York, 1978).
[10]F. J. Wegner, Phys. Rev. B **5**, 4529 (1972).
[11]J. F. Nicoll and P. C. Albright, *Proceedings of the 8th Symposium on Thermophysical Properties,* edited by J. V. Sengers (American Society of Metallurgical Engineers, New York, 1982), p. 377.
[12]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6.
[13]C. Bervillier, Phys. Rev. B **14**, 4964 (1976).
[14]Y. Ikabe and K. Ideura, Prog. Theor. Phys. **66**, 1959 (1981).
[15]M.-C. Chang and A. Houghton, Phys. Rev. Lett. **44**, 785 (1980).
[16]D. J. Wallace and R. K. P. Zia, J. Phys. C **7**, 3480 (1978).
[17]J. Sak, Phys. Rev. B **10**, 3957 (1974).
[18]M. E. Fisher, Phys. Rev. **176**, 257 (1968).
[19]See, for example, E. Brézin, J. C. LeGuillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena,* Ref. 12, Vol. 6, pp. 125—247.
[20]See, for example, J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. **39**, 95 (1077); Phys. Rev. B **21**, 3976 (1980).

[21]C. Bagnuls and C. Bervillier, Phys. Rev. B **24**, 1226 (1981).
[22]C. Bervillier (private communication).
[23]M. Ley-Koo (unpublished).
[24]M. W. Pestak and M. H. W. Chan, Phys. Rev. B **30**, 274 (1984).
[25]J. V. Sengers, in *Phase Transitions: Cargèse 1980,* edited by M. Levy and J. C. Le Guillou (Plenum, New York, 1982).
[26]D. Beysens, A. Bourgou, and P. Calmettes, Phys. Rev. A **26**, 3589 (1982).
[27]A. Aharony and P. C. Hohenberg, Phys. Rev. B **13**, 3081 (1976).
[28]C. Pittman, T. Doiron, and H. Meyer, Phys. Rev. B **20**, 3678 (1979).
[29]I. D. Lawrie, J. Phys. A **14**, 2489 (1981).