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Phase transitions in uniformly frustrated XY models

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We develop the Hubbard-Stratanovich transform for uniformly frustrated XY models both on a square lattice and on a triangular lattice, and construct the Landau-Ginzburg-Wilson Hamiltonians, which reflect the formation of various superlattices according to values of the frustration f. Near the critical point the system $f=\frac{1}{4}$ on a triangular lattice is shown to belong to the same universality class as the fully frustrated ($f=\frac{1}{2}$) system on a square lattice. By decomposing two-mode systems into two coupled XY models and by applying the Migdal-Kadanoff approximation, we show the possibility of Ising-like or three-state Potts-like transitions in addition to the Kosterlitz-Thouless—like ones.

I. INTRODUCTION

The two-dimensional (2D) XY model has been studied quite extensively. At sufficiently low temperatures a combination of spin-wave theory and statistics of induced vortex-antivortex pairs predicts a power-law decay of correlation functions,[1] while above a certain critical temperature bound vortex-antivortex pairs dissociate, constituting the Kosterlitz-Thouless (KT) transition.[2,3] Application of this model to a number of physical systems including superconducting[4] and superfluid films[5] in general gives results in agreement with the experiments. In the case of homogeneous superconducting films an external magnetic field produces mutually repulsive vortices, which form a regular flux lattice at zero temperature. If the superconducting medium has a certain periodicity, however, new effects are expected to be observed because the interaction between the natural periodicity of the flux lattice and the periodicity of the superconductor produces a kind of commensurate-incommensurate effect as the external field is varied.

Indeed, recent experiments[6] on 2D arrays of coupled Josephson junctions have shown novel behavior, such as the oscillatory behavior of the resistance with the magnetic field, whose origin is believed to be the commensurate-incommensurate effect. Since relevant parameters are directly measurable or calculable, these Josephson junction arrays become useful in understanding and probing 2D physics, and especially can serve as a testing ground for statistical mechanics. This observation has created much interest in uniformly frustrated 2D XY models, which can serve as a model for the Josephson junction arrays, which have been studied using both the mean-field analysis[7,8] and numerical simulations.[8,9]

Understanding the statistical mechanics of such systems is a complex problem since at very low temperatures it appears likely that the free energy will form a "devil's staircase" as a function of the frustration f.[10] For certain simple values of f, low-lying excitations may be expressed in terms of fractionally charged "vortices" following Fradkin *et al.*[11]

In this paper we adopt an alternate approach which we believe should be useful at higher temperatures. In this approach we derive an effective Landau-Ginzburg-Wilson (LGW) free-energy functional for specific simple values of f through the use of a Hubbard-Stratanovich transform.[12] Although the resulting expansion (in powers of a coarse-grained order parameter) does not contain a small parameter, it does provide a systematic prescription for generating internal symmetries which are broken at a mean-field transition. Even at the level of mean-field theory, the broken symmetries of the system are a highly discontinuous function of f. Thus the LGW Hamiltonians for systems with different frustrations are expected to differ vastly from each other and to display interesting behavior.

Since the system is two dimensional, the mean-field transitions are suppressed by fluctuations. Our analysis suggests that for simple values of f, these fluctuations may be expressed in terms of coupled XY models. To obtain some idea about the nature of the transitions, we consider the application of the Migdal-Kadanoff approximation[13] to relatively simple cases, leaving more thorough analysis to further studies.

This paper consists of five sections and an appendix. Section II deals with the Hubbard-Stratanovich transform for the uniformly frustrated XY models. Section III constitutes the main part of this paper, and is devoted to the construction of the LGW Hamiltonians for several frustrated systems, both on a square lattice and on a triangular lattice. A natural property which is manifested by this approach is the formation of superlattices, which accounts for the local minima of resistance observed when the flux per cell is a half-integer or higher fraction of the flux quantum.[6] In Sec. IV the Migdal-Kadanoff approximation is discussed for systems described by two degenerate modes to show the possibility of Ising-like or Potts-like transitions as well as the KT-like ones. Finally, the main results are summarized in Sec. V. The appendix provides a derivation of the LGW Hamiltonian for two coupled XY models, which is the basis for the Migdal-Kadanoff approximation used in Sec. IV.

II. THE HUBBARD-STRATANOVICH TRANSFORM

We consider a class of uniformly frustrated XY models described by the Hamiltonian ($K \equiv \beta J \equiv J/kT$)

$$-\beta H = K \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j - A_{ij}) , \quad (2.1a)$$

where ϕ_i is the angle of the XY spin at site i, and A_{ij} is a bond angle such that the plaquette sum is constant over the whole lattice:

$$\sum_P A_{ij} = 2\pi f . \quad (2.1b)$$

With the identification

$$A_{ij} = \frac{2e}{hc} \int_i^j \mathbf{A} \cdot d\mathbf{l} , \quad (2.2)$$

the Hamiltonian given by Eq. (2.1) describes arrays of Josephson junctions in the high capacitance limit. In Eq. (2.2), $\mathbf{A}$ is the vector potential, which may be taken to be that of a uniform transverse magnetic field B in the limit of large penetration depth. The uniform frustration f is then related to B by the relation

$$f = BA_P/\Phi_0 , \quad (2.3)$$

where A_P is the area of a plaquette, and $\Phi_0 \equiv hc/2e$ is the flux quantum.

We write Eq. (2.1) in the compact form

$$-\beta H = \tfrac{1}{2} v^\dagger P v - K_N , \quad (2.4a)$$

where P is a Hermitian and positive definite matrix with elements

$$P_{ij} = K_{ij} e^{-iA_{ij}} + \delta_{ij} \sum_l K_{il} ,$$

$$K_{ij} = \begin{cases} K & \text{for nearest neighbors} , \\ 0 & \text{otherwise} , \end{cases} \quad (2.4b)$$

$K_N \equiv \frac{1}{2} \sum_{i,j} K_{ij}$ is a constant, and v is a column vector with components

$$v_i = e^{-i\phi_i} . \quad (2.4c)$$

To diagonalize the Hamiltonian, it is convenient to work with the Fourier transform

$$v_i = N^{-1/2} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}_i} \tilde{v}_{\mathbf{q}} ,$$

$$P_{ij} = N^{-1} \sum_{\mathbf{q},\mathbf{q}'} e^{i(\mathbf{q}\cdot\mathbf{x}_i - \mathbf{q}'\cdot\mathbf{x}_j)} \tilde{P}_{\mathbf{q},\mathbf{q}'} , \quad (2.5)$$

where $\mathbf{x}_i = (x_i, y_i)$ is the position vector of the site i, and the sums are over the Brillouin zone. Also note that because of A_{ij}, P_{ij} in general does not have translational invariance, i.e., $P_{ij} \neq P_{|i-j|}$. Diagonalization of the Hamiltonian now can be performed by the similarity transform of $\tilde{P}$ with respect to the unitary matrix S whose columns consist of the eigenvectors of $\tilde{P}$. The Hamiltonian reaches its lowest value when $\tilde{v}$ is one of the eigenvectors of $\tilde{P}$ associated with the largest eigenvalue. This may not represent a true ground state of the system since the eigenvectors with the largest eigenvalue may not satisfy the constraint $v_i^* v_i = 1$ or

$$\sum_{\mathbf{p}} \tilde{v}^*_{\mathbf{p}} \tilde{v}_{\mathbf{p}-\mathbf{q}} = \delta_{\mathbf{q},0} . \quad (2.6)$$

This constraint, however, is in general hypothesized to be irrelevant to asymptotic criticality, and consequently the modes associated with the largest eigenvalue are expected to become critical as the critical point is approached from above.[14–16] (There may exist two or more eigenvectors that give the largest eigenvalue. This corresponds to the fact that there exist two or more degenerate critical modes, which is a characteristic of usual frustrated systems.)

To investigate fluctuations from the critical modes, we now apply the Hubbard-Stratanovich transform to Eq. (2.4), and get the partition function in terms of continuous complex spin variables z_i, omitting overall constants,

$$\begin{aligned} Z &\equiv \mathrm{Tr} e^{-\beta H} \\ &= \mathrm{Tr} e^{(1/2) v^\dagger P v} \\ &= \int \prod_{i=1}^{N} dz_i dz_i^* I_0(|z_i|) e^{-(1/2) z^\dagger P^{-1} z} , \end{aligned} \quad (2.7a)$$

where I_0 is the modified Bessel function expanded as

$$I_0(x) = \exp[x^2/4 - x^4/64 + O(x^6)] . \quad (2.7b)$$

Introduction of the Fourier transform defined in Eq. (2.5) allows us to obtain the diagonalized form

$$z^\dagger P^{-1} z = \tilde{z}^\dagger \tilde{P}^{-1} \tilde{z} = \sum_{\mathbf{q}} \psi^*(\mathbf{q}) \lambda_{\mathbf{q}}^{-1} \psi(\mathbf{q}) , \quad (2.8)$$

where $\lambda_{\mathbf{q}}$'s are the eigenvalues of $\tilde{P}$, and $\psi(\mathbf{q})$ is the complex order parameter defined to be

$$\psi \equiv S^\dagger \tilde{z}$$

or (2.9)

$$\psi(\mathbf{q}) \equiv \sum_{\mathbf{q}'} S^\dagger_{\mathbf{q}\mathbf{q}'} \tilde{z}_{\mathbf{q}'} .$$

The partition function given by Eq. (2.7) is now reduced to the desired form

$$Z = \int D\psi \, e^{-F(\psi)} , \quad (2.10)$$

where $D\psi \equiv \prod_{\mathbf{q}} d\Psi(\mathbf{q})$ is the functional integral, and $F(\psi)$ is the LGW Hamiltonian

$$F(\psi) = \tfrac{1}{2} \sum_{\mathbf{q}} (\lambda_{\mathbf{q}}^{-1} - \tfrac{1}{2}) \psi^*(\mathbf{q}) \psi(\mathbf{q}) + \frac{1}{64N} \sum_{\mathbf{q}\text{'s}} \sum_{\mathbf{p}\text{'s}} S_{\mathbf{q}\mathbf{p}} S_{\mathbf{q}'\mathbf{p}'} S_{\mathbf{q}''\mathbf{p}''} S_{\mathbf{q}'''\mathbf{p}'''} \psi^*(\mathbf{p}) \psi^*(\mathbf{p}') \psi(\mathbf{p}'') \psi(\mathbf{p}''') \delta_{\mathbf{q}+\mathbf{q}',\mathbf{q}''+\mathbf{q}'''} + O(\psi^6) . \quad (2.11)$$

The condition that the Hamiltonian reaches the lowest value now reduces to the condition that $\psi(\mathbf{q})$ vanishes unless $\mathbf{q}$ is one of $\mathbf{Q}$'s, where $\lambda_{\mathbf{Q}}$ is the largest eigenvalue. Small but nonzero $\psi(\mathbf{q})$'s for $\mathbf{q}$'s around $\mathbf{Q}$'s then represent fluctuations.

III. CONSTRUCTION OF THE LANDAU-GINZBURG-WILSON HAMILTONIANS

In this section we apply the Hubbard-Stratanovich transform to the cases $f=\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ both on a square lattice and on a triangular lattice, and construct explicitly the LGW Hamiltonian for such systems. The asymmetric gauge $\mathbf{A}=xB\hat{\mathbf{y}}$ will be used throughout this paper. The lattice constant is also set equal to 1 ($a=1$) for simplicity.

A. Square lattices

On a square lattice it is straightforward to evaluate the bond angle given by Eq. (2.2):

$$A_{ij}=\begin{cases}\pm 2\pi f x_i & \text{for } \mathbf{x}_j=\mathbf{x}_i\pm\hat{y}\\ 0 & \text{for } \mathbf{x}_j=\mathbf{x}_i\pm\hat{x}\ .\end{cases} \tag{3.1}$$

Figure 1 shows a square lattice and its Brillouin zone. Equation (3.1) allows one to calculate the inverse transform of (2.5):

$$P_{qq'}=N^{-1}\sum_{i,j}e^{-i(\mathbf{q}\cdot\mathbf{x}_i-\mathbf{q}'\cdot\mathbf{x}_j)}P_{ij}$$
$$=K\delta_{q_2q_2'}R_{q_1q_1'} \tag{3.2}$$

with

$$R_{q_1q_1'}=2(\cos q_1)\delta_{q_1q_1'}+e^{iq_2}\delta_{q_1+2\pi f,q_1'}+e^{-iq_2}\delta_{q_1-2\pi f,q_1'}\ , \tag{3.3}$$

where only the off-diagonal elements of P_{ij} are considered.[17] Thus the infinite matrix $\widetilde{P}$ takes a block-diagonal form, and we need to work with just one block $R(q_2)$ which is also an infinite matrix. Furthermore, when f is a rational number ($f=m/n$ with m and n relatively prime), R can be reduced to an $n\times n$ matrix.

1. $f=\frac{1}{2}$

This is the fully frustrated case studied by Villain,[18] Teitel and Jayaprakash,[9] Domany,[19] etc. These studies have revealed the possibility of an Ising-like transition due to the discrete symmetry in addition to the usual continuous (XY) symmetry. The two degenerate ground states are shown in Fig. 2.

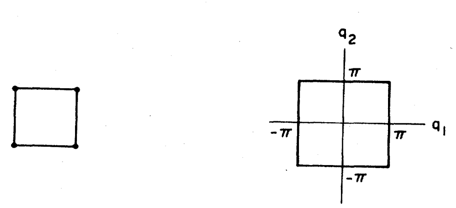

FIG. 1. A unit cell and the Brillouin zone of a square lattice.

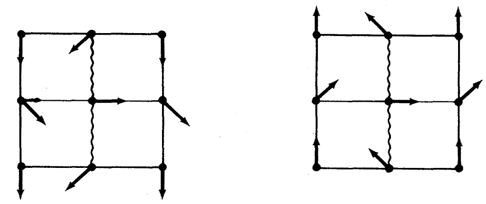

FIG. 2. Doubly degenerate ground states of the fully frustrated system ($f=\frac{1}{2}$) on a square lattice. Straight bonds and wavy bonds represent ferromagnetic and antiferromagnetic couplings, respectively.

The matrix R, given by Eq. (3.3), takes the reduced form

$$R=\begin{bmatrix}2\cos q_1 & 2\cos q_2\\ 2\cos q_2 & -2\cos q_1\end{bmatrix}, \tag{3.4}$$

where two modes (q_1,q_2) and $(q_1+\pi,q_2)$ are coupled to each other. This implies that the Brillouin zone is reduced to half of the original one (the unfrustrated case, $f=0$), or, equivalently, a superlattice whose cell consists of two original cells is formed [Fig. 3(a)]. The 2×2 matrix R has the dominant eigenvalue

$$\lambda_{\mathbf{q}}=(\cos^2 q_1+\cos^2 q_2)^{1/2}\ , \tag{3.5}$$

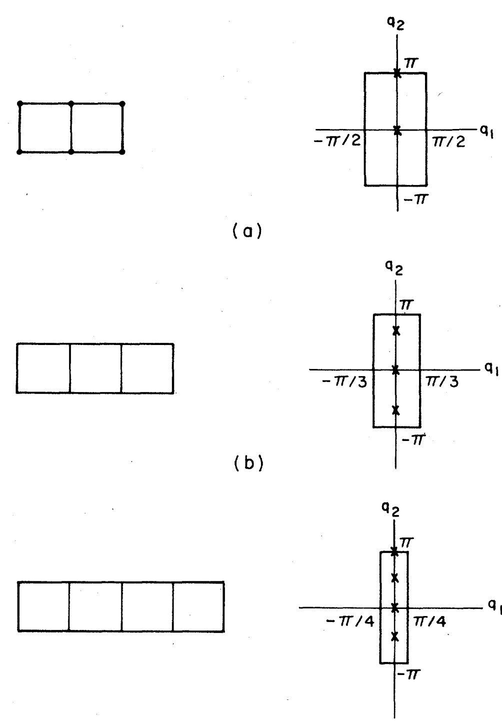

FIG. 3. Unit cells and Brillouin zones of superlattices formed in the cases (a) $f=\frac{1}{2}$, (b) $f=\frac{1}{3}$, and (c) $f=\frac{1}{4}$ on square lattices. Critical modes are denoted by crosses.

which reaches its largest value $\lambda_Q=2\sqrt{2}$ at $\mathbf{Q}_1=(0,0)$ and $\mathbf{Q}_2=(0,\pi)$. The corresponding normalized eigenvectors Φ_1 and Φ_2 are given by the components

$$\Phi_1(\mathbf{q})=a\delta_{\mathbf{q},\mathbf{Q}_1}+b\delta_{\mathbf{q},\mathbf{Q}_1+\pi\hat{\mathbf{q}}},\qquad \Phi_2(\mathbf{q})=a\delta_{\mathbf{q},\mathbf{Q}}-b\delta_{\mathbf{q},\mathbf{Q}+\pi\hat{\mathbf{q}}_1}, \tag{3.6}$$

where $a=(4-2\sqrt{2})^{-1/2}$ and $b=(\sqrt{2}-1)(4-2\sqrt{2})^{-1/2}$ are constants. Since there exist two degenerate modes $\mathbf{Q}_1$ and $\mathbf{Q}_2$ associated with the lowest energy,[20] we need two (complex) order parameters ψ_1 and ψ_2 to describe fluctuations from these two modes, respectively.

With Eqs. (3.5) and (3.6), it is now straightforward to get the LGW Hamiltonian given by Eq. (2.11) in the explicit form

$$\begin{aligned}F(\psi_1,\psi_2)=&-\sum_{\mathbf{q}}(r+eq^2)[\psi_1^*(\mathbf{q})\psi_1(\mathbf{q})+\psi_2^*(\mathbf{q})\psi_2(\mathbf{q})]\\&+\frac{1}{4N}\sum_{\mathbf{q}'s}\{u[\psi_1^*(\mathbf{q})\psi_1^*(\mathbf{q}')\psi_1(\mathbf{q}'')\psi_1(\mathbf{q}''')+\psi_2^*(\mathbf{q})\psi_2^*(\mathbf{q}')\psi_2(\mathbf{q}'')\psi_2(\mathbf{q}''')]+4u'\psi_1^*(\mathbf{q})\psi_2^*(\mathbf{q}')\psi_1(\mathbf{q}'')\psi_2(\mathbf{q}''')\\&+u'[\psi_1^*(\mathbf{q})\psi_1^*(\mathbf{q}')\psi_2(\mathbf{q}'')\psi_2(\mathbf{q}''')+\psi_2^*(\mathbf{q})\psi_2^*(\mathbf{q}')\psi_1(\mathbf{q}'')\psi_1(\mathbf{q}''')]\}\delta_{\mathbf{q}+\mathbf{q}',\mathbf{q}''+\mathbf{q}'''}\end{aligned} \tag{3.7a}$$

where the coefficients are given by

$$\begin{aligned}&r=(k/2\sqrt{2}J)(T-T_0)\quad(kT_0=\sqrt{2}J)\,,\\&e=kT/8\sqrt{2}J\,,\\&u=\tfrac{3}{32}\,,\\&u'=\tfrac{1}{32}\,.\end{aligned} \tag{3.7b}$$

The mean-field transition temperature T_0 is in agreement with that obtained by Shih and Stroud using a self-consistent calculation.[8] In coordinate space the LGW Hamiltonian assumes the form

$$\begin{aligned}F(\psi_1,\psi_2)=\int dx\,dy\Bigg[&\frac{r}{2}(|\psi_1|^2+|\psi_2|^2)+\frac{e}{2}\left(\left|\frac{\partial\psi_1}{\partial x}\right|^2+\left|\frac{\partial\psi_1}{\partial y}\right|^2+\left|\frac{\partial\psi_2}{\partial x}\right|^2+\left|\frac{\partial\psi_2}{\partial y}\right|^2\right)\\&+\frac{u}{4}(|\psi_1|^4+|\psi_2|^4)+u'|\psi_1|^2|\psi_2|^2+\frac{u'}{2}|\psi_1|^2|\psi_2|^2\cos[2(\theta_1-\theta_2)]\Bigg]\,,\end{aligned} \tag{3.8}$$

where $\theta_i(x)$ is the phase of $\psi_i(x)$.[21] The last term in Eq. (3.8) represents the coupling between fluctuations from the two degenerate modes, and is crucial in determining the nature of transition.

2. $f=\frac{1}{3}$

In this case the matrix R given by Eq. (3.3) reduces to the 3×3 form

$$R=\begin{pmatrix}2\cos q_1 & e^{iq_2} & e^{-iq_2}\\ e^{-iq_2} & 2\cos\left(q_1+\frac{2\pi}{3}\right) & e^{iq_2}\\ e^{iq_2} & e^{-iq_2} & 2\cos\left(q_1+\frac{4\pi}{3}\right)\end{pmatrix}, \tag{3.9}$$

which implies that the Brillouin zone is reduced to $\frac{1}{3}$, and the superlattice cell consists of three cells [Fig. 3(b)]. The dominant eigenvalue is

$$\lambda_{\mathbf{q}}=2\sqrt{2}\cos\left[\frac{1}{3}\cos^{-1}\left[\frac{1}{2\sqrt{2}}(\cos3q_1+\cos3q_2)\right]\right], \tag{3.10}$$

leading to the largest value $\lambda_Q=\sqrt{3}+1$ at $\mathbf{Q}_1=(0,0)$, $\mathbf{Q}_2=(0,2\pi/3)$, and $\mathbf{Q}_3=(0,-2\pi/3)$. Therefore three complex order parameters ψ_1, ψ_2, and ψ_3 are deduced, and the LGW Hamiltonian in the coordinate space obtains the form

$$F(\psi_1,\psi_2,\psi_3)=\int dx\,dy f(\psi_1,\psi_2,\psi_3)\,,$$

$$f(\psi_1,\psi_2,\psi_3)=\frac{r}{2}|\psi_1|^2+\frac{e}{2}\left[\left|\frac{\partial\psi_1}{\partial x}\right|^2+\left|\frac{\partial\psi_1}{\partial y}\right|^2\right]+\frac{u}{4}|\psi_1|^4$$
$$+u'|\psi_2|^2|\psi_3|^2+\frac{u'}{2}|\psi_1|^2|\psi_2|\,|\psi_3|\cos(2\theta_1-\theta_2-\theta_3)+\mathcal{T}(\psi_2,\psi_3)\,, \tag{3.11}$$

where $\mathcal{T}$ represents terms of the same form for its arguments from now on, with the mean-field transition temperature $kT_0=[(\sqrt{3}+1)/2]J$ again in agreement with Ref. 8.

3. $f=\frac{1}{4}$

We have the matrix R in the reduced 4×4 form

$$\begin{bmatrix} 2\cos q_1 & e^{iq_2} & 0 & e^{-iq_2} \\ e^{-iq_2} & 2\cos\left[q_1+\frac{\pi}{2}\right] & e^{iq_2} & 0 \\ 0 & e^{-iq_2} & 2\cos(q_1+\pi) & e^{iq_2} \\ e^{iq_2} & 0 & e^{-iq_2} & 2\cos\left[q_1+\frac{3\pi}{2}\right] \end{bmatrix} \tag{3.12}$$

and get the Brillouin zone reduced to a quarter and the superlattice cell consisting of four cells [Fig. 3(c)]. The dominant eigenvalue is

$$\lambda_{\mathbf{q}}=\{4+[12+2(\cos4q_1+\cos4q_2)]^{1/2}\}^{1/2}\,, \tag{3.13}$$

which leads to the largest value $\lambda_{\mathbf{Q}}=2\sqrt{2}$ at $\mathbf{Q}_1=(0,-\pi/2)$, $\mathbf{Q}_2=(0,0)$, $\mathbf{Q}_3=(0,\pi/2)$, and $\mathbf{Q}_4=(0,\pi)$. Thus we obtain four complex order parameters ψ_1, ψ_2, ψ_3, and ψ_4, and get the LGW Hamiltonian in the form

$$f(\psi_1,\psi_2,\psi_3,\psi_4)=\frac{r}{2}|\psi_1|^2+\frac{e}{2}\left[\left|\frac{\partial\psi_1}{\partial x}\right|^2+\left|\frac{\partial\psi_1}{\partial y}\right|^2\right]+\frac{u'}{2}(|\psi_2|^2|\psi_3|^2+|\psi_2|^2|\psi_4|^2+|\psi_3|^2|\psi_4|^2)$$
$$+\frac{u}{4}|\psi_1|^4+u'|\psi_1|^2|\psi_2|\,|\psi_4|\cos(2\theta_1-\theta_2-\theta_4)+\mathcal{T}(\psi_2,\psi_3,\psi_4)$$
$$+2u'\{|\psi_1|^2|\psi_3|^2\cos[2(\theta_1-\theta_3)]+|\psi_2|^2|\psi_4|^2\cos[2(\theta_2-\theta_4)]\}$$
$$+8u'|\psi_1|\,|\psi_2|\,|\psi_3|\,|\psi_4|[\cos(\theta_1+\theta_2-\theta_3-\theta_4)+\cos(\theta_2+\theta_3-\theta_4-\theta_1)] \tag{3.14}$$

with $r=(k/2\sqrt{2}J)(T-T_0)$ and $e=kT/16\sqrt{2}J$. The mean-field transition temperature is given by $kT_0=\sqrt{2}J$, and again agrees with Ref. 8.

B. Triangular lattices

Each site on a triangular lattice has six nearest neighbors, and the bond angle has the value

$$A_{ij}=\begin{cases}\pm(4x_i+1)f\pi & \text{for } \mathbf{x}_j=\mathbf{x}_i+\frac{1}{2}\hat{\mathbf{x}}\pm\frac{\sqrt{3}}{2}\hat{\mathbf{y}}\,,\\ \pm(4x_i-1)f\pi & \text{for } \mathbf{x}_j=\mathbf{x}_i-\frac{1}{2}\hat{\mathbf{x}}\pm\frac{\sqrt{3}}{2}\hat{\mathbf{y}}\,,\\ 0 & \text{for } \mathbf{x}_j=\mathbf{x}_i\pm\hat{\mathbf{x}}\,.\end{cases} \tag{3.15}$$

A triangular lattice and its Brillouin zone are shown in Fig. 4. The inverse transform of (2.5) now becomes

$$\tilde{P}_{\mathbf{q}\mathbf{q}'}=K\delta_{q_2q_2'}R_{q_1q_1'}\,, \tag{3.16}$$

where R again has the off-diagonal elements coupling two or more modes:

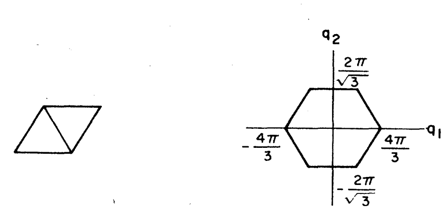

FIG. 4. A unit cell and the Brillouin zone of a triangular lattice.

$$R_{q_1q_1'}=2(\cos q_1)\delta_{q_1q_1'}+2e^{i(\sqrt{3}/2)q_2}\cos\left[f\pi+\frac{q_1}{2}\right]\delta_{q_1+4f\pi,q_1'}+2e^{-i(\sqrt{3}/2)q_2}\cos\left[f\pi-\frac{q_1}{2}\right]\delta_{q_1-4f\pi,q_1'}\ . \tag{3.17}$$

The procedure is thus entirely similar to that for square lattices.

1. $f=\frac{1}{2}$

This fully frustrated case also has been studied recently in the context of the antiferromagnetic *XY* model on a triangular lattice.[22,23] Though interesting and rich critical behavior has been indicated, the nature of transition is still inconclusive.

The matrix R given by Eq. (3.17) then takes the reduced form

$$R=\begin{bmatrix} 2\cos q_1 & -4i\sin\frac{q_1}{2}\sin\frac{\sqrt{3}}{2}q_2 \\ 4i\sin\frac{q_1}{2}\sin\frac{\sqrt{3}}{2}q_2 & 2\cos q_1 \end{bmatrix}. \tag{3.18}$$

Thus two modes (q_1,q_2) and $(q_1+2\pi,q_2)$ are coupled implying a rectangular Brillouin zone and a superlattice whose cell consists of two original unit cells [Fig. 5(a)]. The dominant eigenvalue is

$$\lambda_{\mathbf{q}}=2\cos q_1+4\sin\frac{q_1}{2}\sin\frac{\sqrt{3}}{2}q_2 \tag{3.19}$$

and reaches its largest value $\lambda_{\mathbf{Q}}=3$ at $\mathbf{Q}_1=(\pi/3,\pi/\sqrt{3})$ and $\mathbf{Q}_2=(-\pi/3,\pi/\sqrt{3})$. These two modes satisfy the constraint (2.6), and consequently, describe the doubly degenerate ground states (Fig. 6).

With the two complex order parameters ψ_1 and ψ_2, the LGW Hamiltonian is constructed,

$$f(\psi_1,\psi_2)=\frac{r}{2}|\psi_1|^2+\frac{u}{2}\left[\left|\frac{\partial\psi_1}{\partial x}\right|^2+\left|\frac{\partial\psi_1}{\partial y}\right|^2\right]+\frac{u}{4}|\psi_1|^4+\frac{u_6}{6}|\psi_1|^6+\frac{3}{2}u_6'|\psi_1|^4|\psi_2|^2+\mathscr{T}(\psi_2)$$
$$+u'|\psi_1|^2|\psi_2|^2+\frac{u_6}{3}|\psi_1|^3|\psi_2|^3\cos[3(\theta_1-\theta_2)] \tag{3.20}$$

with $r=k(T-T_0)/3J$ and $e=kT/12J$, where the first coupling term occurs in the sixth order of ψ. Within the Migdal-Kadanoff approximation this term proves to be relevant, and is therefore expected to be crucial in determining the nature of transition. The mean-field transition temperature has the value $kT_0=\frac{3}{2}J$ in agreement with Ref. 8.

2. $f=\frac{1}{3}$

We have the matrix R in the reduced 3×3 form

$$R=2\begin{bmatrix} \cos q_1 & e^{i(\sqrt{3}/2)q_2}\cos\left[\frac{\pi}{3}+\frac{q_1}{2}\right] & e^{-i(\sqrt{3}/2)q_2}\cos\left[\frac{\pi}{3}-\frac{q_1}{2}\right] \\ e^{-i(\sqrt{3}/2)q_2}\cos\left[\frac{\pi}{3}+\frac{q_1}{2}\right] & \cos\left[\frac{2\pi}{3}-q_1\right] & e^{i(\sqrt{3}/2)q_2}\cos\left[\pi+\frac{q_1}{2}\right] \\ e^{i(\sqrt{3}/2)q_2}\cos\left[\frac{\pi}{3}-\frac{q_1}{2}\right] & e^{-i(\sqrt{3}/2)q_2}q_2\cos\left[\pi-\frac{q_1}{2}\right] & \cos\left[\frac{2\pi}{3}+q_1\right] \end{bmatrix}, \tag{3.21}$$

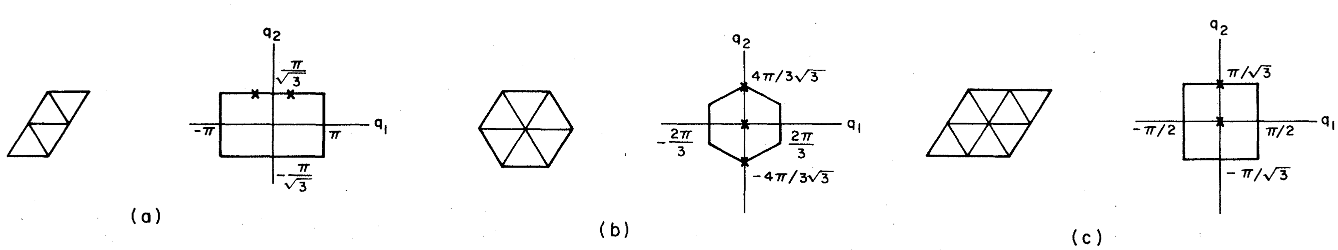

FIG. 5. Unit cells and Brillouin zones of superlattices formed in the cases (a) $f=\frac{1}{2}$, (b) $f=\frac{1}{3}$, and (c) $f=\frac{1}{4}$ on triangular lattices. Crosses denote critical modes.

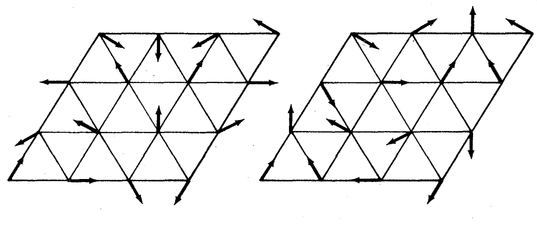

FIG. 6. Doubly degenerate ground states of the fully frustrated system ($f=\frac{1}{2}$) on a triangular lattice corresponding to modes $(-\pi/3,\pi/\sqrt{3})$ and $(\pi/3,\pi/\sqrt{3})$, respectively.

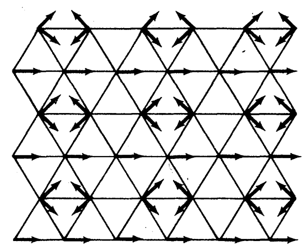

FIG. 7. Infinitely degenerate ground states of the system with $f=\frac{1}{4}$ on a triangular lattice in which every other row can have two configurations $\pm(\pi/4,\frac{3}{4}\pi,\pi/4,\frac{3}{4}\pi,\ldots)$.

which indicates a hexagonal Brillouin zone and a hexagonal superlattice with $\sqrt{3}\times\sqrt{3}$ symmetry [Fig. 5(b)]. We obtain the eigenvalue

$$\lambda_{\mathbf{q}}=2\sqrt{3}\cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{1}{3\sqrt{3}}\left[-3+\cos 3q_1-2\cos\frac{3}{2}q_1\cos\frac{3\sqrt{3}}{2}q_2\right]\right]\right\}, \tag{3.22}$$

and three degenerate modes $\mathbf{Q}_1=(0,2\pi\sqrt{3})$, $\mathbf{Q}_2=(0,2\pi/3\sqrt{3})$, and $\mathbf{Q}_3=(0,-2\pi/3\sqrt{3})$, associated with the largest eigenvalue $\lambda_{\mathbf{Q}}=3$. The LGW Hamiltonian now has the form

$$\begin{aligned}f(\psi_1,\psi_2,\psi_3)=&\frac{r}{2}|\psi_1|^2+\frac{e}{2}\left[\left|\frac{\partial\psi_1}{\partial x}\right|+\left|\frac{\partial\psi_1}{\partial y}\right|\right]+\frac{u}{4}|\psi_1|^4+u'|\psi_2|^2|\psi_3|^2\\&+\frac{u_6}{6}|\psi_1|^6+\frac{3}{2}u_6'(|\psi_2|^4|\psi_3|^2+|\psi_2|^2|\psi_3|^4)+2u_6'|\psi_1|^2|\psi_2|^2|\psi_3|^2\\&+\frac{u_6'}{3}|\psi_2|^3|\psi_3|^3\cos 3(\theta_2-\theta_3)+\mathcal{T}(\psi_2,\psi_3)\end{aligned} \tag{3.23}$$

with the mean-field transition temperature $kT_0=(3/2)J$. Again the first coupling term occurs in the sixth order.

3. $f=\frac{1}{4}$

The matrix R has the form

$$2\begin{bmatrix}\cos(q_1-\pi) & e^{i(\sqrt{3}/2)q_2}\cos\left[\frac{q_1}{2}-\frac{\pi}{4}\right] & 0 & e^{-i(\sqrt{3}/2)q_2}\cos\left[\frac{q_1}{2}-\frac{3\pi}{4}\right]\\ e^{-i(\sqrt{3}/2)q_2}\cos\left[\frac{q_1}{2}-\frac{\pi}{4}\right] & \cos q_1 & e^{i(\sqrt{3}/2)q_2}\cos\left[\frac{q_1}{2}+\frac{\pi}{4}\right] & 0\\ 0 & e^{-i(\sqrt{3}/2)q_2}\cos\left[\frac{q_1}{2}+\frac{\pi}{4}\right] & \cos(q_1+\pi) & e^{i(\sqrt{3}/2)q_2}\cos\left[\frac{q_1}{2}+\frac{3\pi}{4}\right]\\ e^{i(\sqrt{3}/2)q_2}\cos\left[\frac{q_1}{2}+\frac{5\pi}{4}\right] & 0 & e^{-i(\sqrt{3}/2)q_2}\cos\left[\frac{q_1}{2}+\frac{3\pi}{4}\right] & \cos(q_1+2\pi)\end{bmatrix} \tag{3.24}$$

and indicates a rectangular Brillouin zone [Fig. 5(c)]. The eigenvalue has the expression

$$\lambda_{\mathbf{q}}=2(1+\cos^2 q_1+|\cos q_1\cos\sqrt{3}q_2|)^{1/2}, \tag{3.25}$$

and reaches the largest value $\lambda_{\mathbf{Q}}=2\sqrt{3}$ at $\mathbf{Q}_1=(0,0)$ and $\mathbf{Q}_2=(0,\pi/\sqrt{3})$. These two modes do not satisfy Eq. (2.6) and hence do not describe the true ground state of the system. In fact, in the ground state, every other row can have two configurations, and therefore an infinite ground-state degeneracy (in addition to the continuous degeneracy) exists (see Fig. 7). The LGW Hamiltonian has the expression

$$f(\psi_1,\psi_2)=\frac{r}{2}(|\psi_1|^2+|\psi_2|^2)+\frac{e}{2}\left[\left|\frac{\partial\psi_1}{\partial x}\right|+\left|\frac{\partial\psi_1}{\partial y}\right|+\left|\frac{\partial\psi_2}{\partial x}\right|+\left|\frac{\partial\psi_2}{\partial y}\right|\right]$$

$$+\frac{u}{4}(|\psi_1|^4+|\psi_2|^4)+u'|\psi_1|^2|\psi_2|^2+\frac{u'}{2}|\psi_1|^2|\psi_2|^2\cos[2(\theta_1-\theta_2)] \tag{3.26}$$

with the coefficients $r=k/2\sqrt{3}J(T-T_0)$, $e=kT/8\sqrt{3}J$ and the mean-field transition temperature $kT_0=\sqrt{3}J$. Remarkably, the LGW Hamiltonian (3.26) has just the same form as that for the case $f=\frac{1}{2}$ on a square lattice, Eq. (3.8), which leads to the conclusion that the two systems belong to the same universality class near the critical point.

Recent study of this system by Monte Carlo simulations and mean-field calculations reveals two consecutive phase transitions,[8] similar in behavior to that predicted for a stacked antiferromagnetic triangular Ising system,[16] which has also an infinite ground-state degeneracy but two degenerate critical modes.[14] Though the Monte Carlo data are not conclusive, the apparent difference between the result for this case and that for the case $f=\frac{1}{2}$ on a square lattice suggests that there may exist several different types of transitions outside the region of validity of the Hamiltonian (3.26). It should be noted also that the ground state is infinitely degenerate and analogous to the states at the multiphase point in the anisotropic next-nearest-neighbor Ising (ANNNI) model[24] which take on all periodic and nonperiodic configurations.

IV. THE MIGDAL-KADANOFF APPROXIMATION

The remaining task is now to study the statistical mechanics of the LGW Hamiltonians obtained in the preceding section. In this section we consider the application of the Midgal-Kadanoff approximation to relatively simple systems characterized by doubly degenerate modes in order to get some insight into the nature of possible phase transitions in such systems.

Guided by the ferromagnetic case,[1–3] we consider only phase fluctuations and take the amplitudes of order parameters to be equal and constant. This phase-only approximation, which is expected to be accurate at temperatures well below the mean-field transition temperature, allows a generalization of the LGW Hamiltonians (3.8), (3.20), and (3.26) to lattice Hamiltonians of the form[25]

$$-\beta H=\widetilde{K}\sum_{\langle ij\rangle}[\cos(\theta_i^{(1)}-\theta_j^{(1)})+\cos(\theta_i^{(2)}-\theta_j^{(2)})] +\widetilde{h}\sum_i\cos[p(\theta_i^{(1)}-\theta_i^{(2)})]\,, \tag{4.1}$$

where $p=2$ for the Hamiltonians (3.8) and (3.26), and $p=3$ for (3.20). The effective interaction $\widetilde{K}$ and coupling $\widetilde{h}$ are to be given by the comparison of those LGW Hamiltonians with that derived from the lattice Hamiltonian (4.1) (see the Appendix). Thus we obtain two coupled *XY* models, which have been analyzed for $p=2$ in the context of helical *XY* models, to predict two phase transitions, of KT and Ising character, respectively.[25,26] This, however, is not conclusive because in those studies the two types of excitations, KT-like and Ising-like, have been assumed to be independent of each other, which is not true in general.[19]

The application of the MK approximation to the Hamiltonian (4.1) is entirely similar to that described by José *et al.*[3] and is expected to give qualitatively correct results for $p\le 3$. To obtain the recursion law for the coupling $\widetilde{h}$ we approximate the interaction $\widetilde{K}\cos(\theta_i-\theta_j)$ by $-\frac{1}{2}\widetilde{K}_{\text{eff}}(\theta_i-\theta_j)^2$, and carry out a one-dimensional decimation with a bond-moving technique. The result is

$$\widetilde{h}'=\widetilde{h}(1+e^{-p^2/2\widetilde{K}_{\text{eff}}})\,, \tag{4.2}$$

which shows that h is always a relevant variable. This characteristic is also true for the two-dimensional cases, and it is expected that after many iterations the Hamiltonian (4.1) will approach one in which the angle between the two *XY* spins at each site can assume only p discrete values

$$\theta_i^{(1)}-\theta_i^{(2)}=\frac{2\pi n_i}{p}\quad (n_i=0,1,2,\ldots,p-1)\,. \tag{4.3}$$

For the case $p=2$ ($f=\frac{1}{2}$ on a square lattice and $f=\frac{1}{4}$ on a triangular lattice), Eq. (4.3) allows us to define the "Ising-spin" variable $s_i\equiv 2n_i-1$, and to get the Hamiltonian (4.1) in the form[19]

$$-\beta H=\widetilde{K}_\infty\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)(1+s_is_j)\,, \tag{4.4}$$

where $\widetilde{K}_\infty$ must be interpreted as the effective interaction approached after many iterations. This suggests the possibility for an Ising-like transition as well as the KT-like one. As noted at the end of Sec. III, this Hamiltonian is expected to display several different types of transitions.

For $p=3$, which is the case for $f=\frac{1}{2}$ on a triangular lattice, we define a new *XY* spin variable θ_i to be the average of the two original spin variables $\theta_i^{(1)}$ and $\theta_i^{(2)}$, i.e., $2\theta_i=\theta_i^{(1)}+\theta_i^{(2)}$, and obtain the Hamiltonian (4.1) in the form

$$\begin{aligned}-\beta H&=2\widetilde{K}_\infty\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)\cos\frac{\pi}{3}(n_i-n_j)\\&=\widetilde{K}_\infty\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)(3\delta_{n_in_j}-4\delta_{n_i1}\delta_{n_j1}\\&\qquad+2\delta_{n_i1}+2\delta_{n_j1}-1)\,,\end{aligned} \tag{4.5}$$

which contains a three-state Potts interaction with one-site and two-site symmetry-breaking fields.[27] Thus the possibility for a three-state Potts-like transition in addition to the KT-like one is suggested. To obtain a conclusive answer, however, a more thorough analysis is necessary since there could exist alternative possibilities for types of phase transition which this discussion has not revealed.

V. CONCLUDING REMARKS

To investigate the nature of phase transitions at finite temperatures in arrays of coupled Josephson junctions, we have considered uniformly frustrated two-dimensional *XY* models. We have developed the Hubbard-Stratanovich transform for those systems, and constructed the Landau-Ginzburg-Wilson Hamiltonians for cases $f=\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ both on a square lattice and on a triangular lattice. The resulting LGW Hamiltonians naturally reflect the discontinuous variation of the nature of transitions and the formation of superlattices as f is varied. In particular, it is shown that the critical behavior of the system with $f=\frac{1}{2}$ on a square lattice belongs to the same universality class as that with $f=\frac{1}{4}$ on a triangular lattice, though there exists the possibility for different types of transitions outside the region of validity of our approach. Also obtained are the mean-field transition temperatures in complete agreement with previous self-consistent calculations by Shih and Stroud. The LGW Hamiltonians for three relatively simple cases ($f=\frac{1}{2}$ on a square lattice, $f=\frac{1}{2}$ and $\frac{1}{4}$ on a triangular lattice) have been generalized to the lattice Hamiltonian describing two coupled XY models, and the application of the Migdal-Kadanoff approximation has been considered for these systems. In addition to the Kosterlitz-Thouless—like transition, an Ising-like one is shown to be possible in the case $f=\frac{1}{2}$ on a square lattice, while the possibility for a three-state Potts-like transition is suggested in the case $f=\frac{1}{2}$ on a triangular lattice.

Our considerations are, however, far from conclusive, and more thorough analysis of the obtained LGW Hamiltonians or their generalization to the lattice Hamiltonians is necessary. Nevertheless our approach is already sufficient to demonstrate the richness of the behavior of frustrated XY models in connection with artificial structures such as Josephson-junction arrays.

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APPENDIX: LGW HAMILTONIANS FOR COUPLED XY MODELS

We consider two coupled XY models described by the Hamiltonian

$$-\beta H=\widetilde{K}\sum_{\langle ij\rangle}[\cos(\phi_i^{(1)}-\phi_j^{(1)})+\cos(\phi_i^{(2)}-\phi_j^{(2)})]+\widetilde{h}\sum_i\cos[p(\phi_i^{(1)}-\phi_i^{(2)})]\ , \tag{A1}$$

where $\phi_i^{(1)}$ and $\phi_i^{(2)}$ represent two XY spins at site i, respectively, and p is an integer. The Hamiltonian (A1) can be written in a compact form

$$-\beta H=\tfrac{1}{2}(u^\dagger Pu+v^\dagger Pv)+\frac{\widetilde{h}}{2}\mathrm{Re}\sum_i(u_i^*v_1)^p\ , \tag{A2}$$

where u and v are column vectors with components $e^{-i\phi_i^{(1)}}$ and $e^{-i\phi_i^{(2)}}$, respectively, and P is a symmetric matrix with elements $\widetilde{P}_{ij}=\widetilde{K}_{ij}+\delta_{ij}\sum_l\widetilde{K}_{il}$.

We now consider the integral on a complex plane

$$I\equiv\int_{\rm all}\prod_{i=1}^{N}dw_idz_i\exp\left[-\tfrac{1}{2}(w^\dagger P^{-1}w+z^\dagger P^{-1}z)-\frac{\widetilde{h}}{2}\mathrm{Re}\sum_i\left[\sum_{j,k}P_{ij}^{-1}P_{ik}^{-1}w_j^*z_k\right]^p\right]\ , \tag{A3}$$

which can be shown to yield a nondivergent constant. We shift the variables via the relations

$$w=\psi_1+Pu,\quad z=\psi_2+Pv\ , \tag{A4}$$

and write the integral (A3) in terms of ψ_1 and ψ_2:

$$\begin{aligned}I=\int_{\rm all}\prod_i d\psi_{1i}d\psi_{2i}&\exp[-\tfrac{1}{2}(\psi_1^\dagger P^{-1}\psi_1+\psi_2^\dagger P^{-1}\psi_2)-\mathrm{Re}(\psi_1^\dagger u+\psi_2^\dagger v)-\tfrac{1}{2}(u^\dagger Pu+v^\dagger Pv)]\\&\times\exp\left\{-\frac{\widetilde{h}}{2}\mathrm{Re}\sum_i\left[(u_i^*v_i)^p+\left[\sum_{j,k}\psi_{1j}^*\psi_{2k}P_{ji}^{-1}P_{ik}^{-1}\right]^p+C(\psi_1,\psi_2;u,v)\right]\right\}\ ,\end{aligned} \tag{A5}$$

where C represents terms that couple ψ_1,ψ_2 and u,v. Thus we obtain the expression for the partition function

$$Z\equiv\mathrm{Tr}\,e^{-\beta H}\propto\int d\psi_1d\psi_2\exp\left[-\tfrac{1}{2}(\psi_1^\dagger P^{-1}\psi_1+\psi_2^\dagger P^{-1}\psi_2)-\frac{\widetilde{h}}{2}\mathrm{Re}\sum_i\left[\sum_{j,k}\psi_{1j}^*\psi_{2k}P_{ji}^{-1}P_{ik}^{-1}\right]^p\right]e^{f(|\psi_1|,|\psi_2|)}\ , \tag{A6}$$

where

$$e^{f(|\psi_1|,|\psi_2|)} \equiv \operatorname*{Tr}_{u,v} \exp\left[-\mathrm{Re}(\psi_1^\dagger u+\psi_2^\dagger v)-\frac{\tilde{h}}{2}\mathrm{Re}\sum_i C(\psi_1,\psi_2;u,v)\right]$$

depends only on the amplitudes of ψ_{1i} and ψ_{2i}, and is irrelevant to the phase-only approximation.

The Fourier transform then gives the relations

$$\psi_1^\dagger P^{-1}\psi_1+\psi_2^\dagger P^{-1}\psi_2=\sum_{\mathbf{q}} P_{\mathbf{q}}^{-1}[\psi_1^*(\mathbf{q})\psi_1(\mathbf{q})+\psi_2^*(\mathbf{q})\psi_2(\mathbf{q})] \,,$$
$$\sum_i \left[\sum_{i,j}\psi_{1j}^*\psi_{2k}P_{ji}^{-1}P_{ik}^{-1}\right]^p = N^{1-p}\sum_{\mathbf{q}'\mathrm{s}}\prod_{n=1}^{p}[P_{\mathbf{q}_n}^{-1}\psi_1^*(\mathbf{q}_n)P_{\mathbf{q}_n'}^{-1}\psi_2(\mathbf{q}_n')]\delta_{\sum \mathbf{q}_n-\sum \mathbf{q}_n'} \,, \tag{A7}$$

which, with the expression

$$P_q=\begin{cases}2\tilde{K}(\cos q_1+\cos q_2)\approx 4\tilde{K}(1-q^2/4), & \text{square lattice (sq)}\\ 2\tilde{K}\left[\cos q_1+2\cos\tfrac{1}{2}q_1\cos\dfrac{\sqrt{3}}{2}q_2\right]\approx 6\tilde{K}(1-q^2/4), & \text{triangular lattice (tri)}\end{cases} \tag{A8}$$

allows the desired form

$$Z \propto \int d\psi_1 d\psi_2 e^{-F(\psi_1,\psi_2)} \,,$$
$$F(\psi_1,\psi_2)=\int dx\,dy\left[\frac{e}{2}\left(\left|\frac{\partial x_1}{\partial x}\right|^2+\left|\frac{\partial \psi_1}{\partial y}\right|^2+\left|\frac{\partial \psi_2}{\partial x}\right|^2+\left|\frac{\partial \psi_2}{\partial y}\right|^2\right)+\frac{u}{2}|\psi_1|^p|\psi_2|^p\cos[p(\theta_1-\theta_2)]+\tilde{f}(|\psi_1|,|\psi_2|)\right] . \tag{A9a}$$

In Eq. (A8), $\tilde{f}(|\psi_1|,|\psi_2|)$ represents the part that depends only on the amplitudes, and the coefficients are given by

$$e=\frac{1}{16\tilde{K}},\quad u=2(4\tilde{K})^{-2p}\tilde{h},\quad \text{sq}$$
$$e=\frac{1}{24\tilde{K}},\quad u=2(6\tilde{K})^{-2p}\tilde{h},\quad \text{tri} . \tag{A9b}$$

Thus generalization of the LGW Hamiltonians (3.6), (3.20), and (3.26) to the lattice Hamiltonians of the form (A.1) is completed in the phase-only approximation with the relations

$$\tilde{K}=\begin{cases}K/\sqrt{2} & \text{for } f=\tfrac{1}{2},\ \text{sq } (p=2),\\ K/2 & \text{for } f=\tfrac{1}{2},\text{tri } (p=3),\\ K/\sqrt{3} & \text{for } f=\tfrac{1}{4},\ \text{tri } (p=2).\end{cases} \tag{A10}$$

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[21]Equation (3.8) is slightly different in the quartic terms from that obtained by Domany (Ref. 19). It is expected, however, that the critical behavior does not change, and we believe the origin of discrepancy to lie in the problem of independency of each term since our approach gives its coefficient explicitly while the conventional symmetry argument used by Domany does not.

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