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Privman-Fisher hypothesis on finite systems: Verification in the case of the spherical model of ferromagnetism

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The Privman-Fisher hypothesis on the singular part of the free-energy density of a finite system, near the bulk critical point $T=T_c$, is examined in the context of the spherical model of ferromagnetism. A d-dimensional hypercubical lattice (of size $N_1a\times N_2a\times\cdots\times N_da$) is considered and, subject to periodic boundary conditions, explicit expressions are derived for the free energy, the specific heat, and the magnetic susceptibility of the system at temperatures close to T_c. The relevant scaling functions governing the critical behavior of the system are obtained and, with the use of the asymptotic properties of these functions, various predictions of the Privman-Fisher hypothesis are verified. By implication, the passage of the given system towards standard bulk behavior, as $N_j\to\infty$, is also elucidated.

I. INTRODUCTION

In a recent paper Privman and Fisher[1] have argued that the "singular" part of the free-energy density of a finite hypercubical system ($L\times L\times\cdots\times L=L^d$, d being less than the upper critical dimension $d_>$), near the bulk critical point $T=T_c$, may be expressed in the form[2]

$$f^{(s)}(t,h;L)\equiv\frac{F^{(s)}}{Vk_BT}\approx L^{-d}Y(C_1tL^{1/\nu},C_2hL^{\Delta/\nu})\ , \qquad (1)$$

where t and h are the (reduced) temperature and field variables,

$$t=\frac{T-T_c}{T_c},\quad h=\frac{\mu_{\text{eff}}H}{k_BT}\ , \qquad (2)$$

x_1 $(=C_1tL^{1/\nu})$ and x_2 $(=C_2hL^{\Delta/\nu})$ are the appropriate scaled variables, ν and Δ being the familiar bulk indices, while C_1 and C_2 are certain *nonuniversal*, system-dependent scale factors. The function $Y(x_1,x_2)$ is then a *universal* function, common to all systems in the same universality class as the given system. It seems important to emphasize that in expression (1) no nonuniversal metric factor, C_0, appears in front of the function $Y(x_1,x_2)$; moreover, the variable L here denotes the actual physical dimension of the system (and not one scaled in terms of any elementary length appropriate to the situation). As indicated by Privman and Fisher, the above formulation is valid for a cylindrical system ($L^{d-1}\times\infty$) as well; in our investigation, it seems to hold equally well for a system such as ($L^{d^*}\times\infty^{d'}$), where $d^*+d'=d$. Of course, the precise nature of the scaling function $Y(x_1,x_2)$ varies significantly as we move from one geometry to another; the same is true if we alter the set of boundary conditions to which the system is subjected.

Of pivotal importance in expression (1) are the scale factors, C_1 and C_2, whose determination may seem to require an explicit evaluation of the function $f^{(s)}(t,h;L)$ for the given finite system. In reality, such an evaluation is necessary only if one is interested in determining the exact form of the scaling function $Y(x_1,x_2)$; insofar as the scale factors are concerned, they can be determined from a study of the corresponding bulk system instead. As shown by Singh and Pathria[3] in the context of an ideal relativistic Bose gas, this determination can be made with the help of any bulk function, or functions, containing two independent bits of information on the singularity of the problem.

In the present paper we propose to test the scaling hypothesis (1) in the case of the spherical model of ferromagnetism in d dimensions. Using methods developed in earlier papers,[3-6] we derive explicit expressions for various thermodynamic functions of the field-free system, $x_2=0$, of spins on a hypercubical lattice (of size $N_1a\times\cdots\times N_da$) under periodic boundary conditions. While the scale factors C_1 and C_2, and certain asymptotic forms of the scaling function $Y(x_1,x_2)$ and its derivatives, can be determined from the appropriate bulk results,[7,8] our analysis of the finite system enables us to derive the complete mathematical forms of these functions valid for *all* values of x_1. In view of the fact that these functions are characteristic of the geometry of the lattice, finite-size effects in the various thermodynamic properties of the system are also geometry dependent. Although most of the results derived here pertain to $2<d<4$, special cases arising from the most relevant dimension $d=3$, viz., a cube ($d^*=3$), a cylinder ($d^*=2$), and a film ($d^*=1$), are given special consideration. In all cases, the analytical results obtained here are found to be in complete agreement with the ones following from the Privman-Fisher hypothesis.

In Secs. II and III we carry out a detailed investigation of hypothesis (1) and establish a set of results relevant to the subject matter of this paper; this includes the determination of the scale factors C_1 and C_2 on the basis of the bulk results for the spherical model already available in the literature.[7,8] Predictions for the finite system are thereby laid out. Next, these predictions are verified against actual, analytical results derived in Sec. IV for a d-dimensional system which is finite in d^* dimensions ($d^*=1,2,\ldots$) and infinite in the remaining d' $(=d-d^*)$ dimensions; details of the process of verification are given in Sec. V. Wherever possible, a comparison is made with the previous analytical results, which are generally some special cases of the ones reported here.

II. FORMULATION OF THE PROBLEM

In accordance with (1), the "singular" part of the specific heat per unit volume of the system will be given by

$$c^{(s)}(t,h;L)\approx C_1^2 L^{\alpha/\nu} Y_{(1)}(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}) , \tag{3}$$

and that of the magnetic susceptibility by

$$\chi^{(s)}(t,h;L)\approx C_2^2 L^{\gamma/\nu} Y_{(2)}(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}) , \tag{4}$$

where $Y_{(1)}$ and $Y_{(2)}$ are appropriate derivatives of the original function $Y(x_1,x_2)$, while use has been made of the relationships

$$d\nu=2-\alpha,\quad \Delta=\beta+\gamma,\quad \alpha+2\beta+\gamma=2 . \tag{5}$$

It will be noted that Eqs. (3) and (4) are consistent with the standard bulk behavior:

$$c^{(s)}(t,0;\infty)\propto |t|^{-\alpha},\quad \chi^{(s)}(t,0;\infty)\propto |t|^{-\gamma} . \tag{6}$$

In the sequel we shall confine ourselves to the field-free situation ($h=0$); in view of this, the variable x_2 may not be displayed explicitly in the subsequent expressions.

For the determination of C_1 and C_2 for the spherical model, we find it convenient to draw on the bulk behavior of the specific heat and the magnetic susceptibility of the system, namely,

$$c^{(s)}(t;\infty)=\begin{cases}-E_+ t^{-\alpha} & (t>0) , \quad \text{(7a)}\\ 0 & (t<0) , \quad \text{(7b)}\end{cases}$$

and

$$\chi^{(s)}(t;\infty)=\begin{cases}C_+ t^{-\gamma} & (t>0) , \quad \text{(8a)}\\ \infty & (t<0) , \quad \text{(8b)}\end{cases}$$

where

$$\alpha=(d-4)/(d-2),\quad \gamma=2/(d-2) , \tag{9}$$

$$E_+=p_d K_c^{d/(d-2)} a^{-d} , \tag{10}$$

$$C_+=q_d K_c^{-d/(d-2)} a^{-d} , \tag{11}$$

with

$$p_d=\frac{(8\pi)^{d/(d-2)}}{(d-2)\left|\Gamma\left[\frac{2-d}{2}\right]\right|^{2/(d-2)}} ,\qquad q_d=\frac{\left|\Gamma\left[\frac{2-d}{2}\right]\right|^{2/(d-2)}}{(8\pi)^{d/(d-2)}} . \tag{12}$$

Here, K_c stands for the interaction parameter, J/k_BT_c, of the system, a denotes the lattice constant, while d is restricted by the inequality $2<d<4$.

To reproduce (7a) from (3), the scaling function $Y_{(1)}(x_1)$ must behave as

$$-A_+ x_1^{-\alpha} \quad (x_1\to+\infty) , \tag{13}$$

with

$$A_+=E_+/C_1^{2-\alpha} . \tag{14}$$

Similarly, to reproduce (8a) from (4), the scaling function $Y_{(2)}(x_1)$ must behave as

$$G_+ x_1^{-\gamma} \quad (x_1\to+\infty) , \tag{15}$$

with

$$G_+=C_+C_1^{\gamma}/C_2^2 . \tag{16}$$

It follows that

$$C_1=(E_+/A_+)^{1/(2-\alpha)} ,\qquad C_2=(C_+/G_+)^{1/2}(E_+/A_+)^{\gamma/2(2-\alpha)} . \tag{17}$$

We thus obtain, see (9),

$$C_1=(p_d/A_+)^{(d-2)/d} K_c a^{-(d-2)} \tag{18}$$

and

$$C_2=(p_d/A_+)^{1/d}(q_d/G_+)^{1/2} K_c^{-1/2} a^{-(d+2)/2} . \tag{19}$$

For simplicity, we may choose the normalization of the universal function Y such that the coefficients A_+ and G_+ appearing in the asymptotic expressions (13) and (15) are exactly equal to the universal numbers p_d and q_d, respectively; note that, for $d=3$, $p_d=128\pi^2$, and $q_d=1/(128\pi^2)$. With this choice, C_1 and C_2 assume the simplified form

$$C_1=K_c a^{-(d-2)},\quad C_2=K_c^{-1/2} a^{-(d+2)/2} , \tag{20}$$

which is clearly system dependent.

Once C_1 and C_2 are known, no more nonuniversal amplitudes are needed to describe the critical behavior of the system—regardless of whether it is finite or infinite in extent; all amplitudes appearing in the expressions for the various physical properties of the system will be related to C_1 and C_2 through universal factors alone.

III. CONSEQUENCES OF THE PRIVMAN-FISHER HYPOTHESIS

We start with the free-energy density $f^{(s)}(t;L)$, as given by Eq. (1) with $h=0$, and write

$$f^{(s)}(t;L)\approx F_+ t^{2-\alpha} \quad (t>0,\ L\to\infty)\ , \tag{21}$$

where use has been made of the fact that $d\nu=2-\alpha$. Comparing (21) with (7a), we find that

$$F_+ = E_+/(2-\alpha)(1-\alpha)\ . \tag{22}$$

At the same time, we conclude that the scaling function $Y(x_1)$ in (1) must behave as

$$Y_+ x_1^{2-\alpha} \quad (x_1\to+\infty)\ , \tag{23}$$

with the universal coefficient

$$Y_+ = F_+/C_1^{2-\alpha}\ , \tag{24}$$

cf. the corresponding Eq. (14) for the specific-heat function $c^{(s)}$.

For $t<0$ and $L\to\infty$, there are two possibilities of interest in this study:

$$\text{(i)}\quad Y(x_1)\to Y_-\,|x_1|^{\nu(d-\epsilon)} \quad (x_1\to-\infty)\ , \tag{25}$$

so that

$$f^{(s)}(t;L)\approx Y_- C_1^{\nu(d-\epsilon)}\,|t|^{\nu(d-\epsilon)}L^{-\epsilon}\ , \tag{26}$$

the index ϵ is as yet undetermined but is expected to be geometry dependent.

$$\text{(ii)}\quad Y(x_1)\to Y_-^*(\ln|x_1|+\text{const}) \quad (x_1\to-\infty)\ , \tag{25$'$}$$

so that

$$f^{(s)}(t;L)\approx Y_-^*\left[\ln C_1+\ln|t|+\frac{1}{\nu}\ln L+\text{const}\right]L^{-d}\ . \tag{26$'$}$$

In each case, the coefficient Y_- or Y_-^* is universal. The repercussion of this on the specific heat of the system is that, for $\epsilon\neq d$,

$$c^{(s)}(t;L)\propto |t|^{-(\alpha+\nu\epsilon)}L^{-\epsilon} \quad (t<0,\ L\to\infty)\ . \tag{27}$$

The special case $\epsilon\to d$ corresponds to possibility (ii) above, for which $c^{(s)}(t;L)$ is still given by (27), i.e.,

$$c^{(s)}(t;L)\propto |t|^{-2}L^{-d}\ . \tag{27$'$}$$

For $\epsilon=d$, the leading term in $f^{(s)}(t;L)$ would be independent of t, with the result that no L^{-d} term would appear in the specific heat of the system. In passing, we note that the extreme case $\epsilon\to\infty$ would entail a function that vanishes exponentially fast with L.

As regards the susceptibility of the system, we may assume that, for $t<0$ and $L\to\infty$, it diverges as L^{ζ}. The function $Y_{(2)}(x_1)$ must then behave as

$$Y_{(2)}(x_1)\to G_-\,|x_1|^{\nu\zeta-\gamma} \quad (x_1\to-\infty)\ , \tag{28}$$

with the result that

$$\chi^{(s)}(t;L)\approx G_- C_1^{\nu\zeta-\gamma}C_2^2\,|t|^{\nu\zeta-\gamma}L^{\zeta}\ , \tag{29}$$

with G_- universal. Note that the extreme case $\zeta\to\infty$ would now entail a function that diverges exponentially fast with L.

Finally, in the "core" region, where $|x_1|=O(1)$ and hence $|t|=O(L^{1/\nu})$, the functions $f^{(s)}(t;L)$, $c^{(s)}(t;L)$, and $\chi^{(s)}(t;L)$, for a fixed value of x_1, are proportional to L^{-d}, $L^{\alpha/\nu}$, and $L^{\gamma/\nu}$, respectively; see Eqs. (1), (3), and (4). It follows that the quantities

$$U=f^{(s)}(0;L)L^{d}\ , \tag{30}$$

$$U_{(1)}=c^{(s)}(0;L)L^{-\alpha/\nu}C_1^{-2}\ , \tag{31}$$

and

$$U_{(2)}=\chi^{(s)}(0;L)L^{-\gamma/\nu}C_2^{-2}\ , \tag{32}$$

evaluated at the erstwhile critical point ($t=0$), which clearly lies in the core region, must be universal.

This completes the set of predictions, based on the Privman-Fisher hypothesis, which we propose to test at length in the following sections.

IV. THE SPHERICAL MODEL ON A FINITE LATTICE

We consider a system of N spins, s_i, located at sites $\mathbf{r}_i$ $(=\mathbf{n}_i a)$ of a hypercubical lattice [of size $N_1 a\times N_2 a\times\cdots\times N_d a\,(=Na^d)$] interacting through the Hamiltonian[7,8]

$$\mathscr{H}=-J\sum_{\text{NN}} s_i s_j-\mu_{\text{eff}}H\sum_{i=1}^{N}s_i+\lambda\sum_{i=1}^{N}s_i^2\ , \tag{33}$$

where the various symbols have their usual meanings, and NN means nearest neighbors. The spherical field λ, which is conjugate to the quantity $-\sum_i s_i^2$, is introduced so as to satisfy the constraint

$$\sum_{i=1}^{N}\langle s_i^2\rangle\equiv\mathscr{S}^2=N\ . \tag{34}$$

Under periodic boundary conditions, the free energy per spin is given by

$$F(\beta,H,\lambda)=\frac{1}{2N\beta}\sum_q\ln[\beta(\lambda-\mu_q)]-\frac{\mu_{\text{eff}}^2H^2}{4(\lambda-\mu_0)}\ , \tag{35}$$

where $\beta=1/k_BT$, q is a collective symbol for the set of numbers $\{n_1,\ldots,n_d\}$, while the eigenvalues μ_q are given by

$$\mu_q=2J\sum_{j=1}^{d}\cos\left[\frac{2\pi n_j}{N_j}\right]\quad [n_j=0,1,2,\ldots,(N_j-1)]\ , \tag{36}$$

clearly, $\mu_q\le\mu_0=2Jd$. The magnetization per spin is then given by

$$\mathscr{M}(\beta,H)=\mu_{\text{eff}}^2H/2(\lambda-\mu_0) \tag{37}$$

and the susceptibility by

$$\chi(\beta,H)=\mu_{\text{eff}}^2/2(\lambda-\mu_0)\ , \tag{38}$$

while the constraint (34) takes the form

$$\sum_q\frac{1}{\lambda-\mu_q}=2N\beta\left[1-\frac{\mathscr{M}^2(\beta,H)}{\mu_{\text{eff}}^2}\right]\ , \tag{39}$$

Eqs. (37) and (39) together determine λ as a function of β and H.

In zero field, Eqs. (35) and (39) reduce to

$$F(\beta,\lambda)=\frac{1}{2N\beta}\sum_{\{n_j\}}\ln\left\{\beta\left[\lambda-2J\sum_{j=1}^{d}\cos\left[\frac{2\pi n_j}{N_j}\right]\right]\right\} \tag{40}$$

and

$$2K=\frac{1}{N}\sum_{\{n_j\}}\left[\frac{\lambda}{J}-2\sum_{j=1}^{d}\cos\left[\frac{2\pi n_j}{N_j}\right]\right]^{-1}, \tag{41}$$

where $K=\beta J$. Using the representations

$$\ln z=\int_0^\infty (e^{-(1/2)x}-e^{-(1/2)zx})\frac{dx}{x}, \quad z^{-1}=\tfrac{1}{2}\int_0^\infty e^{-(1/2)zx}dx, \tag{42}$$

Eqs. (40) and (41) can be written as

$$F(\beta,\lambda)=\frac{\ln K}{2\beta}+\frac{1}{2N\beta}\sum_{\{n_j\}}\int_0^\infty\left[e^{-(1/2)x}-e^{-(\lambda/2J)x}\prod_{j=1}^{d}e^{x\cos(2\pi n_j/N_j)}\right]\frac{dx}{x} \tag{43}$$

and

$$2K=\frac{1}{2N}\sum_{\{n_j\}}\int_0^\infty e^{-(\lambda/2J)x}\prod_{j=1}^{d}e^{x\cos(2\pi n_j/N_j)}dx. \tag{44}$$

Now, the summations over n_j can be carried out with the help of (a generalization of) the *Poisson summation formula*, viz.

$$\sum_{n=a}^{b}f(n)=\sum_{q=-\infty}^{\infty}\mathscr{F}(q)+\tfrac{1}{2}f(a)+\tfrac{1}{2}f(b), \tag{45}$$

where

$$\mathscr{F}(q)=\int_a^b e^{2\pi iqn}f(n)dn. \tag{46}$$

This leads to the remarkable identity, valid for *all* $N_j\ge 1$,

$$\sum_{n_j=0}^{N_j-1}\exp\left[x\cos\left[\frac{2\pi n_j}{N_j}\right]\right]=N_j\sum_{q_j=-\infty}^{\infty}I_{N_jq_j}(x), \tag{47}$$

where $I_\nu(x)$ denotes a modified Bessel function. Equations (43) and (44) then become

$$F(\beta,\lambda)=\frac{\ln K}{2\beta}+\frac{1}{2\beta}\sum_{\{q_j\}}\int_0^\infty\left[e^{-(1/2)x}\prod_{j=1}^{d}\delta_{q_j,0}-e^{-(\lambda/2J)x}\prod_{j=1}^{d}I_{N_jq_j}(x)\right]\frac{dx}{x} \tag{48}$$

and

$$2K=\tfrac{1}{2}\sum_{\{q_j\}}\int_0^\infty e^{-(\lambda/2J)x}\prod_{j=1}^{d}I_{N_jq_j}(x)dx, \tag{49}$$

here, use has been made of the fact that $\prod_j N_j=N$. It will be noted that terms with $\mathbf{q}=0$ yield standard bulk results, while those with $\mathbf{q}\neq 0$ determine finite-size effects in the system.

Introducing the variable ϕ, where[8]

$$\phi=(\lambda/J)-2d, \tag{50}$$

we may write

$$F(\beta,\phi)=F_B(\beta,\phi)-\frac{1}{2\beta}\sum_{\{q_j\}}{}'\int_0^\infty e^{-(1/2)\phi x}\times\prod_{j=1}^{d}[e^{-x}I_{N_jq_j}(x)]\frac{dx}{x}, \tag{51}$$

where $F_B(\beta,\phi)$ denotes the bulk free energy per spin:

$$F_B(\beta,\phi)=\frac{\ln K}{2\beta}+\frac{1}{2\beta}\int_0^\infty\{e^{-(1/2)x}-e^{-(1/2)\phi x}[e^{-x}I_0(x)]^d\}\frac{dx}{x}. \tag{52}$$

The quantity ϕ is determined by the constraint equation

$$2K=W_d(\phi)+\tfrac{1}{2}\sum_{\{q_j\}}{}'\int_0^\infty e^{-(1/2)\phi x}\prod_{j=1}^{d}[e^{-x}I_{N_jq_j}(x)]dx, \tag{53}$$

where

$$W_d(\phi)=\tfrac{1}{2}\int_0^\infty e^{-(1/2)\phi x}[e^{-x}I_0(x)]^d dx. \tag{54}$$

Note that the primed summations in Eqs. (51) and (53) imply that terms with $\mathbf{q}=0$ are excluded.

Equations (51) and (53) are quite general in respect of the actual values of the numbers N_j. From a practical point of view, however, we may prefer to specify that the system under consideration is finite in d^* dimensions ($d^*=1,2,\ldots$) and infinite in the remaining d' ($=d-d^*$) dimensions, i.e., only N_j with $j=1,2,\ldots,d^*$ are finite, while the remaining $N_j\to\infty$. For the latter dimensions, one can readily verify that $q_j\neq 0$ would make a vanishing contribution to the sums $\sum'$, accordingly, those q_j may be set equal to zero, with the result that only $q_1,\ldots,q_{d^*}$ would make an explicit appearance in $\sum'$.

To evaluate the integral appearing in Eqs. (51) and (53), we make use of the asymptotic expansion[9]

$$I_\nu(x)=\frac{e^x}{\sqrt{2\pi x}}\left[1-\frac{4\nu^2-1}{8x}+\frac{(4\nu^2-1)(4\nu^2-9)}{2!(8x)^2}-\frac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3!(8x)^3}+\cdots\right], \tag{55}$$

which may be recast in the form

$$I_\nu(x)=\frac{e^{x-\nu^2/2x}}{\sqrt{2\pi x}}\left[1+\frac{1}{8x}+\frac{9-32\nu^2}{2!(8x)^2}+\frac{225-928\nu^2+128\nu^4}{3!(8x)^3}+\cdots\right], \quad (56)$$

and the integral[10]

$$\int_0^\infty x^{\nu-1}e^{-tx-\alpha/x}dx=2\left[\frac{\alpha}{t}\right]^{\nu/2}K_\nu(2\sqrt{\alpha t}\,), \quad (57)$$

where $K_\nu(z)$ denotes the other modified Bessel function. Remembering that $K_{-\nu}(z)=K_\nu(z)$, finite-size terms in (53) yield the result

$$(2\pi)^{-d/2}\sum_{\mathbf{q}(d^*)}{}'(\phi^{1/2}/\gamma)^{(d-2)/2}K_{(d-1)/2}(\phi^{1/2}\gamma)\times[1+O(\phi^{1/2}/\gamma)], \quad (58)$$

where

$$\gamma[\mathbf{q}(d^*)]=(N_1^2q_1^2+\cdots+N_{d^*}^2q_{d^*}^2)^{1/2}>0. \quad (59)$$

Assuming, for simplicity, that $N_1=\cdots=N_{d^*}=\tilde{n}$, say, and introducing the *thermogeometric parameter*,[4] y, appropriate to this system, viz.,

$$y=\tfrac{1}{2}\tilde{n}\phi^{1/2} \quad (60)$$

expression (58) takes the form

$$\frac{1}{2\pi^{d/2}}\left[\frac{ya}{L}\right]^{d-2}\mathcal{K}\left[\frac{d-2}{2}\middle|d^*;y\right]\left[1+O\left[\frac{ya^2}{L^2}\right]\right], \quad (61)$$

where L $(=\tilde{n}a)$ denotes the length of the finite dimension(s) of the lattice, while

$$\mathcal{K}(n\,|\,d^*;y)=\sum_{\mathbf{q}(d^*)}{}'\frac{K_n(2yq)}{(yq)^n}. \quad (62)$$

The bulk function $W_d(\phi)$ has been studied in considerable detail;[8] for $2<d<4$, it can be expanded in the form

$$W_d(\phi)=W_d(0)-\frac{1}{(4\pi)^{d/2}}\left|\Gamma\left[\frac{2-d}{2}\right]\right|\phi^{(d-2)/2}+O(\phi^{(d-1)/2}), \quad (63)$$

which prompts one to conclude that the bulk critical point, $T=T_c$, is marked by the relationship

$$2K_c=W_d(0). \quad (64)$$

Equation (53) now takes the form

$$K_c-K=\frac{1}{8\pi^{d/2}}\left[\frac{ya}{L}\right]^{d-2}\left[\left|\Gamma\left[\frac{2-d}{2}\right]\right|-2\mathcal{K}\left[\frac{d-2}{2}\middle|d^*;y\right]\right], \quad (65)$$

correct to the leading order in (a/L). In view of the role it plays in the determination of the parameter $y(\beta,L)$, Eq. (65) is of central importance in our analysis.

A similar operation on Eq. (51) gives

$$F(\beta,\phi)=F_B(\beta,0)+\frac{2}{\beta}W_d(0)\left[\frac{ya}{L}\right]^2-\frac{1}{\beta\pi^{d/2}}\left[\frac{ya}{L}\right]^d\left[\frac{1}{d}\left|\Gamma\left[\frac{2-d}{2}\right]\right|+\mathcal{K}\left[\frac{d}{2}\middle|d^*;y\right]\right], \quad (66)$$

again correct to the leading order in (a/L). Now, Eq. (66) gives us free energy at constant λ; the one at constant $\mathcal{S}$ can be obtained from it through the Legendre transformation[8]

$$A(\beta,\mathcal{S})=F(\beta,\phi)-\frac{\lambda\mathcal{S}^2}{N}=F(\beta,\phi)-2Jd-4J\left[\frac{ya}{L}\right]^2. \quad (67)$$

Combining Eqs. (64)—(67), we get

$$A(\beta,\mathcal{S})=F_B(\beta,0)-2Jd+\frac{1}{\beta\pi^{d/2}}\left[\frac{ya}{L}\right]^d\left[\left[\frac{1}{2}-\frac{1}{d}\right]\left|\Gamma\left[\frac{2-d}{2}\right]\right|-\mathcal{K}\left[\frac{d-2}{2}\middle|d^*;y\right]-\mathcal{K}\left[\frac{d}{2}\middle|d^*;y\right]\right]. \quad (68)$$

The *singular part* of the *reduced free energy per unit volume* is thus given by

$$f^{(s)}(t;L)\equiv\frac{\beta}{a^d}A^{(s)}(\beta,\mathcal{S})=L^{-d}\left[\frac{y}{\sqrt{\pi}}\right]^d\left[\frac{1}{d}\Gamma\left[\frac{4-d}{2}\right]-\mathcal{K}\left[\frac{d-2}{2}\middle|d^*;y\right]-\mathcal{K}\left[\frac{d}{2}\middle|d^*;y\right]\right]. \quad (69)$$

It is now straightforward to show that, to leading order in (a/L), the singular part of the reduced specific heat per unit volume is given by

$$c^{(s)}(t;L)=-\frac{J}{a^d}\left[\frac{\partial\phi}{\partial T}\right]_{h=0}\simeq\frac{8}{a^d}\left[\frac{K_ca}{L}\right]^2\left[y\frac{dy}{dK}\right]=-\frac{32\pi^{d/2}K_c^2}{a^d}\left[\frac{ya}{L}\right]^{4-d}\times\frac{1}{\Gamma\left[\frac{4-d}{2}\right]+2\mathcal{K}\left[\frac{d-4}{2}\middle|d^*;y\right]}, \quad (70)$$

here, use has been made of the recurrence relation

$$\frac{d}{dy}[y^{2n}\mathscr{K}(n\mid d^*;y)]=-2y^{2n-1}\mathscr{K}(n-1\mid d^*;y)\ . \tag{71}$$

In contrast, the reduced susceptibility of the system per unit volume is given by the simple expression, see (38) and (50),

$$\chi^{(s)}(t;L)=\frac{T}{2a^dJ\phi}\simeq\frac{1}{8a^dK_c}\left[\frac{L}{ya}\right]^2, \tag{72}$$

here, for simplicity, μ_{eff} has been set equal to unity.

We are now in a position to verify the set of predictions made in Sec. III.

V. TESTS OF THE PRIVMAN-FISHER HYPOTHESIS

Restricting ourselves to temperatures close to the bulk critical point T_c, Eq. (65) reduces to

$$K_c\left[\frac{L}{a}\right]^{d-2}t=\frac{1}{8\pi^{d/2}}y^{d-2}\left[\left|\Gamma\left[\frac{2-d}{2}\right]\right|-2\mathscr{K}\left[\frac{d-2}{2}\middle|d^*;y\right]\right] \tag{73}$$

$$(\,|t|\ll 1)\ .$$

Recalling expression (20) for C_1, the left-hand side of (73) assumes the form $C_1L^{d-2}t$ and, since the index ν for the system under study is equal to $1/(d-2)$, it reduces to simply x_1, where x_1 $(=C_1tL^{1/\nu})$ is the scaled variable appropriate to the present problem. Expression (69) is thus manifestly in conformity with the hypothesis (1), for

$$f^{(s)}(t;L)=L^{-d}Y(x_1)\ , \tag{74}$$

where $Y(x_1)$ is given by the *parametric equations*

$$Y(y)=\left[\frac{y}{\sqrt{\pi}}\right]^d\left[\frac{1}{d}\Gamma\left[\frac{4-d}{2}\right]-\mathscr{K}\left[\frac{d-2}{2}\middle|d^*;y\right]-\mathscr{K}\left[\frac{d}{2}\middle|d^*;y\right]\right], \tag{75}$$

$$x_1(y)=\frac{y^{d-2}}{8\pi^{d/2}}\left[\left|\Gamma\left[\frac{2-d}{2}\right]\right|-2\mathscr{K}\left[\frac{d-2}{2}\middle|d^*;y\right]\right], \tag{76}$$

among which y is supposed to be eliminated; note that no nonuniversal metric factor, C_0, appears in front of the scaling function $Y(x_1)$ in Eq. (74). It is not surprising that expressions (70) and (72) for the specific heat and the magnetic susceptibility of the system are also in conformity with the corresponding formulas, (3) and (4), with

$$Y_{(1)}(y)=-\frac{32\pi^{d/2}y^{4-d}}{\Gamma\left[\frac{4-d}{2}\right]+2\mathscr{K}\left[\frac{d-4}{2}\middle|d^*;y\right]} \tag{77}$$

and

$$Y_{(2)}(y)=1/(8y^2)\ . \tag{78}$$

We shall now examine the behavior of the scaling functions Y, $Y_{(1)}$, and $Y_{(2)}$ in different regimes of t and L, and for different geometries of the lattice.

(a) $t>0$, $L\to\infty$. In this regime, $x_1\to+\infty$, with the result that y diverges while the functions $\mathscr{K}(y)$ vanish exponentially. Equations (75) and (76) then give

$$Y(x_1)\simeq\frac{\Gamma[(4-d/2)]}{d}\times\left[\frac{8\pi}{|\Gamma[(2-d)/2]|}\right]^{d/(d-2)}x_1^{d/(d-2)}\ ,$$

which conforms to the requirement (23), with $\alpha=(d-4)/(d-2)$, as given by Eq. (9), and with Y_+ as given by Eqs. (24), (22), (10), (20), and (12), i.e.,

$$Y_+=\frac{(d-2)^2E_+}{2dC_1^{d/(d-2)}}=\frac{(d-2)^2p_d}{2d}=\frac{(d-2)(8\pi)^{d/(d-2)}}{2d\left|\Gamma\left[\frac{2-d}{2}\right]\right|^{2/(d-2)}}\ .$$

At the same time we find that the functions $Y_{(1)}(x_1)$ and $Y_{(2)}(x_1)$, as given by Eqs. (76)–(78), reproduce exactly the bulk results (13) and (15), with appropriate values of α and γ, and with A_+ and G_+ as determined by Eqs. (14) and (16)—coupled with Eqs. (10)–(12).

(b) $t<0$, $L\to\infty$. In this regime, $x_1\to-\infty$ with the result that y tends to zero while the functions $\mathscr{K}(y)$ diverge. It is not difficult to show that, for $y\to 0$,

$$\mathscr{K}(n\mid d^*;y)\to\begin{cases}\frac{1}{2}\pi^{d^*/2}\Gamma\left[\frac{d^*}{2}-n\right]y^{-d^*}\quad(n<\frac{1}{2}d^*)\ , & \text{(79a)}\\ \pi^{d^*/2}y^{-d^*}[\ln(1/y)+\text{const}]\quad(n=\frac{1}{2}d^*)\ , & \text{(79b)}\\ \frac{1}{2}\Gamma(n)\sum_{\mathbf{q}(d^*)}{}'\,q^{-2n}y^{-2n}\quad(n>\frac{1}{2}d^*)\ . & \text{(79c)}\end{cases}$$

Equation (76) then gives

$$y(x_1) \simeq \begin{cases} \left[\frac{1}{8\pi^{d'/2}} \Gamma \left[\frac{2-d'}{2} \right] \right]^{1/(2-d')} |x_1|^{-1/(2-d')} \quad (d'<2) , & (80a) \\ \text{const} \times \exp(-4\pi |x_1|) \quad (d'=2) , & (80b) \end{cases}$$

where d' ($=d-d^*$) is the number of dimensions in which the system is infinite. For $d'>2$, we encounter a crossover to the new critical point, $T=T_c(L)$, a study of which would require a somewhat closer examination of the functions $\mathscr{K}(n \mid d^*;y)$ for $n>\frac{1}{2}d^*$; we hope to return to this aspect of the problem in a subsequent investigation. For a related discussion for $d>4$, see Ref. 8.

Focusing our attention on the specific heat and the magnetic susceptibility, we now find that, for $d'<2$,

$$Y_{(1)}(x_1) \simeq -A_- |x_1|^{-(4-d')/(2-d')} , \tag{81}$$

where

$$A_- = \frac{8}{(2-d')} \left[\frac{\Gamma[(2-d')/2]}{8\pi^{d'/2}} \right]^{2/(2-d')} . \tag{82}$$

This leads to an expression for the specific heat, $c^{(s)}(t;L)$, consistent with the prediction (27), with

$$\epsilon = 2(d-d')/(2-d') . \tag{83}$$

Passage to the bulk behavior (7b) thus takes place through a *power law* which reduces to the form $L^{-\epsilon}|t|^{1-\epsilon}$, where $\epsilon=3$ for a finite cube and 4 for an infinite cylinder of finite cross section. Only for $d'=2$, an example of which is provided by a film in three dimensions, do we expect an *exponential* behavior. To see it more explicitly, we go back to Eqs. (76) and (77) and set $d=3$, $d^*=1$; we obtain in a closed form

$$\mathscr{K}(\tfrac{1}{2} \mid 1;y) = -\frac{\sqrt{\pi}}{y} \ln(1-e^{-2y}) ,$$
$$\mathscr{K}(-\tfrac{1}{2} \mid 1;y) = \frac{\sqrt{\pi}}{e^{2y}-1} , \tag{84}$$

whence,

$$Y_{(1)}(x_1) = -32\pi y \tanh y , \tag{85}$$

with

$$y(x_1) = \sinh^{-1}(\tfrac{1}{2}e^{4\pi x_1}) . \tag{86}$$

For $x_1 \to -\infty$, $y \simeq \frac{1}{2}e^{-4\pi|x_1|}$, with the result that $c^{(s)}(t;L)$ vanishes as $L^{-1}\exp(-8\pi C_1 L |t|)$.

The limiting behavior of the scaling function, $Y_{(2)}(x_1)$, for the susceptibility of the system in this regime turns out to be, again for $d'<2$,

$$Y_{(2)}(x_1) \simeq G_- |x_1|^{2/(2-d')} , \tag{87}$$

where

$$G_- = \left[\frac{8\pi^{d'/2}}{\Gamma[(2-d'/2)]} \right]^{2/(2-d')} . \tag{88}$$

This leads to an expression for the susceptibility, $\chi^{(s)}(t;L)$, consistent with the prediction (29), with

$$\zeta = 2(d-d')/(2-d') , \tag{89}$$

the same as ϵ. It follows that, in this case too, passage to the corresponding bulk behavior (8b) takes place through a power law which reduces to the form $L^{\zeta}|t|^{\zeta-2}$ for $d=3$. Again, only for $d'=2$ do we obtain an exponential behavior. In the special case $d=1$, $d^*=3$, we find that

$$Y_{(2)}(x_1) \simeq \tfrac{1}{2}e^{8\pi|x_1|} \quad (x_1 \to -\infty) , \tag{90}$$

with the result that $\chi^{(s)}$ diverges as $L^2\exp(8\pi C_1 L |t|)$—in agreement with the corresponding result obtained earlier by Barber and Fisher.[8]

(c) Finally, at the erstwhile critical point ($t=0$), Eq. (76) reduces to

$$\mathscr{K}\left[\frac{d-2}{2} \middle| d^*;y_0 \right] = \frac{1}{2} \left| \Gamma\left[\frac{2-d}{2} \right] \right| , \tag{91}$$

which yields the universal number y_0. Since $y_0=O(1)$, in most cases it has to be obtained numerically. Once y_0 is known, one readily obtains the universal numbers $U=Y(y_0)$, $U_{(1)}=Y_{(1)}(y_0)$, and $U_2=Y_{(2)}(y_0)$, as defined in Eqs. (30)—(32). Again, in the special case $d=3$, $d^*=1$, y_0 is known exactly,[11] i.e., $y_0=\ln[\frac{1}{2}(\sqrt{5}+1)]=0.4812\ldots$. The values of y_0 for $d=3$ and $d^*=2,3$ have been obtained numerically by Pajkowski and Pathria[12] in the context of Bose-Einstein condensation in restricted geometrics.

In conclusion we find that the various predictions of the Privman-Fisher hypothesis on the hyperuniversality of finite systems are fully borne out in the case of the spherical model of ferromagnetism. Our analysis has enabled us to derive the set of scaling functions that govern the behavior of the system in the vicinity of the bulk critical point $T=T_c$. Making use of the asymptotic properties of these functions in different regimes of t and L, we have been able to study the size dependence of the various thermodynamic properties of the system both above and below T_c. This, in turn, has elucidated the manner in which the system under study approaches its bulk behavior as $L\to\infty$.

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